XXVIII.—An Investigation into the Effects of Errors in Surveying. By Henry Briggs, B.Sc., A.R.S.M. Communicated by Principal A. P. LAURIE.

(MS. received January 7, 1911. Read February 6, 1911. Issued separately April 18, 1911.)

This paper discusses the effects of errors in linear and angular measurements on the accuracy of surveys.

It is necessary to state at the outset that, as the investigation is based on the theory of probability, the conclusions drawn are only advanced in the hope that they will serve as guides in practical surveying, and must not be taken as of rigid precision. Perhaps it is superfluous to state that there is no method of determining beforehand the error which will accumulate in any survey; the utmost that can be done is to assign some mean value to the error which will serve as a useful criterion to which actual errors can be referred.

In dealing with probability far more difficulty is encountered in laying down trustworthy premises than in building mathematical structures upon them, and it would seem that the most satisfactory method of testing whether the premises are sound is to compare one's results as often as possible with those derived from experience. At several points the results in this paper have been so tested, and indeed some of them have turned out to be nothing else than algebraic representations of facts which have long been known in a general and more or less shadowy way to surveyors. In the last section of the paper the practical bearing of the results is emphasised, partly to show to what extent they conform with empirical conclusions.

The Average Error.

As it is impossible to deal with actual errors in a general investigation, choice has to be made of some form of mean error as a representative value. The *average error*, or *average deviation* as it would better be called, is selected for the purpose, firstly, because it is the standard of comparison most serviceable in a discussion of the relative accuracy of different results and processes; secondly, because it is more readily determined in practice than either the probable error or the mean-square error, and also because it is more easily understood by those unfamiliar with the theory of errors.

The average error in a series of equally trustworthy measurements of the same thing, whether a length or an angle, is defined as the arithmetical mean of the separate errors, taken either all with positive or all with negative signs.*

* To write the plus-or-minus sign before average errors, as is done throughout the paper, is unorthodox. An average error is usually defined as the arithmetical mean of the separate errors taken regardless of their signs, and is expressed in books as $\frac{\Sigma(+v)}{n}$. It is, however, equally logical to express it as $\frac{\Sigma(-v)}{n}$, and hence it appears to have the same right to the double sign as, say, the probable error. Moreover, to give an average error the positive sign only, and then to apply the usual theory of errors to it (as is generally done) seems to the writer inconsistent, since there is, in such an application, a tacit admission that the error is minus as often as plus.

The paper is divided into six sections, thus :—

Section I. The Sum of Vector Errors ;

Section II. The Average Error due to Imperfect Centring ;

Section III. The Relative Effects of Errors in Centring and those of Sighting and Reading ;

Section IV. The Propagation of Errors in Traversing ;

Section V. The Propagation of Errors in Minor Triangulation ; and

Section VI. A Summary of Results.

Notation.

The following notation is employed, all errors having the plus-or-minus sign :—

$x_1, x_2.$	Vector errors.
$R.$	The sum of two or more vector errors.
$v.$	The average error, due to sighting and reading, in an angle measured by the theodolite ; expressed in radians, unless otherwise stated.
$u.$	The average error in a bearing taken with a compass-instrument ; expressed in radians, unless otherwise stated.
$r.$	The displacement in centre due to the imperfect setting of a theodolite over a station ; expressed in feet, unless otherwise stated.
$T, T_1, T_2 \dots T_n.$	Traverse angles.
$t, t_1, t_2 \dots t_n.$	Average errors affecting traverse angles.
$S.$	The angle which is actually measured (in place of T) when the theodolite is displaced in centre by an amount r .
$\beta_1, \beta_2, \beta_3 \dots \beta_n.$	The bearings of traverse lines.
$\beta'_1, \beta'_2, \beta'_3 \dots \beta'_n.$	The average errors in the bearings of traverse lines.
$L, L_1, L_2, L_3 \dots L_n.$	The lengths of traverse lines, in feet.
$l, l_1, l_2, l_3 \dots l_n.$	The average errors in the lengths of traverse lines.
$W.$	The total length of a traverse.
$k.$	A coefficient, equal to $l \div \sqrt{L}$.
k_1 and $k_2.$	Values of k for the chain and steel-band respectively.
$A, B, C, \dots$	Triangulation angles.
$a, b, c, \dots$	The lengths of the sides of triangles, being respectively opposite $A, B, C, \dots$ The base of the scheme is the line c .
$a_1, b_1, c_1, \dots$	Average errors in the sides $a, b, c, \dots$ respectively.
$\frac{a_1}{a}, \frac{b_1}{b}, \frac{c_1}{c}, \dots$	Average fractional errors in the triangulation lines.
$z.$	A check-base of a triangulation scheme.
$z_A, z_B.$	The calculated and measured lengths of a check-base respectively.
$z_1, z_2.$	The average error in z_A and z_B respectively.
$z_3.$	The average value of the difference of z_A and z_B .
$n.$	The number of triangles in a triangulation scheme.
$1, 2, 3 \dots N.$	Triangulation stations.
$Z_N.$	The distance of a triangulation station, N , from station 1, the origin of the survey.

SECTION I. THE SUM OF VECTOR ERRORS.

Theorem.—The average error in the position of a point influenced by two or more vector errors is equal to the square root of the sum of the squares of the average magnitudes of the vector errors, and is independent of their relative clinures.

average sighting error over an exceptionally short line, say one 15 feet in length, is greater than for one of average length, say 200 feet; yet, on the other hand, there would not seem to be any sensible difference in accuracy of sighting as between lines, say, 200 and 400 feet long. Moreover, atmospheric unsteadiness and haze, which interfere to a greater extent with long lines, tend to nullify the increase in precision resulting from the finer definition of an object sighted at the extremity of a long draft, and will indeed outweigh it under certain atmospheric conditions. It would therefore be incorrect to assume that sighting errors shrink as the lines increase in length, and we shall approximate closely to the truth if we take this class of error as having a constant average value for all lines except the very shortest.

Let the average value of the combined errors of sighting, reading, etc., be $\pm v$ radians.

Average Error in Traverse Angles.

Leaving exceptionally short lines out of consideration, the error, t , in a traverse angle, T , is thus compounded of a constant component, v , and a variable component, namely the error due to imperfect centring.

Combining these, we have :

$$t = \pm \sqrt{\left\{ v^2 + \frac{4r^2}{\pi^2} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} - \frac{2 \cos T}{L_1 L_2} \right) \right\}} \quad (7)$$

The values of v and r vary greatly with different observers and instruments. To a surveyor it is not of the first importance to obtain results in which *general* average values of these quantities are assumed; in order to make fullest use of equation (7)—or, indeed, of any relation in this paper,—it is necessary that he should determine, as nearly as possible, his own average errors using his own instruments.

