Review

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line is the excess of the number of positive maxima over the number of positive minima in the series of quantities

$$\begin{aligned} & (-S_n), S_1, S_2, \dots, S_n, (-S_1), \\ \text{where} \quad & S_1 = -t_1 + t_2 + \dots + t_n = \Sigma t - 2t_1, \\ & S_2 = -t_1 - t_2 + \dots + t_n = S_1 - 2t_2, \\ & \dots\dots\dots \\ & S_n = -t_1 - t_2 - \dots - t_n = S_{n-1} - 2t_n. \end{aligned}$$

By a maximum (or minimum) quantity is meant one which is greater (or less) than each of the adjacent quantities. The rule is applied to $S_1, S_2, \dots, S_n$ only, S_1 and S_n being counted as minima if less than the adjacent quantity, but not as maxima if greater than the adjacent quantity, since $S_1 < -S_n$, $S_n < -S_1$.

Thus if there are $2n+1$ equal forces alternately pushing and pulling, there are n real positions of equilibrium on each side of the line.

If $t_1, t_2, \dots, t_n$ are the absolute magnitude of the forces, and positive, the total number of real positions of equilibrium for all possible combinations of thrust and tension is the excess of the total number of positive signs over the total number of negative signs in all the expressions $-t_1 \pm t_2 \pm \dots \pm t_n$ which have positive values. This result (like the last) is independent of the distances between the given points, and (unlike the last) remains unchanged if the order of the points is changed.

The greatest possible number of real positions of equilibrium is found by subtracting each coefficient in the expansion of $(1+x)^{n-1}$ from the greatest coefficient, and adding all the results. This is counting the solutions on both sides of the line; and may be compared with the total number of solutions, real and imaginary, viz. $(n-1)(n-2)2^{n-3}$. The ratio is roughly that of 3 to $n^{\frac{3}{2}}$.

F. S. MACAULAY.

REVIEWS.

The Theory of Sets of Points. By W. H. YOUNG, M.A., Sc.D., and GRACE CHISHOLM YOUNG, Phil. Doc. (Gött.). (Cambridge: at the University Press. 1906. Pp. xii + 316.)

The theory of aggregates may roughly be divided into two parts: the one, which arose out of the profound study of integration, is associated with the names of Riemann, Hankel, Smith, Harnack, P. du Bois-Reymond, Lebesgue, and Young; while the other, which arose from the need for a secure basis of the theory of analytic functions, has, as characteristics, Weierstrass' and Cantor's theories of irrationals, Cantor's theory of derivatives of point aggregates, Cantor's theory of the 'powers' (*Mächtigkeiten*) of aggregates, Cantor's theory of transfinite numbers, and the modern developments of this last theory. The two parts are by no means disconnected; thus, Cantor has greatly

contributed to the theory of the "content" of an aggregate (the essential notion in the conception of integration), and Dr. Young, whose past and present work has been mainly concerned with what may be called the Riemann-direction, has given some excellent chapters on Cantor's ideas and numbers.

To quote the author's preface: "In subjects as wide apart as Projective Geometry, Theory of Functions of a Complex Variable, the Expansions of Astronomy, Calculus of Variations, Differential Equations, mistakes have in fact been made by mathematicians of standing, which even a slender grasp of the Theory of Sets of Points would have enabled them to avoid." This should raise interest in the theory even among mathematicians who do not concern themselves with fundamental questions, while, for those who do, this theory has, of course, shown itself to be indispensable.

On account of both the novelty of the subject as material for a textbook and the independence and originality of its treatment, the work of Dr. and Mrs. Young deserves the highest praise; in this notice, I have thought it would be more useful, instead of praising the many excellences of the book in detail, to direct attention to those few parts which, in my opinion, require modification or discussion.*

The correspondence between the real numbers and the points of a straight line is dealt with on pp. 9-15. By a fairly well-known extension of von Staudt's method of forming a projective number-scale, combined with the Cantor-Dedekind axiom, the result is reached (p. 14) that the continuum of real numbers and the linear continuum can be brought into a (1, 1)—correspondence, maintaining the order. Two points should be remarked. Firstly, Cantor stated his axiom (the date on p. 14 should be 1872 instead of 1878) in a form applying to the *metrical* geometry of the straight line. Secondly, all we know of that projective scale is that it is of type $\theta, \dagger$ —and series of *some* (which are not *all*) real numbers can be of this type (Harnack, Bettazzi $\ddagger$). Thus it seems that purely ordinal characteristics do not describe our notion of the straight line, but that a line (as a substratum to a point-set) is a set of points in *metrical* (not merely ordinal) correspondence with the system of real numbers. The difference between sets of points on a substratum and a set of intrinsically determined order is emphasised on pages 22-23, 127, and 287.

