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Multiple-Attribute Group Decision-Making Method under a Neutrosophic Number Environment

Abstract: As a neutrosophic number, which consists of a determinate part and an indeterminate part, can more easily and better express incomplete and indeterminate information that exists commonly in real situations, the main purposes of this paper are to provide a neutrosophic number tool for group decision-making problems with indeterminate information under a neutrosophic number environment and to develop a de-neutrosophication method and a possibility degree ranking method for neutrosophic numbers from the probability viewpoint as a methodological support for group decision-making problems. In group decision-making problems with neutrosophic numbers, through the de-neutrosophication and possibility degree ranking order of neutrosophic numbers, the ranking order of alternatives is performed well as the possibility degree ranking method has the intuitive meaning from the probability viewpoint, and then the best one(s) can be determined as well. Finally, two illustrative examples show the applications and effectiveness of the proposed method.

Keywords: Neutrosophic number, de-neutrosophication, possibility degree ranking method, group decision making.

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1 Introduction

Because of the ambiguity of people's thinking and the complexity of objective things in the real world, it is difficult to express people's judgments about some objective things by using crisp numbers. For example, the evaluation of people's morality cannot be always expressed by crisp numbers, while neutrosophic numbers proposed originally by Smarandache [4–6] may express it as a neutrosophic number consisting of a determinate part and an indeterminate part. Therefore, it can more easily and better express incomplete and indeterminate information that exists commonly in real situations. The neutrosophic number can be represented by $N = a + bI$, where a is the determinate part and bI is the indeterminate part. In the worst scenario, N can be unknown, i.e., $N = bI$. In the best scenario (when there is no indeterminacy related to N), $N = a$. Obviously, it is very suitable for the expression of incomplete and indeterminate information in complex decision-making problems. The evaluation information of attributes for alternatives given by decision makers is often incomplete and indeterminate. Therefore, the neutrosophic number can effectively handle the decision-making problem with incomplete and indeterminate information. Although neutrosophic numbers have been defined in neutrosophic probability since 1996 [4], little progress has been made in processing indeterminate problems by using neutrosophic numbers in scientific and engineering applications thereafter.

On the other hand, the neutrosophic set presented by Smarandache [4] can handle indeterminate information and inconsistent information and is a powerful general formal framework that generalizes the

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concept of the classic set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, paraconsistent set, dialetheist set, paradoxist set, and tautological set [4], and then it is represented by a truth-membership degree, an indeterminacy-membership degree, and a falsity-membership degree independently.

However, existing decision-making methods based on neutrosophic sets [2, 3, 10, 11], including their subclasses – single valued neutrosophic sets, interval neutrosophic sets, and simplified neutrosophic sets – cannot handle decision-making problems with neutrosophic numbers, as the neutrosophic set and the neutrosophic number are two subclasses of neutrosophy [4] and indicate different information forms and concepts. Then, the neutrosophic sets have been applied to decision making [2, 3, 10, 11], medical diagnosis [12], clustering analysis [9], image processing [1], etc., while little research in existing literature has been done on the application of neutrosophic numbers. To break through the applied predicament of neutrosophic numbers, this paper proposes a method for processing group decision-making problems with neutrosophic numbers, including a de-neutrosophication process and a possibility degree ranking method for neutrosophic numbers.

The main purposes of this paper are to provide a neutrosophic number tool for group decision-making problems with indeterminate information under a neutrosophic number environment and to develop a de-neutrosophication method and a possibility degree ranking method for neutrosophic numbers as a methodological support for group decision making. To do so, the remainder of the paper is organized as follows. In Section 2, we introduce some basic concepts related to neutrosophic numbers and a possibility degree ranking method for interval numbers. Section 3 proposes a possibility degree ranking method for neutrosophic numbers from the probability viewpoint, as the possibility degree ranking method has the intuitive meaning from the probability viewpoint. Section 4 develops a handling method for multiple-attribute group decision-making problems with indeterminate information under a neutrosophic number environment. In Section 5, two illustrated examples are provided to demonstrate the applications and effectiveness of the proposed group decision-making method, and then results and discussion are given. Section 6 gives the conclusions and future research directions.