Experience in triangulation will be the best guide as to the magnitude of the sighting-and-reading error, v ,—for a triangulation line (as will shortly be shown) is almost always so long that the angular effect of a small centring displacement is negligible; hence, in measuring a triangulation angle, the second term on the right-hand side of (7) disappears, leaving v as the error in angle.

If, after considerable experience in minor triangulation, using, say, a reliable 5-inch transit theodolite, and employing three reiterations on each “face” in measuring the angles, it were found that the average error of summation of the three angles of a triangle was 12 seconds, the average error in each angle could be taken as $12 \div \sqrt{3}$, or 7 seconds. Now, in traversing it is not usual, or necessary, to employ the “reiteration” method of measuring angles, a traverse angle being more generally determined by averaging a single face-left and a single face-right observation; therefore, having assumed three reiterations were used in triangulation, the value of v for a traverse

angle so measured would be $\frac{12}{\sqrt{3}} \times \sqrt{3}$, or 12 seconds, for the particular observer and instrument.

Nor, if the surveyor carefully examine his own practice, is the displacement r difficult to ascertain—at any rate approximately. Thus, if he were in the habit, in ordinary traversing, of centring to within a quarter of an inch, he would disallow all displacements greater than that amount, but accept, as being sufficiently near, any of a smaller magnitude. Now, if it were true that there was an equal likelihood of the occurrence of any deviation from centre between zero and a quarter of an inch, the average deviation would be one-eighth of an inch; yet, if the actual method of centring be borne in mind—in which the instrument is moved until the point of the plumb-bob comes within an imaginary circle, in this case a quarter of an inch in radius—it will be evident that there is a greater chance of displacements approaching the maximum than of those approaching zero. Probably by taking three-quarters (rather than half) of the maximum deviation a value for r will be obtained approximating closely to the true average. In the present example this would give three-sixteenths of an inch as the average r .

Having determined, by the methods outlined, his own average centring displacement, r , and sighting-and-reading error, v , the surveyor can make use of relation (7) to ascertain whether he is justified in neglecting errors of these magnitudes in the case of any traverse angle he may select.

The relative effects of v and r are best shown by means of curves, and this process of analysis is perhaps most clearly elucidated by plotting curves for specific values of v and r . In selecting three-sixteenths of an inch, or 0·016 foot, as a suitable average amount for r , the writer believes himself safe from the accusation of having assumed a degree of precision in centring beyond that secured in ordinary practice; while 12 seconds probably approximates closely to a general average value of v for traversing, when a reliable 5-inch theodolite is used, and when the angle is obtained by taking the mean of single face-left and face-right observations.

The curves of fig. 3 are constructed from relation (7), making use of these values of v and r , and taking the lines enclosing the traverse angles as being of equal length, *i.e.* $L_1 = L_2$. Curve A is plotted for a traverse angle of 180° , B for one of 120° , and C for one of 60° , while the straight line D is for a traverse angle of 0° .

These graphs, besides illustrating the decrease in importance of centring displacements as the traverse angle becomes smaller, also show that while the centring error has a preponderant influence when the lines are short, its effect rapidly diminishes as they become longer.

The necessity of close centring on short lines is generally admitted, and in emphasising this fact, especially for traverse angles approaching 180° , lies the chief practical value of these curves. They illustrate, for example, that with a traverse angle of about 180° , and lines of less than 100 feet, nothing will be gained in accuracy by an endeavour to reduce the average sighting-and-reading error, so long as the

average centring displacement remains $\frac{3}{16}$ inch, and that greater precision can only be attained by closer centring.

FIG. 3.

In relation (7), when either v , or the average angular error due to centring becomes equal to or less than one-third of the other, it may be considered negligible, so far as its effect on t is concerned.*

* Compare with S. J. HOLMAN'S *Discussion of the Precision of Measurements*, New York, 1901, second edition, p. 69.

If this criterion of negligibility be first applied to determine for what length of traverse line, L , a sighting-and-reading error of 12 seconds (5.8×10^{-5} rads.) may be neglected when $r = 0.016$ foot, and $T = 180^\circ$, we have :

$$v \leq \frac{1}{3} \left\{ \frac{4r}{\pi L} \right\},$$

giving $L \geq 114$ feet as the result we require. A further justification, or, properly, an excellent excuse, may here be deduced for taking v as constant for all values of L , or, in other words, for drawing D , fig. 3, as a straight line; for though v , strictly speaking, must increase to some extent as L becomes shorter than 100 feet, it cannot do so at the same rate that the curve shows the centring error to increase. Even if v were to become 60 seconds for traverse lines 24 feet long, or five times the amount selected in constructing the curves, it would nevertheless be almost negligible under the above criterion when $r = 0.016$ foot and $T = 180^\circ$. It would therefore be of no immediate consequence to determine—were it possible—the rate of increase of v as the lines become very short, since the increase can have very little, if any, sensible effect.

The same criterion also allows of a test being applied on the legitimacy of the assumption, already made, to the effect that triangulation lines are usually of such a length that a small centring displacement has no tangible influence. An average sighting-and-reading error of 12 seconds will be reduced to $\frac{12}{\sqrt{3}}$, or 7 seconds, by three reiterations. Consider the case of two sides of a triangle of equal length, L , embracing an angle of about 60° . If $r = 0.016$ foot, L will be such that the average angular centring error is equal to or less than one-third of 7 seconds (expressed in radians) when $L \geq$ about 900 feet. As the sides of the main triangles in a minor triangulation scheme generally exceed 300 yards in length, and, moreover, since it is usual—though evidently not strictly necessary—to centre with greater care in triangulation than in traversing, we may safely conclude that centring errors in triangulation are negligible in their effects when compared with those we have termed sighting-and-reading errors.

Therefore we are justified in assuming, as we shall do when discussing the influence of errors in triangulation, that the average error in each of the angles of a triangle is of the same magnitude, being, in point of fact, equal to v , the sighting-and-reading error.

SECTION IV. THE PROPAGATION OF ERRORS IN TRAVERSING.

