THE INTRODUCTION OF DEMONSTRATIVE GEOMETRY.¹

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"When you come to the geometry class you must leave your common sense behind," is the most astounding statement that I have ever heard a teacher make to students. Some time ago I was visiting a geometry class in one of the larger high schools of this state. The class was in the first week of geometry study; after the class had gone through formally the usual proof of one of the earlier theorems, a bright youngster remarked, "I don't see the use of all that talk. I knew that theorem was true before by my common sense." Then came this most startling statement from the teacher, "Fred, when you come to the geometry class you must leave your common sense behind." No wonder there was great mental confusion and little, if any, interest evident in that room.

I was introduced to demonstrative geometry by an attempt to prove the obvious theorem, "All straight angles are equal," and remember my distress at the fuss made over the truth of so simple a statement when its truth seemed to me self-evident to anyone with ordinary sense.

These are not uncommon experiences, judging from the many people who tell of the long time it took them to see just what the purpose and value of geometry was. Just this week an excellent teacher told me he believed that success or failure in geometry was very largely determined by the character of the first few weeks' work in the subject. Widespread dissatisfaction with the old formal methods and appreciation of the importance and difficulty of a good introduction to demonstrative geometry have roused interest in the question before us for discussion, "What is an effective way to introduce class work in geometry?"

Before beginning a course in demonstrative geometry it is desirable that the students take some preliminary work in observational geometry. For example, such a course as that in concrete geometry recommended by the N. E. A. Committee of Ten on Secondary Studies (Report 1894, p. 110), or as that given in Hedrick's little book on *Constructive Geometry* (Macmillan Co., 1916). But since such courses are found in few of our schools we will assume in this discussion no more preliminary

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work in geometry than that which is gotten from drawing courses, arithmetic, and general daily experience. We will assume further that the work here outlined is for the ordinary class of second-year high school students, since demonstrative geometry is usually begun in the second year in most of our high schools. We offer no arguments for an informal beginning since there is quite a general agreement that such an introduction is necessary. That this need has long been appreciated in Germany, France, and Italy is shown by the numerous preliminary, or "propaedeutic," courses offered in their elementary and secondary schools, such as those outlined by Veronese, Holzmüller, and Simon, and by the able discussions of the merits of such courses as found in Reid's *Anleitung Zum Mathematischen Unterricht* (Berlin, 1906), in Schotten's *Inhalt and Methode des planimetrischen Unterricht* (Leipzig, 1890), and in Simon's *Methodik Des Rechnens und der Mathematik*.

There are three aims in this preliminary work: (1) to review, clarify, and informally develop from previous experience the fundamental notions of geometry; (2) to introduce the technical language of geometry, and (3) to create a feeling of need for proofs by slowly bringing students to the point of searching for reasons. We are too apt to forget that students beginning geometry already have much geometric knowledge which only needs to be clarified and translated into the technical language of geometry. Further, students will never appreciate what "geometry is all about" until they feel a real need for seeking reasons. The old and, unfortunately, too often the present practice of beginning by attempting to prove statements which seem perfectly obvious makes the whole subject of geometry seem trivial to the students and discourages interest. Carson well says, "one of the few really certain facts about the juvenile mind is that it revels in exploration of the unknown, but loathes analysis of the known" (*Mathematical Education*, Ginn & Co., 1913, p. 10). Klein is thinking of the same quality of the juvenile mind when he says, "At the beginning I pay no attention whatever to proofs, but am satisfied with the 'Behold' of the Hindus, until by very slow degrees I have brought my pupils to the point of searching for reasons" (*Jahresbericht der deutsch, Math. Ver.*, 1904).

What principles will guide us in attaining these aims? (1) There must be either a direct or an indirect basis of sense perception: (2) that which is new or general must be made

real by application to or connection with past experience; (3) the method must be that of analysis and induction rather than deduction; (4) there must be motivation through the early creation of a desire to find reasons for beliefs.