2 Preliminaries

2.1 Basic Concepts of Neutrosophic Numbers and Some of Their Operations

Smarandache first proposed the concept of a neutrosophic number in neutrosophic probability [4–6], which consists of a determinate part and an indeterminate part. It is usually denoted as

$$N = a + bI,$$

where a and b are real numbers, and I is indeterminacy, such that $I^2 = I$, $0 \cdot I = 0$, and I/I = undefined.

For example, assume that a neutrosophic number is $N = 6 + I$, where $I \in [0, 0.3]$. Thus, it is equivalent to $N \in [6, 6.3]$; for sure, $N \geq 6$. This means that the determinate part of N is 6, while the indeterminate part of N is $I \in [0, 0.3]$, which means the possibility for number N to be a little bigger than 6.

Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two neutrosophic numbers. Then, Smarandache [4–6] gave the following operational relations of neutrosophic numbers:

1. $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$;
2. $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$;
3. $N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$;
4. $N_1^2 = (a_1 + b_1 I)^2 = a_1^2 + (2a_1 b_1 + b_1^2)I$;
5. $\dfrac{N_1}{N_2} = \dfrac{a_1 + b_1 I}{a_2 + b_2 I} = \dfrac{a_1}{a_2} + \dfrac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} \cdot I$ for $a_2 \neq 0$ and $a_2 \neq -b_2$;

6. $$\sqrt{N_1}=\sqrt{a_1+b_1I}=\begin{cases}\sqrt{a_1}-(\sqrt{a_1}+\sqrt{a_1+b_1})I\\ \sqrt{a_1}-(\sqrt{a_1}-\sqrt{a_1+b_1})I\\ -\sqrt{a_1}+(\sqrt{a_1}+\sqrt{a_1+b_1})I\\ -\sqrt{a_1}+(\sqrt{a_1}-\sqrt{a_1+b_1})I\end{cases}$$

Example 1. Let us have two neutrosophic numbers $N_1 = 4 + 2I$ and $N_2 = 6 + 4I$. Then, according to the above operational relations of neutrosophic numbers, we can calculate them as the following results:

1. $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I = 4 + 6 + (2 + 4)I = 10 + 6I$;
2. $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I = 4 - 6 + (2 - 4)I = -2 - 2I$;
3. $N_1 \times N_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I = 4 \times 6 + (4 \times 4 + 2 \times 6 + 2 \times 4)I = 24 + 36I$;
4. $N_1^2=a_1^2+(2a_1b_1+b_1^2)I=4^2+(2\times4\times2+2^2)I=16+20I$,
   $N_2^2=a_2^2+(2a_2b_2+b_2^2)I=6^2+(2\times6\times4+4^2)I=36+64I$;
5. $\frac{N_1}{N_2}=\frac{a_1}{a_2}+\frac{a_2b_1-a_1b_2}{a_2(a_2+b_2)}\cdot I=\frac{4}{6}+\frac{6\times2-4\times4}{6(6+4)}I=0.6667-0.0667I$;
6.

$$\sqrt{N_1}=\sqrt{a_1+b_1I}=\begin{cases}\sqrt{a_1}-(\sqrt{a_1}+\sqrt{a_1+b_1})I\\ \sqrt{a_1}-(\sqrt{a_1}-\sqrt{a_1+b_1})I\\ -\sqrt{a_1}+(\sqrt{a_1}+\sqrt{a_1+b_1})I\\ -\sqrt{a_1}+(\sqrt{a_1}-\sqrt{a_1+b_1})I\end{cases}$$
$$=\begin{cases}\sqrt{4}-(\sqrt{4}+\sqrt{4+2})I=2-4.4495I\\ \sqrt{4}-(\sqrt{4}-\sqrt{4+2})I=2+0.4495I\\ -\sqrt{4}+(\sqrt{4}+\sqrt{4+2})I=-2+4.4495I\\ -\sqrt{4}+(\sqrt{4}-\sqrt{4+2})I=-2-0.4495I\end{cases}$$
$$\sqrt{N_2}=\sqrt{a_2+b_2I}=\begin{cases}\sqrt{a_2}-(\sqrt{a_2}+\sqrt{a_2+b_2})I\\ \sqrt{a_2}-(\sqrt{a_2}-\sqrt{a_2+b_2})I\\ -\sqrt{a_2}+(\sqrt{a_2}+\sqrt{a_2+b_2})I\\ -\sqrt{a_2}+(\sqrt{a_2}-\sqrt{a_2+b_2})I\end{cases}$$
$$=\begin{cases}\sqrt{6}-(\sqrt{6}+\sqrt{6+4})I=2.4495-5.6118I\\ \sqrt{6}-(\sqrt{6}-\sqrt{6+4})I=2.4495+0.7128I\\ -\sqrt{6}+(\sqrt{6}+\sqrt{6+4})I=-2.4495+5.6118I\\ -\sqrt{6}+(\sqrt{6}-\sqrt{6+4})I=-2.4495-0.7128I\end{cases}$$