Two classes of errors have to be taken into account in determining the average error at the end of a traverse consisting of several lines—namely, *errors in bearing*, and *errors in linear measurement*. Before it is possible to deduce expressions compounding these errors, it is first necessary to consider them separately :—

Then the bearing β_m , of any line, is given by—

$$\begin{aligned}\beta_m &= \beta_1 + T_1 + T_2 \dots + T_{m-1} - 180^\circ(m-1), \text{ and} \\ \beta'_m &= \pm \sqrt{\{t_1^2 + t_2^2 \dots + t_{m-1}^2\}}\end{aligned}\quad (9)$$

From (7) we have that the average error in any angle—

$$t_m = \pm \sqrt{\left\{v^2 + \frac{4r^2}{\pi^2} \left(\frac{1}{L_m^2} + \frac{1}{L_{m+1}^2} - \frac{2 \cos T_m}{L_m L_{m+1}} \right) \right\}} \quad (10)$$

Combining expressions (9) and (10)—

$$\beta'_m = \pm \sqrt{\left\{ (m-1)v^2 + \frac{4r^2}{\pi^2} \left(\frac{1}{L_1^2} + \frac{2}{L_2^2} + \frac{2}{L_3^2} \dots + \frac{2}{L_{m-1}^2} + \frac{1}{L_m^2} \right) - \frac{8r^2}{\pi^2} \left(\frac{\cos T_1}{L_1 L_2} + \frac{\cos T_2}{L_2 L_3} \dots + \frac{\cos T_{m-1}}{L_{m-1} L_m} \right) \right\}} \quad (11)$$

This may also be written—

$$\beta'_m = \pm \sqrt{\left\{ \beta_{m-1}^2 + v^2 + \frac{4r^2}{\pi^2} \left(\frac{1}{L_{m-1}^2} + \frac{1}{L_m^2} - \frac{2 \cos T_{m-1}}{L_{m-1} L_m} \right) \right\}} \quad (12)$$

The latter form is of more service in studying in detail the propagation of angular error in an actual traverse, since the average error of the bearings are then worked out in succession, commencing with β'_2 .

By means of equations (11) or (12) the average error in summation of the angles of a closed traverse, or polygon, can be determined.

(2) *Errors in Linear Measurement.*

If a line were 660 feet long it would require a 66-foot chain or steel-band to be applied ten times to cover it; hence if the error of a single application of the chain or band were α , that for the whole line would be $\alpha\sqrt{10}$. Therefore, if L be the length of a line, and l the average error in that distance.

$$l = \pm k\sqrt{L} \quad (13)$$

The coefficient, k , is often taken as a constant for a given kind of instrument (chain or tape), but, strictly speaking, its value is dependent also on the length of the chain or band used, being smaller for a long than for a short one.

Average Total Error at the End of the nth Line of a Traverse.

(a) *In the Case of a Simple Compass Traverse.*—Consider the case of a traverse in which bearings are taken by a compass and lengths by an ordinary chain. Let u be the average error in each bearing, and k_1 the value k assumes for a chain. Let L_m be any line of the traverse, and l_m the average error in its length; then, from (13) we have—

$$l_m = \pm k_1 \sqrt{L_m} \quad (14)$$

Owing to the error in bearing, the end of any line L_m will be displaced by the average amount $L_m u$. Therefore the displacement of the point will be due to the resultant of the vector errors l_m and $L_m u$: that is to say, it will be $\pm \sqrt{k_1^2 L_m + L_m^2 u^2}$, by equation (1).

Now, this average displacement will be passed on to the next line of the traverse, and, indeed, to all subsequent lines; hence the end-point of the traverse will be affected by an average error compounded of all such errors as $\sqrt{k_1^2 L_m + L_m^2 u^2}$. Thus, by (1)—

$$\left. \begin{array}{l} \text{Average total error at the} \\ \text{end of the compass traverse} \end{array} \right\} = \pm \sqrt{\{k_1^2(L_1 + L_2 \dots + L_n) + u^2(L_1^2 + L_2^2 \dots + L_n^2)\}} \quad (15)$$

If the total length of the traverse, namely $(L_1 + L_2 \dots + L_n)$, is constant, a choice can often be made between performing the work by a few long lines or by a greater number of shorter ones. Now, the magnitude of $(L_1^2 + L_2^2 \dots + L_n^2)$ diminishes as the number of lines increases; therefore, in simple compass traversing, numerous short lines are preferable to few long ones when the average total error is the question of chief importance. This fact is pretty well known, but the graphical proof sometimes attempted by writers, though plausible at first sight, is unsound and incapable of withstanding careful scrutiny.

(b) *In the Case of a Theodolite Traverse.*—By means of relation (12) the average errors in the bearings of the traverse-lines are successively ascertainable; similarly, the average errors affecting the lengths may be determined from (13), writing k_2 in place of k , where k_2 is the value the coefficient assumes for a steel tape. Using the same notation as before, we have—

$$\left. \begin{array}{l} \text{Average total error at the end} \\ \text{of the theodolite traverse} \end{array} \right\} = \pm \sqrt{\{k_2^2(L_1 + L_2 \dots + L_n) + (L_2^2 \beta_2^2 + L_3^2 \beta_3^2 \dots + L_n^2 \beta_n^2)\}} \quad (16)$$

The total error may be analysed in the following way:—

The co-ordinates of the end point of the traverse, with reference to the first point as origin, are—

$$\begin{aligned} \text{Latitude} &= L_1 \cos \beta_1 + L_2 \cos \beta_2 \dots + L_n \cos \beta_n; \text{ and} \\ \text{Departure} &= L_1 \sin \beta_1 + L_2 \sin \beta_2 \dots + L_n \sin \beta_n. \end{aligned}$$

Therefore, by applying the theory of errors, we obtain—

$$\begin{aligned} \text{Average error in the latitude of the end point} \\ = \pm \sqrt{\{k_2^2(L_1 \cos^2 \beta_1 + L_2 \cos^2 \beta_2 \dots + L_n \cos^2 \beta_n) + (L_2^2 \beta_2^2 \sin^2 \beta_2 + L_3^2 \beta_3^2 \sin^2 \beta_3 \dots + L_n^2 \beta_n^2 \sin^2 \beta_n)\}} \quad (17) \end{aligned}$$

$$\begin{aligned} \text{Average error in the departure of the end point} \\ = \pm \sqrt{\{k_2^2(L_1 \sin^2 \beta_1 + L_2 \sin^2 \beta_2 \dots + L_n \sin^2 \beta_n) + (L_2^2 \beta_2^2 \cos^2 \beta_2 \dots + L_n^2 \beta_n^2 \cos^2 \beta_n)\}} \quad (18) \end{aligned}$$

It is seen that (16) may be obtained by compounding (17) and (18).

