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IX. *On Algebraical Couples.* By ARTHUR CAYLEY, Esq.,
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IT is worth while, in connection with the theory of quaternions and the researches of Mr. Graves (Phil. Mag., No. 173), to investigate the properties of a couple $\iota x + jy$, in which ι, j are symbols such that

$$\begin{aligned}\iota^2 &= \alpha \iota + \epsilon j \\ \iota j &= \alpha^1 \iota + \epsilon^1 j \\ j \iota &= \gamma \iota + \delta j \\ j^2 &= \gamma^1 \iota + \delta^1 j.\end{aligned}$$

If $\overline{\iota x + jy} \overline{\iota x_1 + jy_1} = \iota X + j Y$, then

$$X = \alpha x x_1 + \alpha^1 x y_1 + \gamma x_1 y + \gamma^1 y y_1,$$

$$Y = \epsilon x x_1 + \epsilon^1 x y_1 + \delta x_1 y + \delta^1 y y_1.$$

Imagine the constants $\alpha, \epsilon \dots$ so determined that $\iota x + jy$ may have a modulus of the form $K (x + \lambda y) (x + \mu y)$; there results one of the four following essentially independent systems:—

$$\begin{aligned}\text{A.} \quad \iota^2 &= \frac{1}{\lambda \mu} (\delta \lambda \mu + \gamma \overline{\lambda + \mu}) \iota - \frac{\gamma}{\lambda \mu} j \\ \iota j &= j \iota = \gamma \iota + \delta j \\ j^2 &= -\lambda \mu \delta \iota + (\gamma + \overline{\lambda + \mu} \delta) j \\ \begin{cases} X + \lambda Y = \frac{1}{\lambda} (\gamma + \lambda \delta) (x + \lambda y) (x_1 + \lambda y_1) \\ X + \mu Y = \frac{1}{\mu} (\gamma + \mu \delta) (x + \mu y) (x + \mu y_1). \end{cases}\end{aligned}$$

The couple may be said to have the *two linear moduli*,

$$\frac{1}{\lambda} (\gamma + \lambda \delta) (x + \lambda y) : \frac{1}{\mu} (\gamma + \mu \delta) (x + \mu y);$$

as well as the quadratic one,

$$\frac{1}{\lambda \mu} \overline{\gamma + \lambda \delta} \overline{\gamma + \mu \delta} \overline{x + \lambda y} \overline{x + \mu y},$$

the product of these, which is the modulus, and the only modulus in the remaining systems.

$$\begin{aligned}\text{B.} \quad \iota^2 &= -\delta \iota + \frac{1}{\lambda \mu} (\gamma + \delta \overline{\lambda + \mu}) j \\ \iota j &= j \iota = \gamma \iota + \delta j \\ j^2 &= (\overline{\lambda + \mu} \gamma + \lambda \mu \delta) \iota - \gamma j\end{aligned}$$

* Communicated by the Author.

$$\begin{cases} X + \lambda Y = \frac{1}{\mu} (\gamma + \lambda \delta) (x + \mu y) (x_1 + \mu y_1) \\ X + \mu Y = \frac{1}{\lambda} (\gamma + \mu \delta) (x + \lambda y) (x_1 + \lambda y_1). \end{cases}$$

C. $i^2 = \frac{1}{\lambda \mu} (\delta \lambda \mu + \gamma \overline{(\lambda + \mu)}) i - \frac{\gamma}{\lambda \mu} j$
 $i j = \left(\frac{\mu^2 + \mu \lambda + \lambda^2}{\mu \lambda} \gamma + \overline{\mu + \lambda \delta} \right) i + \left(-\delta - \frac{\lambda + \mu}{\lambda \mu} \gamma \right) j$
 $j i = \gamma i + \delta j$
 $j^2 = (\lambda + \mu \gamma + \lambda \mu \delta) i - \gamma j$
 $\begin{cases} X + \lambda Y = \frac{1}{\lambda} (\gamma + \lambda \delta) (x + \lambda y) (x_1 + \mu y_1) \\ X + \mu Y = \frac{1}{\mu} (\gamma + \mu \delta) (x + \mu y) (x_1 + \lambda y_1). \end{cases}$