2.2 Possibility Degree Ranking Method for Interval Numbers

Let $\tilde{a}_1=[a_1^L, a_1^U]$ and $\tilde{a}_2=[a_2^L, a_2^U]$ be two interval numbers. To compare the two interval numbers, Xu and Da [8] defined the possibility degree of $\tilde{a}_1\ge\tilde{a}_2$ as

$$p([a_1^L, a_1^U]\ge[a_2^L, a_2^U])=\max\left\{1-\max\left(\frac{a_2^U-a_1^L}{a_1^U-a_1^L+a_2^U-a_2^L},0\right),0\right\}. \tag{1}$$

Assume that there are n interval numbers of $\tilde{a}_i=[a_i^L, a_i^U]$ ($i = 1, 2, \ldots, n$). Then, each interval number $\tilde{a}_i$ (i=1, 2, ..., n) compared with all interval numbers of $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) is expressed as follows:

$$p_{ij} = p(\tilde{a}_i \geq \tilde{a}_j) = \max\left\{1 - \max\left(\frac{a_j^U - a_i^L}{a_i^U - a_i^L + a_j^U - a_j^L}, 0\right), 0\right\}. \tag{2}$$

Then, the matrix of possibility degrees can be constructed as $p = (p_{ij})_{n\times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, and $p_{ii} = 0.5$. The value of $\tilde{a}_i$ ($i = 1, 2, \ldots, n$) for ranking order is given by r_i [7, 8]:

$$r_i = \frac{\left(\sum_{j=1}^{n} p_{ij} + \frac{n}{2} - 1\right)}{n(n-1)}. \tag{3}$$

3 Possibility Degree Ranking Method for Neutrosophic Numbers

The possibility degree ranking methods proposed in Refs. [7, 8] have the intuitive meaning from the probability viewpoint and are important tools for handling decision-making problems. In this section, the possibility degree ranking method of interval numbers is extended to neutrosophic numbers to propose a possibility degree ranking method for neutrosophic numbers.

Assume that $N_i = a_i + b_i I$ with $I \in [\beta^-, \beta^+]$ ($i = 1, 2, \ldots, n$) is any neutrosophic number for a_i, b_i, β^-, $\beta^+ \in R$, where R is all real numbers. Then, the neutrosophic number N_i is equivalent to $N_i \in [a_i + b_i\beta^-, a_i + b_i\beta^+]$. To compare N_i with N_j ($i, j = 1, 2, \ldots, n$), we can give the possibility degree $P_{ij} = P(N_i \geq N_j)$:

$$P_{ij} = P(N_i \geq N_j) = \max\left\{1 - \max\left(\frac{(a_j + b_j\beta^+) - (a_i + b_i\beta^-)}{(a_i + b_i\beta^+) - (a_i + b_i\beta^-) + (a_j + b_j\beta^+) - (a_j + b_j\beta^-)}, 0\right), 0\right\}. \tag{4}$$

Thus, the matrix of possibility degrees can be yielded as $P = (P_{ij})_{n\times n}$, where $P_{ij} \geq 0$, $P_{ij} + P_{ji} = 1$ and $P_{ii} = 0.5$. Then, the value of N_i ($i = 1, 2, \ldots, n$) for ranking order is given as follows:

$$q_i = \frac{\left(\sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1\right)}{n(n-1)}. \tag{5}$$

Hence, the neutrosophic numbers of N_i ($i = 1, 2, \ldots, n$) can be ranked in a descending order according to the values of q_i ($i = 1, 2, \ldots, n$).