Of these last three equations (16) is the simplest and most useful; it allows of an assessment being made of the final error which might reasonably be expected in any theodolite traverse, whether “closed” or “open.” Relations (17) and (18) are of value when attention is directed to the error in some particular direction, for the co-ordinate axes can purposely be arranged so that the direction in question is either north-and-south, or east-and-west, with reference to them.

It follows from (16), since $\beta'_2, \beta'_3 \dots \beta'_n$ all attain their maximum values when $T_1, T_2, T_3 \dots T_n$ are all 180° , that, other things being equal, a *straight traverse* will

be affected with a greater average total error than any other traverse with the same number and lengths of lines.

The relative Accuracy of Compass and Theodolite Traversing.

Consider a straight traverse of n equal lines, running between two fixed points W feet apart. In these circumstances $nL = W$ (L being the length of any of the lines), and $T_1 = T_2 \dots = T_{n-1} = 180^\circ$. Therefore (15) reduces to—

$$\text{Average total error in a simple compass traverse} = \pm \sqrt{\{k_1^2 W + u^2 L W\}} \quad (19)$$

In like manner (16) reduces to—

$$\text{Average total error in a theodolite traverse} = \pm \sqrt{\left\{ k_2^2 W + \frac{W(W-L)}{2} \left(v^2 + \frac{16r^2}{\pi^2 L^2} \right) \right\}} \quad (20)$$

Now, while (19) diminishes as L diminishes, (20) increases as L diminishes; thus keeping W constant, it follows that there must be some value of L for which the average total error of the theodolite traverse equals the average total error of the compass traverse, and this condition will be secured when—

$$k_1^2 W + u^2 L W = k_2^2 W + \frac{W(W-L)}{2} \left(v^2 + \frac{16r^2}{\pi^2 L^2} \right).$$

In these circumstances L will be short, and unless the fixed points between which the traverses run are very close together, it is sufficiently exact to assume ($W - L$) as equal to W ; when the above simplifies to the following cubic equation:—

$$L^3 + L^2 \left\{ \frac{2(k_1^2 - k_2^2) - Wv^2}{2u^2} \right\} - \frac{8Wr^2}{\pi^2 u^2} = 0 \quad (21)$$

To illustrate the value of this result, let us make a comparison between the miner's dial (used as a simple compass instrument) and a 5-inch theodolite, when our experience with the instruments has led us to conclude that k_1 (for the 100-foot chain) = 0.0096, k_2 (for the 100-foot steel band) = 0.0063, $v = 12$ seconds, and $u = \frac{1}{8}^\circ$. Instead of taking $r = \frac{3}{16}$ inch, as has been done in former examples, we may profit by the results attained in an earlier part of the paper, and, now that we are dealing with exceptionally short lines, reduce the average centring displacement to $\frac{1}{8}$ inch. Substituting these values in (21), and taking W , the total length of the traverse as 1000 feet, L is found to be approximately 23 feet. In other words, under the stated conditions, if the lines of a straight traverse, all assumed equal, are more than 23 feet long, the theodolite may be expected to give more accurate results than the miner's dial; yet if they should be shorter than this figure the compass instrument will have the advantage, providing local magnetic attraction is absent. It also follows from (21) that this limiting length of line increases as W increases; that is to say, the *relative* accuracy of the two modes of traversing depends on the total length of the survey, being more in favour of the compass method with long traverses than with short ones.

Before leaving this section of the subject the writer desires to make it quite clear

that such a result as that just obtained (*viz.*, $L = 23$ feet) can only be a very rough one even under the stated conditions, and must by no means be taken as being generally correct. In this instance it is the method illustrated which possesses value, and not the result derived. There are so many factors to be taken into account that it is not possible to discuss the general case completely. In the above example we have, if anything, favoured the theodolite in assuming β'_1 the error in bearing of the first line of the traverse as being zero. Were a value to be taken for β'_1 , equation (8) would need to be applied to include its effect, $W\beta'_1$ being written instead of x_1 , and the "total error" from (20) instead of x_2 . Now, it is evidently quite impossible to give any specific value to β'_1 such as would serve, even approximately, as a general average; the magnitude of β'_1 depends on things at present outside our knowledge—namely, on what occurred prior to the commencement of the traverse under discussion. Questions such as the following immediately arise in reference to this initial error: Does the theodolite traverse commence from a triangulation station and become oriented by means of a preliminary sight over a triangulation line? If so, may it be assumed that the triangulation was conducted with such a degree of accuracy that the assumption $\beta'_1 = 0$ is sufficiently near the truth for our purpose? Or, was the traverse oriented by taking the magnetic bearing of the first line using, say, the trough compass?—in which case β'_1 can by no means be neglected. These, among other possible cases, will serve to show the difficulty of dealing with the problem in a general way; yet, in making a comparison of the compass and theodolite under any actually existing set of conditions, all questions such as the above are quickly answered.

When the comparison is made between a theodolite traverse referred to the true meridian and a compass traverse referred to the magnetic meridian, another factor of the first importance enters, namely, the accuracy by which the magnetic meridian has been determined. In the case where the compass can be sighted over a line—such as a triangulation line—whose true bearing has been carefully computed, the declination may be taken as being determined with an average error of u ; the average total error in the compass traverse must then be adjusted by means of (8), x_1 being Wu , and x_2 the value given by (19).

In short, while it is a matter of the greatest difficulty to make any serviceable *general* comparison between the accuracy of compass and theodolite traversing, it is not at all so difficult to make a comparison to suit any set of conditions which may arise in practice.

SECTION V. THE PROPAGATION OF ERRORS IN MINOR TRIANGULATION.

To solve any triangle of a triangulation scheme it is necessary to know the angles of the triangle and the length of one side. In the first triangle of the scheme the length of one side is actually measured, and this line is termed the *base* of the scheme. Every angle in every main triangle is measured by the theodolite; it is only in

By the theory of errors—

$$\begin{aligned} a_1 &= \pm \sqrt{\left\{ \left(\frac{\delta a}{\delta c} \right)^2 c_1^2 + \left(\frac{\delta a}{\delta A} \right)^2 v^2 + \left(\frac{\delta a}{\delta C} \right)^2 v^2 \right\}} \\ &= \pm \sqrt{\left\{ \left(\frac{\sin A}{\sin C} \cdot c_1 \right)^2 + \frac{c^2 v^2 \cos^2 A}{\sin^2 C} + \frac{c^2 v^2 \sin^2 A \cos^2 C}{\sin^4 C} \right\}} \\ &= \pm \sqrt{\left\{ \frac{\sin^2 A}{\sin^2 C} \cdot c_1^2 + a^2 v^2 \cot^2 A + a^2 v^2 \cot^2 C \right\}}. \end{aligned}$$

Therefore the average fractional error in α is determined by—

$$\frac{a_1}{a} = \pm \sqrt{\left\{ v^2(\cot^2 A + \cot^2 C) + \left(\frac{c_1}{c} \right)^2 \right\}}. \quad (24)$$

FIG. 4.