"There is nothing in the understanding which has not been first in the senses," is an old pedagogical maxim. Any idea we have if carefully analyzed may be run back to sense perceptions. This basis is shown by the difficulties in the education of such defectives as Helen Keller. So there must be some use for models and figures in beginning geometry as well as some measuring of lines and angles. But care should be taken not to overdo this work. Dewey says, "It is foolish to insist upon the observation of objects if the student is so familiar with the objects that he could just as well recall the facts independently" (*Democracy and Education*, p. 185). Frank H. Hall used pins stuck into flat cushions to give his blind students something in place of our figures. He said his blind students would use his pincushion figures only a few days because by that time they could imagine the figures and thought the pincushions a nuisance. It is deadening to interest to continue the use of models and concrete work beyond the point of usefulness. Basic sense perceptions are necessary in new experiences, but their continued use in familiar situations is both distasteful and detrimental.

"Vague generality" is a common term of criticism. General terms and general statements too frequently lack reality and do not carry meaning. Students seldom feel the need of following up definitions with one or two concrete illustrations or special cases. Teachers generally fail in developing the habit of illustration in their students, which habit is necessary evidence of reality or true comprehension.

Since textbooks in geometry almost universally present proofs in the deductive form there is grave danger that students will fail to see the true methods of originating proofs. Their great difficulty with originals is evidence of a lack of training in methods of attack or discovery. Have your students ever said, "Yes, I understand that proof and see that it is all true, but how did the author ever think of doing it that way?" Failure to develop the analytic and inductive methods of attack and too early emphasis upon deductive forms tends to simple memorization of proofs originated by others. "Scientific method applied to the teaching of solving mathematical problems means the method of discovery, the method of induction, the method of

analysis" (*Scientific Method*, F. W. Westaway, p. 412). Deduction is the method for presenting final results where the false starts, the wrong suggestions, the study of special cases, and the unsuccessful trials of discovery have been left out; in presenting proofs in deductive form all the scaffolding of discovery has been removed. If students see only polished deductive proofs in the textbooks, and neither originate a proof by themselves nor witness the teacher discover a proof, there should be no surprise at their lack of ability in the discovery of proofs.

As suggested at the beginning of this paper, the first great difficulty in beginning demonstrative geometry is to make the students see the purpose of it, to make them feel what we are driving toward. The poorest possible beginning is in attempting to prove theorems which seem obviously true to the students. We should early make them see that we are trying to give valid reasons for our beliefs about geometric things. Simply stated, a proof consists in the production of facts as sufficient reasons to support some conclusion, implicitly challenging a denial of either the truth or the relevance of these reasons. The first step is to get the students to make some statements which they believe to be true about some figures they have drawn; these can be brought out by good questions from the teacher. The second step is to call for reasons for their belief, and the third step is for the teacher to attack and show insufficient some reason given. In the beginning, obvious reasons should be accepted without discussion, meanwhile watching for an incorrect reason to attack. Probably the first wrong inference will be that some lines are equal or parallel because they look equal or parallel. That appearances are often deceptive may well be shown by certain optical illusions, which the teacher keeps at hand for such purpose. (See *1908 Proceedings of the Central Association of Science and Mathematics Teachers*, p. 177, and Wentworth-Smith *Plane Geometry*, p. 15, for useful optical illusions.) Likewise measurement always appeals to beginners as a good reason for the truth of equalities in geometry. Although often using measurements to suggest truths, yet students must early be convinced of the difficulty of making accurate drawings, that measurements are at best only approximations, and that more simple and precise tests for equality are needed. The writer feels that the difficulty of making accurate measurements, their approximate character, and the feeling of repulsion which the abler students manifest for too much measuring, is not sufficiently

appreciated by teachers. In an informal, conversational way students can be led to see weakness in their old methods of reasoning and to feel a need for better reasons for their beliefs; under a little skilful guiding they soon find delight in seeking reasons for their beliefs which will stand up against all attacks upon either their truth or relevance. Only after the students have attained some appreciation of the purpose of proofs should they be led to put their final results into the formal, precise, and elegant deductive form.

Having considered general principles, we shall now give them concrete application. The students should come to the first class provided with paper, pencil, ruler, compass, and protractor. Through oral directions teach the following constructions:

- (1) To bisect a given line segment.
- (2) To bisect a given arc of a circle.
- (3) To bisect an angle.
- (4) To erect a perpendicular to a line at a point in the line.
- (5) To drop a perpendicular to a line from a point without the line.
- (6) To trisect a right angle.
- (7) To inscribe a circle in a triangle.
- (8) To pass a circle through three points not in a straight line.
- (9) To draw a line parallel to a given line through an exterior point.