Example 2. Let $N_1 = 4 + I$, $N_2 = 3 + 3I$, $N_3 = 3 + 2I$, and $N_4 = 4 + 3I$ with $I \in [0.3, 0.5]$ be four neutrosophic numbers. By the proposed ranking method, we can calculate the matrix of the possibility degree by Eq. (4) as follows:

$$P = \begin{bmatrix} 0.50 & 0.75 & 1.00 & 0.00 \\ 0.25 & 0.50 & 0.90 & 0.00 \\ 0.00 & 0.10 & 0.50 & 0.00 \\ 1.00 & 1.00 & 1.00 & 0.50 \end{bmatrix}.$$

By Eq. (5), the values of q_i ($i = 1, 2, 3, 4$) are obtained as follows:

$q_1 = 0.2708$, $q_2 = 0.2208$, $q_3 = 0.1333$, and $q_4 = 0.3750$.

As $q_4 > q_1 > q_2 > q_3$, the ranking order of the four neutrosophic numbers is $N_4 > N_1 > N_2 > N_3$.

4 Group Decision-Making Method with Neutrosophic Numbers

In this section, we present a handling method for multiple-attribute group decision-making problems with neutrosophic numbers.

In a multiple-attribute group decision-making problem with neutrosophic numbers, let $G = \{G_1, G_2, \ldots, G_m\}$ be a discrete set of alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ be a set of attributes, and $D = \{D_1, D_2, \ldots, D_s\}$ be a set of decision makers. If the kth ($k = 1, 2, \ldots, s$) decision maker provide an evaluation value for the alternative G_i ($i = 1, 2, \ldots, m$) under the attribute C_j ($j = 1, 2, \ldots, n$) by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, the evaluation value can be represented by the form of a neutrosophic number $N_{ij}^k = a_{ij}^k + b_{ij}^k I$ for $a_{ij}^k, b_{ij}^k \in R$ ($k = 1, 2, \ldots, s$; $j = 1, 2, \ldots, n$; $i = 1, 2, \ldots, m$). Therefore, we can elicit the kth neutrosophic number decision matrix N^k:

$$N^k = \begin{bmatrix} N_{11}^k & N_{12}^k & \cdots & N_{1n}^k \\ N_{21}^k & N_{22}^k & \cdots & N_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ N_{m1}^k & N_{m2}^k & \cdots & N_{mn}^k \end{bmatrix}.$$

If the weights of attributes and decision makers are considered as the different importance of each attribute C_j ($j = 1, 2, \ldots, n$) and each decision maker D_k ($k = 1, 2, \ldots, p$), the weighting vector of attributes is given by $W = (w_1, w_2, \ldots, w_n)^T$ with $w_j \geq 0$, $\sum_{j=1}^n w_j = 1$, and the weighting vector of decision makers is $V = (v_1, v_2, \ldots, v_s)^T$ with $v_k \geq 0$, $\sum_{k=1}^s v_j = 1$.

Then, the steps of the decision-making problem are described as follows:

Step 1: According to the decision matrix N^k ($k = 1, 2, \ldots, s$) provided by decision makers, by calculating $\bar{N}_{ij} = a_{ij} + b_{ij} I = \sum_{k=1}^s v_k \left(a_{ij}^k + b_{ij}^k I\right)$, a collective neutrosophic number decision matrix is obtained as follows:

$$\bar{N} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} & \cdots & \bar{N}_{1n} \\ \bar{N}_{21} & \bar{N}_{22} & \cdots & \bar{N}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{N}_{m1}^k & \bar{N}_{m2}^k & \cdots & \bar{N}_{mn} \end{bmatrix}.$$

Step 2: The product between the neutrosophic number decision matrix $\bar{N}$ and the weighting vector W is given as follows:

$$\begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_m \end{bmatrix} = \bar{N} \times W = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} & \cdots & \bar{N}_{1n} \\ \bar{N}_{21} & \bar{N}_{22} & \cdots & \bar{N}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{N}_{m1} & \bar{N}_{m2} & \cdots & \bar{N}_{mn} \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

where $G_i = (a_i + b_i I) = w_1(a_{i1} + b_{i1} I) + w_2(a_{i2} + b_{i2} I) +, \ldots, + w_n(a_{in} + b_{in} I)$ for $i = 1, 2, \ldots, m$.

Step 3: We propose a de-neutrosophication process in the decision-making problem. It is based on max-min values of I. A neutrosophic number can be transformed to an interval with two values, which are the maximum and minimum values for I, i.e., the lower limit is β^- and the upper limit is β^+. Thus, the neutrosophic number G_i is equivalent to $G_i \in [a_i + b_i\beta^-, a_i + b_i\beta^+]$.