It is evident then, that the fractional error in α can be made to alter by varying the shape of the triangle, keeping the length of the base, c , and the fractional error in the base, $\frac{c_1}{c}$, constant. Substituting from (22) we obtain, for an isosceles triangle—

$$\frac{a_1}{a} = \pm \sqrt{\left\{ v^2(\cot^2 A + \cot^2 2A) + \left(\frac{c_1}{c} \right)^2 \right\}}. \quad (25)$$

and this is a minimum when $(\cot^2 A + \cot^2 2A)$ is a minimum. By differentiating the latter expression with respect to A and equating the result to zero, it is found that $A = 56^\circ 15'$ gives (25) a minimum value.

That is to say, the theoretically perfect triangle for triangulation purposes is an isosceles one, having the angles at the base $56^{\circ} 15'$ and that at the apex $67^{\circ} 30'$.

If a triangulation could be laid out composed entirely of triangles of this perfect shape, the lengths of the sides would gradually decrease as one proceeded further and further from the base. Now, this would not be desirable, since, as the sides shorten, centring errors would in time cease to have a negligible influence, and the ultimate effect would thus be to increase the average error in the angles. Again, with a scheme in which the triangles were gradually being reduced in size, more triangles would be needed to cover a given area of ground. Therefore, although the result last deduced possesses a considerable theoretical interest, there is no reason to depart from the opinion generally held, to the effect that the best shape of triangle for practical purposes is the equilateral.

Attempts are sometimes made to prove the equilateral to be the best-conditioned triangle by a graphical method; the premises, however, are generally defective, and in some cases quite unreal. The writer believes the problem to be too intricate to be treated adequately by any graphical mode.

The curves, fig. 5, are constructed from equation (25); they show the relation between the average fractional error, $\frac{\alpha_1}{\alpha}$, in the side, α , of an isosceles triangle, ABC , in which the apical angle, C , assumes all values between 0° and 180° . For curve A the fractional error in the base, namely $\frac{c_1}{c}$, is assumed negligible; for curve B it is taken as $1 \div 20,000$, or 5×10^{-5} , and for curve C as $1 \div 5000$, or 2×10^{-4} .

In all three cases the average error, v , in angle, is taken as 10 seconds (4.85×10^{-5} radians).

These curves supply a considerable amount of information of practical importance. They provide, first of all, a further justification in taking the equilateral as the perfect shape, the difference in $\frac{\alpha_1}{\alpha}$ being inconsiderable as between $C = 67^{\circ} 30'$ and $C = 60^{\circ}$.

Secondly, curve B , which is constructed with $\frac{c_1}{c}$ and v of similar magnitudes—a case pretty common in practice—illustrates how rapidly $\frac{\alpha_1}{\alpha}$ increases when C exceeds 120° or becomes less than 30° , but also shows that the *rate* of increase is not so rapid when C assumes unduly large as when it assumes unduly small values; it may therefore be taken to demonstrate the well-known rule to the effect that no important triangle should have an angle less than 30° or greater than 120° . On the other hand, curve C shows that, when the error in base greatly exceeds that in angle, these limits may be set much further apart without any appreciable reduction in accuracy. Indeed, when $v = 12$ seconds, and $\frac{c_1}{c} = 1 \div 5000$ the angle C may be given any value from 15° to 145° without risk.

We learn, then, that whether a triangle is to be considered permissible for triangulation

purposes does not depend merely on its shape, but also on the relative accuracy of the base and angles.

Now, since the lines of a triangulation become affected with a greater average error the further one proceeds from the base (see equation (30)), and as each of these lines virtually serves as base for the triangle depending on it, it follows that it is permissible in practice to use triangles towards the end of a scheme departing more from the perfect shape than those near the beginning. We can also conclude, that, whenever circum-

FIG. 5.

stances are such as to preclude accurate base-measurement, the best aim in laying out the scheme is not to arrange the triangles as nearly equilateral as possible, but purposely to elongate them so as to cover the area with as few triangles as possible.

Referring again to curve *B*, it is seen that even when the base is measured with an accuracy commensurate with that of the angles, too great a stress may be laid in practice on striving after the equilateral shape, for isosceles triangles in which the apical angles lie anywhere between 50° and 90° are all almost equally well-conditioned. This fact simplifies the subsequent work, for we can now proceed to consider schemes built of equilateral triangles, knowing that the results attained will be of almost equal practical value for triangles departing considerably from that shape.

greater part of them will run from one triangulation station to the next. Hence—to use a criterion of negligibility already employed (p. 856)—if the distance between two triangulation stations is known with an average error equal to or less than one-third of that accruing in a traverse connecting them, the closing error of the traverse can be assessed without there being need to take the triangulation error into account. For example, if the maximum accuracy in one of these traverses is likely to be 1 in 4000, a suitable accuracy for a triangulation line would be 1 in 12,000. Should it then be found, on completion of the triangulation, that the two values for the check-base agree to within one-twelve-thousandth of its length, it could safely be concluded that the required degree of precision has been attained.

When the actual fractional error in a check-base has been determined it is probable that the fractional error in all preceding lines will be less than that amount.

From considerations such as those just discussed a surveyor decides on the degree of accuracy he requires in the lengths of the triangulation lines, and the verification base allows him to tell if that precision has been reached.

One of the most important and most difficult questions in triangulation arises before the survey commences and after it has been decided what precision is required, and is, How carefully must the base and angles be measured in order that there may be a fair chance of this degree of accuracy being achieved?