In the definition (p. 16)§ of what is meant by a 'set of points,' we read: "a linear set of points consists of points of a straight line determined by a certain law which is such that (1) every point of the straight line either belongs to the set or does not, but not both, nor neither; . . ." This condition seems so self-evident to the student that its meaning cannot be grasped at this stage. The importance of the condition lies in the fact that we can frame certain 'definitions' which *appear* to define valid classes ('sets'), and yet such a 'set' both

* There is (pp. 1-5) a short account of the system of real numbers as defined by (Cantor's) 'sequences.' I hope to give, in the near future, a brief discussion of what one can mean by the phrase 'definition of a number by a sequence' in this *Gazette* (cf. Russell, *The Principles of Mathematics*, vol. i., 1903, pp. 270-286).

† Cf. Russell, *op. cit.*, pp. 353, 384 *sqq.*

‡ *Ann. di mat.*, t. XVI. (1888-9), pp. 49-60.

§ An analogous definition is on p. 170.

is and is not a member of itself. Such 'sets' are Russell's 'class of the x 's such that x is not a member of itself' and the class known as the 'cardinal number of all the ordinal numbers.'*

It is logically better to separate the idea of 'potency' (power, or cardinal number) from that of measurement (cf. pp. 33, 76), for, as Russell† has shown, the latter idea is irrelevant to this part of mathematics, at least.

In the description (pp. 124-127) of Bernstein's diagrammatic representation of simply-ordered enumerable series, reference should, perhaps, be made to the prototype of this,—Schröder's representation by a 'matrix' of a binary relative.‡

There seems to be some confusion in the note on p. 159. The number ω^ω is not the number ω exponentiated by ω in any sense analogous to exponentiation as defined for cardinal numbers. The definitions of ordinal exponentiation given by Hausdorff and myself,§ on the contrary, are.

In Ex. 5 on p. 152, the symbolism $Lt n!$ is wrong; for the meaning of this is: The least cardinal which is greater than all the (finite) cardinals $n!$; and this is a ; whereas what we want is the product of all the finite cardinals, and this is equal to c .

In the discussions of the law of arbitrary choice and the multiplicative axiom (pp. 147, 289-291), it appears to me that Bernstein's conception of 'multiple equivalence' is irrelevant. Given any not-null class u , we can state certain propositions beginning ' x is a u ,' and it is irrelevant in how many ways x can be chosen: the multiplicative axiom is necessary in order to be able to say that ' $x, y, z, \dots$ (an infinity) are different members of u ' has any meaning at all (given, of course, that u is infinite). A case of this axiom, it should be stated, is involved in the definition (p. 151) of the product of an infinity of cardinals.

There are some small misprints: p. 19 (note), for 1881 read 1882, and for 1892 read 1882; pp. 124, 134, 301, 'Huntington' is spelt wrongly; on p. 155, for 2ω read $\omega \cdot 2$, and so on; on page 191 (note) the reference to Bolzano should be "Rein analytischer Beweis...", 1817 (lately republished, with notes, in Ostwald's *Klassiker der exakten Wissenschaften*); on p. 295, 'Lebesgue' is spelt wrongly, and on p. 299, 'Beiträge'; and (p. 155) a number is not a 'symbol.'

In the admirable Bibliography (pp. 295-309), it should, I think, be remarked that the French translations of Cantor in vol. 2 of *Acta Math.* are not always full; that the work mentioned in the 10th reference to Cantor is included in that mentioned in the 9th; and references to Dedekind's "Stetigkeit und irrationale Zahlen" of 1872, G. H. Hardy's important paper on absolutely convergent series in *Proc. Lond. Math. Soc.*, for 1905, Pringsheim's article on irrational numbers in the *Encykl. der Math. Wiss.*, and Russell's papers in *Mind* of 1905 (on "Denoting") and *Proc. Lond. Math. Soc.*, for 1906, should be given.

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* Cf. pp. 145-147, especially the suggestive ll. 16-25 on p. 146. † *Op. cit.*, pp. 111-116.

‡ *Algebra und Logik der Relative*, Leipzig, 1895, pp. 42-68.

§ *Mess. of Math.*, May, 1906, pp. 13-16.