Step 4: By applying Eq. (4), the possibility degree $P_{ij} = P(G_i \geq G_j)$ can be given by

$$P_{ij} = P(G_i \geq G_j) = \max\left\{1 - \max\left(\frac{(a_j + b_j\beta^+) - (a_i + b_i\beta^-)}{(a_i + b_i\beta^+) - (a_i + b_i\beta^-) + (a_j + b_j\beta^+) - (a_j + b_j\beta^-)}, 0\right), 0\right\}, \tag{6}$$

So, the matrix of possibility degrees is structured as $P = (P_{ij})_{m \times m}$.

Step 5: The values of q_i ($i = 1, 2, \ldots, m$) for ranking order are calculated by using Eq. (5).

Step 6: The alternatives are ranked according to the values of q_i ($i = 1, 2, \ldots, m$), and then the best one(s) is obtained.

Step 7: End.

5 Illustrative Examples

In this section, two illustrative examples for multiple-attribute group decision-making problems with neutrosophic numbers are presented to demonstrate the applications and effectiveness of the proposed decision-making method in realistic scenarios.

5.1 Example 1

An illustrative example about investment alternatives for a multiple-attribute group decision-making problem adapted from Reference [10] is provided to show the applications of the proposed decision-making method under a neutrosophic number environment. An investment company wants to invest a sum of money for the best option. Four possible alternatives are given to invest the money: (i) G_1 is a car company; (ii) G_2 is a food company; (iii) G_3 is a computer company; and (iv) G_4 is an arms company. The investment company must take a decision according to the three attributes: (i) C_1 is the risk factor; (ii) C_2 is the growth factor; and (iii) C_3 is the environmental factor. Assume that the weighting vector of the attributes is $W = (0.35, 0.25, 0.4)^T$. If three experts are required in the evaluation process and their weighting vector is $V = (0.37, 0.33, 0.3)^T$, the kth (k = 1, 2, 3) expert evaluates the four possible alternatives of G_i (i = 1, 2, 3, 4) with respect to the three attributes of C_j (j = 1, 2, 3) by the form of neutrosophic numbers $N_{ij}^k = a_{ij}^k + b_{ij}^k I$ for $a_{ij}^k, b_{ij}^k \in R$ (k = 1, 2, 3; j = 1, 2, 3; i = 1, 2, 3, 4).

For example, the first expert gives the neutrosophic number of an attribute C_1 on an alternative G_1 as $N_{11}^1 = 4 + I$ by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, which indicates that the mark of the alternative G_1 with respect to the attribute C_1 is the determinate degree 4 with an indeterminacy I. Thus, when the four possible alternatives with respect to the above three attributes are evaluated by the three experts, we can establish the following three neutrosophic number decision matrices:

$$N^1 = \left(N_{ij}^1\right)_{4\times 3} = \begin{bmatrix} 4+I & 5 & 3+I \\ 6 & 6 & 5 \\ 3 & 5+I & 6 \\ 7 & 6 & 4+I \end{bmatrix},$$

$$N^2 = \left(N_{ij}^2\right)_{4\times 3} = \begin{bmatrix} 5 & 4 & 4 \\ 5+I & 6 & 6 \\ 4 & 5 & 5+I \\ 6+I & 6 & 5 \end{bmatrix},$$

$$N^3 = \left(N_{ij}^3\right)_{4\times 3} = \begin{bmatrix} 4 & 5+I & 4 \\ 6 & 7 & 5+I \\ 4+I & 5 & 6 \\ 8 & 6 & 4+I \end{bmatrix}.$$

Thus, we utilize the proposed method for group decision-making problems with neutrosophic numbers to get the most desirable alternative(s). The computing procedure of the proposed method is described as follows:

Step 1: According to the above three decision matrices of N^k (k = 1, 2, 3), by calculating $\bar{N}_{ij} = a_{ij} + b_{ij} I = \sum_{k=1}^{3} v_k \left(a_{ij}^k + b_{ij}^k I\right)$ (i = 1, 2, 3, 4; j = 1, 2, 3), the collective neutrosophic number decision matrix can be obtained as follows:

$$\bar{N} = \left(\bar{N}_{ij}\right)_{4\times 3} = \begin{bmatrix} 4.33+0.37I & 4.67+0.3I & 3.63+0.37I \\ 5.67+0.33I & 6.3 & 5.33+0.3I \\ 3.63+0.3I & 5+0.37I & 5.67+0.33I \\ 6.97+0.33I & 6 & 4.33+0.67I \end{bmatrix}.$$