Make no attempt here to give the usual proofs of these constructions; if students question their truth give them approximate verification by measurement. Next have the students write a description of each construction in a notebook. Thus for (3) they would write something like the following:

To bisect any triangle BAC.

With A as a center and any radius r , describe an arc intersecting the angle sides in M and L. With M as a center describe an arc of a circle; with L as a center, and with the same radius describe another arc intersecting the latter arc in a point K. Join K to A and the line AK is the required bisector.

While learning these constructions and writing up their descriptions the students are introduced to the symbolism of geometry; they learn how to read lines and angles; use the simpler single letter notation wherever possible. While this work is being done in class periods under the direction of the teacher, assign for home work related construction problems, such as,

"Construct a square," "Inscribe a square in a circle," "Construct an inscribed hexagon by bisecting the sides of an inscribed triangle," etc. Geometric designs involving the fundamental constructions may be used for home work. (Hedrick's *Constructive Geometry* will suggest home work to the teacher.)

During all this work the teacher should have before him a list of geometric terms which he expects to fix in the students' minds; such as point (having position only), straight line (length only, no breadth or thickness), plane, solid, rectilinear figure, parallel lines, curves, circle, radius, diameter, chord, angle, right angle, acute, obtuse, complement, supplement, interior, exterior, vertical angles, bisect, trisect, etc. Develop the meaning of each of these terms by means of questions and illustrations; ask the students to point out examples of these terms on their figures; ask them to find examples in objects about them; translate old expressions, like "corners fit," "square corner," etc., into geometric language; compare their definitions with dictionary and text definitions; make these terms meaningful by continued and careful use. The originality and questioning skill of the teacher will direct this work of fixing fundamental notions and correlating them with their daily experiences.

If we are to avoid attempting to prove that which seems obvious to the students, there must be a broader foundation of assumptions than is common in beginning geometry. So the teacher should also have before him a carefully prepared list of obvious geometric facts which he expects to use as assumptions in later proofs, such as the following: (1) All straight angles are equal; (2) All right angles are equal; (3) The shortest path between two points is the line segment joining the points; (4) Two distinct points determine a straight line; (5) Any side of a triangle is less than the sum of the other two sides; (6) A diameter bisects a circle; (7) A straight line intersects a circle at most in two points; (8) Complements (or supplements) of equal angles are equal; (9) Two lines parallel to the same line are parallel to each other, etc. (The introductory chapter in Young and Schwartz's, or Wentworth-Smith's *Geometry*, or the *Report of the National Committee of Fifteen on Geometry*, p. 20, will assist a teacher in making such a list of fundamental geometric assumptions.) In most cases the emphatic statement of these fundamental facts is sufficient to bring conviction of their truth: informal discussion will bring acceptance of the truth of all; draw figures illustrating these fundamental facts; seek illustra-

tions in everyday experiences. After oral discussion has fixed these facts they should be carefully listed in the students' notebooks for future reference.

In order to make progress toward proving theorems, next develop with the class proofs for the simple theorems: (1) Vertical angles are equal; and (2) The bisectors of vertical angles form one and the same straight line.

Next take up the construction of triangles, from given data, as from two sides and their included angle, considering the usual four cases. Consider also their application to right triangles with necessary modifications in statement.

From this consideration of triangle construction make the transition to the formal proofs of congruent triangles as given in most texts, and begin from this point on to make use of the textbooks. By certain judicious omissions it is believed that this introductory oral work can be here connected up with any one of the textbooks now in common use.

Within the space here available, it is impossible to give all the details of the plan for introductory work here presented, but it is hoped that enough suggestions have been given to enable the teacher to complete all the details and adapt the scheme to his particular class and text. The length of time given to such an introduction must be determined by the preparation and the particular needs of each class. It may be quite short if students have had preliminary courses in constructive or concrete geometry. The prolonging of such informal work when not needed will tend to disgust the abler students.

The writer knows of no text on demonstrative geometry which gives a sufficiently informal introduction, and doubts whether such work can be put into a text with as effective results as when presented in class by the teacher. Some texts (Wentworth-Smith, Wells-Hart, Young-Schwartz, Long-Brenke, for illustration) however have greatly improved the introductory work.

This is one way of introducing demonstrative geometry. I hope the suggestions will assist the skilful teacher in making the study of geometry real, interesting, and valuable.