A very satisfactory answer to this question can be obtained by the aid of relation (27), which, when the triangles are approximately equilateral, reduces to the following form—

$$\frac{z_1}{z} = \pm \sqrt{\left\{ \frac{2n}{3} v^2 + \left(\frac{c_1}{c} \right)^2 \right\}} . \quad (30)$$

By substituting what we may roughly term the “accuracy ratio” (1 in 12,000, 1 in 10,000, or whatever it may be) for $\frac{z_1}{z}$ in this equation, suitable values for v and $\frac{c_1}{c}$ may be ascertained. Theoretically the most equitable arrangement is for $\frac{c_1}{c}$ to equal $v \sqrt{\frac{2n}{3}}$; but this condition cannot always be satisfied. Circumstances may be such that it is easier to attain a low value of v than a low value of $\frac{c_1}{c}$, in which case $\frac{c_1}{c}$ may profitably be allowed to assume a higher, and v a correspondingly lower, value than those the strictly equitable arrangement would give;—for it must be borne in mind that the necessary degree of precision is best secured when the expenditure of labour is a minimum. In this connection it is possible that the following table, calculated from (30) will be useful as a guide. The table is computed on the assumption that an accuracy of 1 in 12,000 is desired in the check-base—a sufficiently high degree of precision for most minor triangulations—and it gives suitable values of v and $\frac{c_1}{c}$ for different numbers of roughly equilateral triangles:—

<i>Number of Triangles in the Scheme.</i>	<i>Average Error in Angle in Secs.</i>	<i>Average Error in Base.</i>	<i>Suggested Mode of Measuring Angles.</i>
<i>n.</i>	<i>v.</i>	<i>c₁ ÷ c.</i>	
1	12	1:14,500	Single F.L. and F.R. observation.
3	8·5	1:17,000	2 reiterations on each face.
5	8·5	1:28,000	Ditto.
	or 7	1:18,000	3 reiterations on each face.
7	7	1:25,500	Ditto.
9	6	1:23,500	4 reiterations.

The mode of measuring the angles entered in the last column refers to the use of a reliable modern 5-inch transit theodolite, with two verniers to the horizontal limb each reading to 20" of arc. Dogmatism would be very out of place in dealing with the mode of measurement best to employ to attain a certain average error in angle, surveyor's methods and instruments differing as they do; the methods named in column 4 of the table merely express the writer's individual experience with instruments of the above-named kind.

It could be urged that this process of determining suitable values for v and $\frac{c_1}{c}$ is faulty in one important respect, namely, that if the values stated in the table were used they would only ensure that the *average* error in the check-base would be 1 in 12,000, and not that the actual error would be *less than* that ratio. In other words, it would seem as though we had only secured roughly an even chance* of the actual error being below 1 in 12,000. In answer to this very logical objection, attention is called to the fact that the values for $\frac{c_1}{c}$ in the third column have purposely been kept considerably below those which will actually be obtained if the base is over 500 feet long, and if it is measured with a standardised steel tape by an approved method. While few will be willing to go so far as one writer on surveying, who states that the difference between successive measurements of a base should not exceed $1 \div 160,000$ of the length, it is pretty widely admitted that, when the temperature correction is applied, the standard tension given, and the slope determined, an accuracy of 1 in 50,000 can be secured.†

To show that the latter ratio is by no means excessive, and also to illustrate the method of ascertaining the accuracy of a base measurement, the following example is given. Attention is called to the fact that the discrepancies between the readings are, if anything, greater than those one usually expects in practice:—

Example.—After applying the necessary corrections, the results of six measurements of the length of a base were obtained as: 1014·52, 1014·59, 1014·63, 1014·50, 1014·60,

* More exactly, a chance of about 9 to 7, owing to the average error being equal to 1·18 of the probable error.

† Compare with J. B. JOHNSON'S *Theory and Practice of Surveying* (Wiley, New York).

and 1014·58 feet, of which the mean is 1014·57 feet. By subtracting each result from the mean, the *residual* errors are respectively found to be : ·05, ·02, ·06, ·07, ·03, ·01 ft., their signs being disregarded. By taking the arithmetical mean of these—which is found to be ·04 ft.—the *apparent* average error of a single measurement is obtained. Assuming that the same line of argument can be applied to apparent as to real errors,* the theory of errors states that the real average error of the mean can be determined from the apparent error of a single measurement by dividing the latter by $\sqrt{n-1}$, where n is the number of observations. Hence in this case the real average error of the mean equals $\cdot04 \div \sqrt{5}$, or ·018 ft. The average fractional error is therefore $\cdot018 \div 1014$, or about 1 in 56,300.

Besides the fact that reasonable care will give a degree of precision in the base measurement very superior to any stated in the third column of the above table, a further slight increase of accuracy will result from the distribution of error in the angles.† If the values of v in the table are adhered to as nearly as possible, the probability that the error in the check-base will work out under 1 in 12,000 is therefore in actual circumstances considerably more than one-half.

The Accuracy of Triangulation as a Method of Transmitting Distance.

Having determined how the various lines of a triangulation scheme are affected by error, it remains to discuss in what way these several errors combine in disturbing the positions of the stations.

Fig. 4 shows a chain of approximately equilateral triangles springing from a measured base, c . Let station 1 be taken as origin of co-ordinates, and line c as arbitrary meridian. An error in any point is stated with regard to station 1 as a fixed point and to the direction of line c as a fixed direction.

In dealing with the average total error affecting, say, the point 7 it is necessary to regard separately the influence of angular and of base error. The effect of the latter is easy to assess, since a fractional error in the base will cause the whole scheme to

* An assumption often made. In this case it is evidently of no use unless the tape has recently been standardised, for if not, constant error of comparatively large magnitude may be introduced which would render nugatory any such calculation as that above. The writer has seen calculations for probable error made on two measurements; the result (though believed to be valuable) was of course quite worthless. A base should be measured quite five times before the error can be analysed—and the mathematician will probably say that this number is insufficient.

The method of evaluating the average error in angle, given earlier in the paper, and based on the summation of the angles of a triangle, makes an attempt to assess the *real* error. Hence the need of obtaining the real average error in the base, as distinct from the apparent error, since the linear and angular errors are compounded in many of the expressions derived. From the strictly mathematical point of view there are objections to compounding errors determined in so different a way; however, the conclusions reached will be sufficiently near the truth to be of service in practice.

† The usual tendency is, perhaps to exaggerate the value of distributing error. It can be shown that the average angular error, v , is only reduced to $v \sqrt{\frac{2}{3}}$, or $0\cdot8v$, by adjusting the angles of a triangle by means of one equation of condition. (See CHAS. L. CRANDALL'S *Text-book on Geodesy and Least Squares*, 1907.)

If Z_N be the distance of station N from the origin, the average *fractional error* in that distance, due to angular error in the scheme, is :—

$$\frac{Z'_N}{Z_N} = \pm \frac{vc}{Z_N} \sqrt{\left\{ \begin{array}{l} \text{Sum of the squares of the coefficients of } vc \text{ for} \\ \text{stations } (N-1) \text{ and } (N-2) \text{ plus } \frac{4}{3} (N-2) \end{array} \right\}} \quad (39)$$

We have seen already that the average fractional error in Z_N due to an error, c_1 , in base measurement is $\pm \frac{c_1}{c}$, and we are now in a position to compound these errors to obtain :—

$$\left. \begin{array}{l} \text{Average fractional error in } Z_N \text{ due to} \\ \text{linear and angular errors combined} \end{array} \right\} = \pm \sqrt{\left\{ \left(\frac{Z'_N}{Z_N} \right)^2 + \left(\frac{c_1}{c} \right)^2 \right\}} \quad (40)$$

In this result it will be noticed that, while the effect of the angular error increases as the number of triangles increase, the influence of the fractional error in base measurement is constant; hence where greater precision of distance-transference is desired in an extensive scheme, it will generally pay better to devote attention to improving the angular rather than the linear measurements.