Step 2: The product between the neutrosophic number decision matrix $\bar{N}$ and the weighting vector W is as follows:

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \bar{N} \times W = \begin{bmatrix} 4.33+0.37I & 4.67+0.3I & 3.63+0.37I \\ 5.67+0.33I & 6.3 & 5.33+0.3I \\ 3.63+0.3I & 5+0.37I & 5.67+0.33I \\ 6.97+0.33I & 6 & 4.33+0.67I \end{bmatrix} \times \begin{bmatrix} 0.35 \\ 0.25 \\ 0.4 \end{bmatrix},$$

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} 4.135+0.3525I \\ 5.6915+0.2355I \\ 4.7885+0.3295I \\ 5.6715+0.3835I \end{bmatrix}.$$

Step 3: For de-neutrosophication in the decision-making problem, assume that the lower limit is taken as $\beta^- = 0$ and the upper limit is taken as $\beta^+ = 0.5$ to consider the maximum and minimum values for indeterminacy I, which are determined by the decision makers' preference or requirements in real situations. Thus, the neutrosophic number G_i is equivalent to $G_i \in [a_i, a_i + 0.5b_i]$ for $i = 1, 2, 3, 4$.

Step 4: By applying Eq. (6), the matrix of the possibility degree $P_{ij} = P(G_i \geq G_j)$ can be given as follows:

$$P = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.5000 & 1.0000 & 0.4451 \\ 1.0000 & 0.0000 & 0.5000 & 0.0000 \\ 1.0000 & 0.5549 & 1.0000 & 0.5000 \end{bmatrix}.$$

Step 5: By Eq. (5), the values of q_i ($i = 1, 2, 3, 4$) for ranking order are obtained as follows:
$q_1 = 0.125$, $q_2 = 0.3288$, $q_3 = 0.2083$, and $q_4 = 0.3379$.

Step 6: As $q_4 > q_2 > q_3 > q_1$, the ranking order of the four alternatives is $G_4 > G_2 > G_3 > G_1$. Hence, the alternative A_4 is the best choice among all the alternatives in the specific indeterminate range.

On the other hand, if we consider different ranges of the indeterminate degree for I, by Steps 3–6, we can obtain different results, as shown in Table 1.

5.2 Example 2

Let us consider a decision-making problem of alternatives in the flexible manufacturing system. Suppose a set of four alternatives for the flexible manufacturing system is $G = (G_1, G_2, G_3, G_4)$. We must take a decision according to the three attributes: (i) C_1 is the improvement in quality; (ii) C_2 is the market response; and (iii) C_3 is the manufacturing cost. The four possible alternatives under the above three attributes are to be evaluated by a group of three decision makers corresponding to the evaluation values of neutrosophic numbers. Assume the weighting vector of the three attributes is $W = (0.38, 0.3, 0.32)^T$, and the weighting vector of the three decision makers is $V = (0.36, 0.38, 0.26)^T$.

Table 1: Decision Results Selecting Different Indeterminate Ranges for *I*.

I	$\{q_1, q_2, q_3, q_4\}$	Ranking Order
$I = 0$	/	$G_2 > G_4 > G_3 > G_1$
$I \in [0, 0.2]$	{0.1250, 0.3368, 0.2083, 0.3298}	$G_2 > G_4 > G_3 > G_1$
$I \in [0, 0.4]$	{0.1250, 0.3301, 0.2083, 0.3366}	$G_4 > G_2 > G_3 > G_1$
$I \in [0, 0.6]$	{0.1250, 0.3279, 0.2083, 0.3388}	$G_4 > G_2 > G_3 > G_1$
$I \in [0, 0.8]$	{0.1250, 0.3267, 0.2083, 0.3399}	$G_4 > G_2 > G_3 > G_1$
$I \in [0, 1]$	{0.1250, 0.3261, 0.2083, 0.3406}	$G_4 > G_2 > G_3 > G_1$