D. $i^2 = -\delta i + \frac{1}{\lambda \mu} (\gamma + \delta \overline{\lambda + \mu}) j$
 $i j = (-\gamma - \overline{\lambda + \mu \delta}) i + \left(\frac{\lambda + \mu}{\lambda \mu} \gamma + \frac{\lambda^2 + \lambda \mu + \mu^2}{\lambda \mu} \delta \right) j$
 $j i = \gamma i + \delta j$
 $j^2 = -\lambda \mu \delta i + (\gamma + \overline{\lambda + \mu \delta}) j$
 $\begin{cases} X + \lambda Y = \frac{1}{\mu} (\gamma + \lambda \delta) (x + \mu y) (x_1 + \lambda y_1) \\ X + \mu Y = \frac{1}{\lambda} (\gamma + \mu \delta) (x + \lambda y) (x_1 + \mu y_1). \end{cases}$

The formulæ are much simpler and not essentially less general, if $\mu = -\lambda$. They thus become

A'. $i^2 = \delta i + \frac{\gamma}{\lambda^2} j$
 $i j = j i = \gamma i + \delta j$
 $j^2 = \lambda^2 \delta i + \gamma j$
 $X \pm \lambda Y = \pm \frac{1}{\lambda} (\gamma \pm \lambda \delta) \cdot (x \pm \lambda y) (x_1 \pm \lambda y_1).$

(Two linear moduli.)

B'. $i^2 = -\delta i - \frac{\gamma}{\lambda^2} j$
 $i j = j i = \gamma i + \delta j$
 $j^2 = -\lambda^2 \delta i - \gamma j$
 $X \pm \lambda Y = \mp \frac{1}{\lambda} (\gamma \pm \lambda \delta) (x \mp \lambda y) (x_1 \mp \lambda y_1).$

$$\begin{aligned}
 C'. \quad i^2 &= -\delta i + \frac{\gamma}{\lambda^2} j \\
 ij &= -\gamma i - \delta j \\
 ji &= \gamma i + \delta j \\
 j^2 &= -\lambda^2 \delta i - \gamma j \\
 X \pm \lambda Y &= \pm \frac{1}{\lambda} (\gamma \pm \lambda \delta) (x \pm \lambda y) (x_1 \mp \lambda y_1).
 \end{aligned}$$

$$\begin{aligned}
 D'. \quad i^2 &= -\delta i - \frac{\gamma}{\lambda^2} j \\
 ij &= -\gamma i - \delta j \\
 ji &= \gamma i + \delta j \\
 j^2 &= \lambda^2 \delta i + \gamma j \\
 X \pm \lambda Y &= \mp \frac{1}{\lambda} (\gamma \pm \lambda \delta) (x \mp \lambda y) (x_1 \pm \lambda y_1).
 \end{aligned}$$

There is a system more general than (A.) having a *single* linear modulus $q(\theta X + Y)$: this is

$$\begin{aligned}
 E. \quad i^2 &= \alpha(i - \theta j) + \theta^2 qj \\
 ij &= \alpha'(i - \theta j) + \theta qj \\
 ji &= \gamma(i - \theta j) + \theta qj \\
 j^2 &= \gamma'(i - \theta j) + qj \\
 \theta X + Y &= q(\theta x + y)(\theta x_1 + y_1);
 \end{aligned}$$

or, without real loss of generality,

$$\begin{aligned}
 E'. \quad i^2 &= \alpha i \\
 ij &= \alpha' i \\
 ji &= \gamma i \\
 j^2 &= \gamma' i + qj. \\
 Y &= qy y_1.
 \end{aligned}$$

To complete the theory of this system, one may add the identical equation

$$X + \frac{1}{\theta - qM} Y = \frac{q(\theta^2 - M\alpha)}{\theta - qM} \left(x + \frac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'} y \right) \left(x_1 + \frac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma'} y_1 \right),$$

$$\text{where} \quad M = \frac{\theta(\gamma - \theta\gamma') - (\alpha - \theta\alpha')}{\alpha'\gamma - \alpha\gamma'}.$$

By determining the constants, so that $\frac{1}{\theta - qM} = \frac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'}$
 $= \frac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma'}$, the system would reduce itself to the form A.

Cambridge, April 23, 1845.