Another fact of the first importance established by the last result is the necessity of using as *few* triangles as possible in the scheme to cover the area under survey. As accuracy in distance-transmission is desired of every survey, this conclusion is hardly second in importance to the one that the triangles should be well-conditioned.

The Value of the Check-base as a means to Assessing the error in Distance Transmission.

The value of the check-base in allowing of the error in the *sides* of the triangles being roughly assessed has already been shown; we have yet to examine whether the wider claim—that the accuracy of the triangulation *as a whole* is determined by the discordance (expressed as a ratio) between the measured and calculated length of the line—is justifiable.

If the average fractional error in the check-base were exactly equal to the fractional error in Z_N (the distance of station N from station 1), it would follow that the right-hand sides of equations (29) and (40) would be equal, and it is evident that this is not generally the case. Since, then, the above-stated claim cannot be established in full, it remains to determine between what limits it is admissible.

The table below is drawn up to compare the average fractional error in Z_N with the average fractional difference between the calculated and measured values of the check-base, z , for the scheme of fig. 4. It is assumed that the check-base forms a side of the triangle whose apex is station N .

Station N .	Average Fractional Error in Z_N . (X).	Average Fractional Difference between Measured and Calculated Check-base. (Y).	$X^2 - Y^2$.
Apex of 3rd triangle	$\sqrt{\left\{ 2.33v^2 + \left(\frac{c_1}{c}\right)^2 \right\}}$	$\sqrt{\left\{ 2v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}$	$0.33v^2 - \left(\frac{z_2}{z}\right)^2$
Do. 5th do.	$\sqrt{\left\{ 3.85v^2 + \left(\frac{c_1}{c}\right)^2 \right\}}$	$\sqrt{\left\{ 3.33v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}$	$0.52v^2 - \left(\frac{z_2}{z}\right)^2$
Do. 7th do.	$\sqrt{\left\{ 6.58v^2 + \left(\frac{c_1}{c}\right)^2 \right\}}$	$\sqrt{\left\{ 4.67v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}$	$1.91v^2 - \left(\frac{z_2}{z}\right)^2$
Do. 9th do.	$\sqrt{\left\{ 11.79v^2 + \left(\frac{c_1}{c}\right)^2 \right\}}$	$\sqrt{\left\{ 6v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}$	$5.79v^2 - \left(\frac{z_2}{z}\right)^2$
Do. 11th do.	$\sqrt{\left\{ 22.08v^2 + \left(\frac{c_1}{c}\right)^2 \right\}}$	$\sqrt{\left\{ 7.33v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}$	$14.75v^2 - \left(\frac{z_2}{z}\right)^2$

The condition to be satisfied in order that the fractional error in the position of, say, the apex of the ninth triangle shall be equal to the proportional difference between the measured and calculated check-base is that $5.79v^2$ shall be equal to $\left(\frac{z_2}{z}\right)^2$. That

is to say, if v were 6 secs. (see table, p. 869), $\frac{z_2}{z}$ would have to be about 1 in 14,250—a

degree of accuracy inferior to that usually attained in practice in the measurement of a check-base for a scheme of this size. We may therefore conclude that if the triangles were arranged as in fig. 4, it would not be safe to consider the actual degree of agreement between the two values for the length of the check-base as a measure of the accuracy in position of the apex of the ninth triangle. Again, fig. 4 shows an exceptionally favourable arrangement of triangles for the rapid transference of distance; almost every other arrangement of the same triangles would result in a diminished Z_N , and therefore in an increased fractional error in Z_N . So far, then, this investigation has shown that the average fractional error in the check-base is always less than, and in many cases will be much less than, the average fractional error in distance-transmission when the number of triangles is nine. On the other hand, it can be shown in like manner that if the scheme is composed of only two or three triangles the accuracy in distance-transmission is superior, on the average, to that in the check-base. There must therefore be a value of n , the number of triangles, at which the average fractional error in each of these quantities is about equal, and, to be on the safe side, the writer places this at $n=5$.

Thus we are led to conclude that the fractional error in the verification base, as determined by comparing its measured and calculated length, may be taken, in general, as a reliable check on the accuracy of distance-transmission only if the number of triangles is less than five; but that when the triangles exceed five in number the check ceases to have value, owing to the rate of increase of the error in Z_N being more rapid than that in the check-base.

In this connection lies the chief utility of this portion of the investigation, for it provides a criterion on the accuracy of distance-transmission when the check-base ceases to afford a safe measure of it.

SECTION VI. SUMMARY OF RESULTS.

The conclusions reached in the foregoing portions of the paper are here collected in an abridged form under two heads, (*A*) and (*B*). Under the first of these heads are placed those results which may be considered as already recognised by surveyors, and under the second are given those which the writer believes to be new. The numbers in brackets call attention to the formulæ deduced in the previous sections, from which the various conclusions are derived :—

(*A*) (*a*) Displacements in centre have a greater influence on short than on long lines ; hence the necessity for closer centring on short lines. (See (7) and the curves, fig. 3, which allow this question to be studied fully.)

(*b*) Centring errors in triangulation are generally negligible.* [(7) and p. 857.]

(*c*) In compass traversing (loose-needle work) short lines are advisable [(15) and (19)], and in theodolite traversing, long ones [(16) and (20)]. The fact that a *superior* accuracy may be attainable by a compass instrument than by a theodolite when the traverse lines are very short is admitted by some.† A proof of this is given (p. 861), and it is shown that the length of line for which the accuracy becomes equal for the two kinds of instrument is capable of being roughly calculated.

(*d*) The best shape of triangle for triangulation purposes is the equilateral. It is shown [(25) and curves, fig. 5] that theoretically the best shape is an isosceles triangle having an angle of $67^{\circ} 30'$ at the apex ; fig. 5 indicates, however, that in this respect there is no appreciable difference between an equilateral triangle and one of this theoretically best shape, and for other reasons stated (p. 865) it is concluded that the equilateral may be considered the most suitable form for practical purposes.