Suppose we invite three decision makers to make judgments and to give evaluation values of neutrosophic numbers according to a scale from 1 (less fit) to 10 (more fit) with indeterminacy I. Then, the evaluation values of an alternative G_i ($i = 1, 2, 3, 4$) on an attribute C_j ($j = 1, 2, 3$) are given from the three decision makers, as listed in the following three neutrosophic number decision matrices:

$$N^1 = \begin{bmatrix} 5.5 & 5.6+I & 2.5+I \\ 5.5 & 6.2 & 4.5 \\ 5+I & 6.5 & 5.5 \\ 6.8 & 6.3+I & 4.5 \end{bmatrix},$$

$$N^2 = \begin{bmatrix} 5+I & 5.6 & 3.5 \\ 4.5 & 6.5 & 4.3+I \\ 6 & 6.4+I & 5.6 \\ 6.5 & 7.2 & 5+I \end{bmatrix},$$

$$N^3 = \begin{bmatrix} 5.3+I & 5.8 & 3+I \\ 5+I & 6.5 & 4.6 \\ 6.5 & 6.5 & 6+I \\ 5.6 & 6.5+I & 5.5 \end{bmatrix}.$$

Then, we employ the developed approach to yield the ranking order of the alternatives and the most desirable one(s), which can be described as the following steps:

Step 1: by calculating $\bar{N}_{ij} = a_{ij} + b_{ij} I = \sum_{k=1}^{3} v_k \left(a_{ij}^k + b_{ij}^k I\right)$ ($i = 1, 2, 3, 4$; $j = 1, 2, 3$) according to the above three decision matrices of N^k ($k = 1, 2, 3$), the collective neutrosophic number decision matrix is obtained as follows:

$$\bar{N} = \left(\bar{N}_{ij}\right)_{4\times 3} = \begin{bmatrix} 4.33+0.37I & 4.67+0.3I & 3.63+0.37I \\ 5.67+0.33I & 6.3 & 5.33+0.3I \\ 3.63+0.3I & 5+0.37I & 5.67+0.33I \\ 6.97+0.33I & 6 & 4.33+0.67I \end{bmatrix}.$$

Step 2: The product between the neutrosophic number decision matrix $\bar{N}$ and the weighting vector W is as follows:

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \bar{N} \times W = \begin{bmatrix} 5.258+0.64I & 5.652+0.36I & 3.01+0.62I \\ 4.99+0.26I & 6.392 & 4.45+0.38I \\ 5.77+0.36I & 6.462+0.38I & 5.668+0.26I \\ 6.374 & 6.694+0.62I & 4.95+0.38I \end{bmatrix} \times \begin{bmatrix} 0.38 \\ 0.3 \\ 0.32 \end{bmatrix},$$

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} 4.6568+0.5496I \\ 5.2378+0.2204I \\ 5.945+0.334I \\ 6.0143+0.3076I \end{bmatrix}.$$

Step 3: For de-neutrosophication in the decision-making problem, assume that the lower limit is taken as $\beta^- = 0$ and the upper limit is taken as $\beta^+ = 1$ to consider the maximum and minimum values for indeterminacy I, which are determined by the decision makers' preference or requirements in real situations. Thus, the neutrosophic number G_i is equivalent to $G_i \in [a_i, a_i + b_i]$ for $i = 1, 2, 3, 4$.

Step 4: By applying Eq. (6), the matrix of the possibility degree $P_{ij} = P(G_i \geq G_j)$ can be yielded as follows:

$$P = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.5000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 0.5000 & 0.4125 \\ 1.0000 & 1.0000 & 0.5875 & 0.5000 \end{bmatrix},$$

Step 5: By Eq. (5), the values of q_i ($i = 1, 2, 3, 4$) for ranking order is obtained as follows:
$q_1 = 0.125$, $q_2 = 0.2083$, $q_3 = 0.326$, and $q_4 = 0.3406$.
Step 6: As $q_4 > q_3 > q_2 > q_1$, the ranking order of the four alternatives is $G_4 > G_3 > G_2 > G_1$. Hence, the alternative A_4 is the best choice among all the alternatives in the specific indeterminate range.

Similarly, if one considers different ranges of the indeterminate degree for I, by Steps 3–6, one can obtain different results, as shown in Table 2.

5.3 Results and Discussion

For Example 1, we can see from Table 1 that the ranking orders of the four alternatives are shown as $G_2 > G_4 > G_3 > G_1$ for $I \in [0, 0.2]$ and $G_4 > G_2 > G_3 > G_1$ for $I \in [0, 1]$. For Example 2, we can see from Table 2 that the ranking orders of the four alternatives are shown as $G_4 > G_3 > G_2 > G_1$ for $I \in [0, 2]$ and $G_4 > G_3 > G_1 > G_2$ for $I \in [0, 5]$. The two illustrative examples demonstrate that different ranges of indeterminate degrees for I result in different ranking orders of alternatives. Then, the group decision-making method proposed in this paper can deal with the decision-making problems with indeterminate information. If we do not consider the indeterminacy I in neutrosophic numbers (i.e., $I = 0$), then this group decision-making method reduces to the classical one with crisp values.

Furthermore, as different ranges of indeterminate degrees for I can affect the ranking order of alternatives in the group decision-making problems, the method proposed in this paper can provide a more general and more flexible way of selecting for decision makers when the indeterminate degree for I is assigned different ranges in the de-neutrosophication process. Therefore, the decision makers can select some ranges of indeterminate degrees for I according to their preference and/or real requirements and have flexibility in real decision-making problems.

Finally, let us compare the decision-making method based on neutrosophic numbers in this paper with existing decision-making methods based on neutrosophic sets [2, 3, 10, 11]. First, the decision-making method in this paper use the information of the neutrosophic number that consists of a determinate part and an indeterminate part, while existing neutrosophic decision-making methods [2, 3, 10, 11] use the information of the neutrosophic set (including its subclasses: single valued neutrosophic set, interval neutrosophic set, and simplified neutrosophic set), which consists of truth-membership degree, indeterminacy-membership degree, and falsity-membership degree independently. Obviously, existing decision-making methods based on neutrosophic sets [2, 3, 10, 11] cannot handle decision-making problems with neutrosophic numbers because the neutrosophic set and the neutrosophic number indicate different information forms and concepts. Then, little research in existing literature has been done on decision-making problems with neutrosophic numbers. Therefore, this paper provides a new decision-making method for solving decision-making problems with neutrosophic numbers. Furthermore, the method proposed in this paper can provide not only a simpler and more flexible algorithm for decision makers under an indeterminate environment but also a new application to break through the applied predicament of neutrosophic numbers. Therefore, the developed method will be more suitable for dealing with decision-making problems with indeterminacy and demonstrate its advantage.

Table 2: Decision Results Choosing Different Indeterminate Ranges for I.

I	$\{q_1, q_2, q_3, q_4\}$	Ranking Order
$I = 0$	/	$G_4 > G_3 > G_2 > G_1$
$I \in [0, 1]$	{0.1250, 0.2083, 0.3260, 0.3406}	$G_4 > G_3 > G_2 > G_1$
$I \in [0, 2]$	{0.1530, 0.1803, 0.3305, 0.3361}	$G_4 > G_3 > G_2 > G_1$
$I \in [0, 3]$	{0.1843, 0.1698, 0.3207, 0.3252}	$G_4 > G_3 > G_1 > G_2$
$I \in [0, 4]$	{0.2107, 0.1763, 0.3048, 0.3093}	$G_4 > G_3 > G_1 > G_2$
$I \in [0, 5]$	{0.2265, 0.1836, 0.2938, 0.2961}	$G_4 > G_3 > G_1 > G_2$

6 Conclusion

In this paper, we provided a neutrosophic number tool for group decision-making problems with indeterminate information under a neutrosophic number environment, and then developed a de-neutrosophication process and a possibility degree ranking method for neutrosophic numbers from the probability viewpoint as a methodological support for the group decision-making problems. In group decision-making problems with neutrosophic numbers, through the de-neutrosophication and possibility degree ranking order of neutrosophic numbers, the ranking order of alternatives is performed well as the possibility degree ranking method has the intuitive meaning from the probability viewpoint, and the best one(s) can be determined as well. Finally, two illustrative examples show the applications and effectiveness of the proposed method.

The proposed neutrosophic number multiple-attribute group decision-making method is very suitable for decision-making problems with indeterminacy and shows its advantage. A significant note is that the methods proposed in this paper will be extended to other applications, such as medical diagnosis and clustering analysis, which are our future research directions.

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