(*e*) No angle in a triangulation triangle should be more than 120° or less than 30° . This rule, given in most text-books and forming an article of faith for almost all surveyors, is shown to be of value only under certain conditions. It is, however, always on the safe side, and in that respect may be taken as demonstrated. The question is dealt with more fully below.

(*f*) The accuracy of a triangulation may be tested by a comparison of the measured and calculated length of the check-base. Such a comparison is usually viewed as the most rigorous test that can be applied on the accuracy of a survey. It is generally assumed that if the difference between the two values for the length of the check-base is, say, $\frac{1}{12,000}$ of that distance, the accuracy of the survey as a whole is expressible by that

* See MIDDLETON and CHADWICK'S *Treatise on Surveying*, 1904, Part I., p. 193, for instance.

† For example, see "Notes on Railway Surveying," by C. J. ALBRECHT, *Min. Proc. I.C.E.*, cliv. p. 262, where a compass instrument is advised for running trial lines in country overgrown by forest, or in broken ground.

fraction. This is proved true only in part (p. 872). It is shown that such a ratio is of value in assessing the probable accuracy of the *individual lines* of the survey (p. 867), but not necessarily in determining the accuracy of the triangulation in distance-transmission. This latter point is further discussed below. It will also be evident that if the main and verification bases are measured by the same steel-tape, the latter will provide no check on *cumulative* errors in measurement, such as that resulting from the tape being incorrect in length.

(g) In a triangulation scheme covering a certain area of ground, the fewer the triangles the better (40). The importance of this fact is emphasised.

(h) It is permissible to use triangles towards the end of a scheme departing more from the perfect shape than those near the beginning.

(B) (a) Though essentially a mathematical theorem, the mode of summing vector errors (1) perhaps deserves first place in this series of new results, since much of the work—particularly the study of the accuracy of triangulation in distance-transmission—would have been very difficult, if not impossible, without it.

(b) A method of assessing the average error due to imperfect centring is given (Section II.), allowing of this class of angular error being studied in relation to that due to imperfect sighting and reading of a surveying instrument (Section III.).

(c) It is shown that the average error due to centring depends on the value of the traverse-angle, being a maximum when that angle is 180° (6).

(d) Formulæ are derived [(12), (15), and (16)] which allow of the average total error being computed in any traverse, whether “closed” or “open,” and whether run by a theodolite or compass instrument. By their application it is possible to determine if the error of closure in a “closed” traverse is less or greater than the average, and thus they provide a criterion of accuracy more satisfactory than that given by expressing the closing error as a fraction of the total length of the traverse. The chief objection to the formulæ is their cumbrousness, and the apparent impossibility of reducing their size by any process of approximation without at the same time nullifying any value they may possess. Trial will show, however, that computation by means of them is not such a lengthy process as would at first seem (providing tables of squares, square-roots, and reciprocals are at hand), since round numbers only need be employed for the lengths and angles. In obtaining the average value of the total error of a theodolite traverse, (12) is first used, and the results afterwards substituted in (16).

Perhaps the chief utility of these results lies in the fact that they enable the error *which might reasonably be expected* in any “open” traverse to be ascertained.

(e) By formulæ (17) and (18) the average error respectively in the total latitude and total departure of the end point of a traverse can be computed. These formulæ are of use when it is desired to find the error likely to occur in any given direction, one co-ordinate axis being, for the purpose of the calculation, arranged in that direction. For instance, in the well-known problem in practical surveying to determine the length and bearing of the closing side of a polygon, the average error in both these unknown

quantities could be studied by means of (17) and (18), roughly the direction of the closing side being taken, say, as arbitrary north for co-ordinates. (A preliminary plotting with protractor and scale, or perhaps a measurement from an old plan, will enable the direction of the missing line to be obtained sufficiently closely for this purpose.) Then the result as given by (17) would evaluate the average error in the closing side, while that given by (18) divided by the length of the line would express the average error in bearing in radians. Considering that this mode of analysis could be of great service, particularly in mine-surveying, it is a pity that it is so tedious to carry out and that it involves the calculation of co-ordinates, which, after the analysis is completed, would not usually be of service in plotting, since they are computed with reference to no definite meridian.

(f) A traverse running more or less in a straight line will on the average be affected by a greater error than any other—a closed traverse, for instance—having the same number and lengths of lines (16).

(g) Though equation (24) is not new,* the writer believes the inference drawn from it, to the effect that theoretically the best shape of a triangle for triangulation purposes is an isosceles one with an apical angle of $67^{\circ} 30'$, has not previously been published.

(h) With reference to the question as to what may be considered a *permissible* shape for a triangle, it is shown [(25) and curves, fig. 5] that the answer depends on the relative accuracy of the base and angles. The limits usually given— 30° and 120° —for angles of a main triangle are satisfactory when the base and angles are measured with a *similar* degree of precision, but should the accuracy of angular measure be considerably superior to that of the base, these limits may be set further apart with advantage (p. 865). A consideration of (40) will show that to measure the angles more precisely than the base is not so illogical a proceeding as might at first be supposed.

(j) Too great a stress can be laid on considerations of shape of triangles; even when the base and angles are measured with the same order of precision, isosceles triangles in which the apical angles lie between 50° and 90° are all almost equally well-conditioned (p. 866 and curves, fig. 5).

(k) Values for the average error in the calculated length of a check-base are obtained [(27) and (30)], and also for the average difference between the calculated and measured length [(28) and (29)]. From the former, suitable average values are deduced for the errors in angular and linear measurements so as to result in a given average error in a check-base. These values are set forth in tabular form (p. 869).

(l) Having regard to the relative accuracy in base and angles usually attained in practice, it is shown to be generally more desirable to increase the accuracy of the angular rather than of the linear measurements when a greater precision in distance-transmission is desired of a triangulation survey (40).

(m) It is dangerous to consider that the fractional error in the check-base, as determined from the measured and calculated values, expresses even approximately the

* Vide W. WEITBRECHT'S *Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate*, p. 63.

fractional error in distance-transmission of the scheme as a whole. Such an assumption will only be on the safe side when the number of triangles is less than five, and will become rapidly more incorrect the more that number exceeds five [(40) *et seq.*]. A table is given (p. 873) which allows of the difference between the accuracy of distance-transmission and that of the check-base being studied for different numbers of approximately equilateral triangles, when arranged in the form of a straight chain.

No matter what may be the number of triangles, it is possible, after the fractional difference between the measured and calculated length of the check-base has been computed, to assess roughly a value for the error in distance-transmission for any scheme. This may be done by the aid of (29) and (40).