

The Teaching of Easy Calculus to Boys

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Source: *The Mathematical Gazette*, Vol. 7, No. 108 (Dec., 1913), pp. 201-208

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603182>

Accessed: 22-06-2016 09:06 UTC

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As a labour-saving device the value of considering the step or increment becomes obvious. When the study of the graphs of algebraic expressions is commenced the same method of working with the increment will save much of that tedious and brutalising arithmetic which Dr. Filon, in a communication which you will all recall, denounced as one of the vilest accompaniments of graphical work.

GRAPHS.

To digress for a moment. Many people have been disappointed to find that the enormous amount of time apparently spent on graphs has not yielded as a harvest a general appreciation of the notions of functionality, continuity, and so forth.

But if we look at the facts from the boy's point of view, which for this purpose is the right point of view, can we be surprised? He has spent fifteen minutes in tedious and dull arithmetic—three minutes in the tedious plotting of points. Then he is ready to start on another example without wasting time. I see here, fifteen minutes on arithmetic, three minutes plotting. What time *has* the boy spent on studying graphs? None. If we can save time on the arithmetic and devote it to a careful drawing of the curve—a figuring of the scales, a free use of colour—so as to produce a work of art, five minutes' opportunity to look at the finished work, we may be entitled to say that the boy has been studying graphs. Time spent on devising a suitable brief and expressive heading is exceedingly well spent.

The English lad who devised the advertisement of the London tube railways:

Underground	Quickest way
Anywhere	Cheapest fare

and the American lawyer who received \$1200 for drafting the notice: "STOP, LOOK, LISTEN" for the level crossings of the railways in the States earned their money well. The English firm above one of whose gates appears in imperishable metal the legend: "No way in at this entrance," and the railway companies who do not really wish to insist that "Passengers must cross the line by the bridge" were less happily inspired. In art, I believe that the painter who printed: "This is a lion" underneath his picture, did not succeed in founding a school of painting. Let his name, if it be known, be commemorated as a founder of the true school of graphs. Indeed, a student is often engaged in what is to him tedious and absorbing arithmetic or algebra when his teacher thinks that mechanics or economics is being studied. A teacher ought to be able to give his pupils a little leisure to contemplate results quietly, even though in some cases this may lead to a patent waste of five minutes, instead of a latent waste, to which no one would object, of the whole hour.

C. S. JACKSON.

(*To be continued.*)

THE TEACHING OF EASY CALCULUS TO BOYS.

THE majority of the boys who enter the Technical Day School of the Borough Polytechnic Institute come from elementary schools, a few only come from secondary schools. The general age of entry is about 13 years, and the boys are about up to the level of the VIIIth standard of a good Council School.

As far as we are able, we select for admission boys who have a mechanical or constructive bias. All the boys who enter are expected to take the full three-years' course of work. At the end of that time they go into engineering workshops.

They study in succession the regular geometrical solids, with the lines, surfaces and angles which we meet with in connection with them. Mathematical operations and methods of working are acquired and made use of as the need for them arises, and the geometrical properties are observed and noted as they occur. Thus we usually commence with the geometry and mensuration of the cube, which involves the straight line, right angle and square, and then take in succession the various prisms and pyramids, the cylinder and the cone. Upon commencing the third year's work of the course the boys have an acquaintance with the equation of the straight line, the manipulation and solution of a simple equation, and the use of Mathematical Tables; and it is at this point that the first introduction to the methods of the Calculus takes place.

The third year's work of the course deals with the parabola, cubic curve, sine curve, and the logarithmic curve, each being taken both graphically and numerically. The calculus work, which is of the simplest kind, is taken as it is naturally required in connection with the above curves, and has a very strong bias on the graphical side. It consists of differentiation and integration of the equations of these curves, finding areas of figures contained by the curves, volumes of solids of revolution, maximum and minimum values and sometimes finding centre of gravity.

The first introduction to the methods of the calculus is the showing that $\frac{\delta y}{\delta x}$ measures the slope of a line, and that in a straight line $\frac{\delta y}{\delta x}$ is constant, while in the parabola the value of $\frac{\delta y}{\delta x}$ is not constant.

This is shown in three ways:

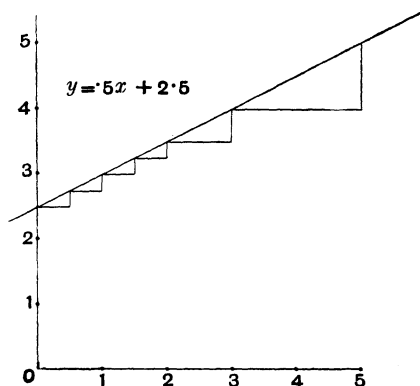
- (1) Numerically (by finite differences).
- (2) Graphically.
- (3) In general terms (*i.e.* algebraically).

(1) The equation of a particular straight line is taken, and by setting out in tabular form coordinate values of x , y , δx , δy , and $\frac{\delta y}{\delta x}$, it is shown that the value of $\frac{\delta y}{\delta x}$ is constant.

$$y = .5x + 2.5.$$

δx	x	y	δy	$\frac{\delta y}{\delta x}$
	0	2.5		
.5	.5	2.75	.25	.5
.5	1	3.0	.25	.5
.5	1.5	3.25	.25	.5
.5	2.0	3.5	.25	.5
1.	3	4.0	.5	.5
2.	5	5.0	1.0	.5

This is then represented by a graph in which the lines correspond to the numbers in the preceding tabular form.



Then each boy works such an exercise independently, choosing his own equation. They thus verify, both graphically and numerically, that in a straight line the value of $\frac{\delta y}{\delta x}$ is constant.

As a result of this we have a number of independent tests, all of which show that the $\frac{\delta y}{\delta x}$ of an equation of the form $y = a + bx$ is constant.

Note.—They may have found out the general rule.

The next step is to find whether this is true generally. This is shown by selecting a particular point (x, y) on the line $y = a + bx$, and taking points $(x - \delta x, y - \delta y)$ and $(x + \delta x, y + \delta y)$ on each side of it ;

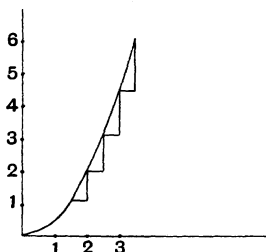
then

$$y - \delta y = a + b(x - \delta x)$$

and

$$y + \delta y = a + b(x + \delta x) ;$$

$$\therefore \frac{\delta y}{\delta x} = b.$$



A curve of the form $y = kx^2$ is next taken (e.g. $y = 5x^2$), and, as before, coordinate values of x , y , δx , δy , and $\frac{\delta y}{\delta x}$ are set out in tabular form, and it is shown that in this case the value of $\frac{\delta y}{\delta x}$ is not constant.

$$y = .5x^2.$$

δx	δx_1	x	y	$\frac{\delta y_1}{\delta y_2}$	$\frac{\delta y}{\delta x}$	
1	{	1.5	1.125	.875	2	NOTE : $x=2$; $x-\delta x_1=1.5$ $x+\delta x_1=2$. $y-\delta y_1=1.125$, $\delta y_1=.875$, $\delta y_2=1.125$.
		2	2	1.125		
		2.5	3.125			
1	{	3	4.5	1.375	3	
		3.5	6.125	1.625		

$$\delta x = 2 \cdot \delta x_1 ; \delta y = \delta y_1 + \delta y_2.$$

(2) The curve is plotted, and it is shown from the graph that the successive values of $\frac{\delta y}{\delta x}$ are not equal, but increase as x increases. (In this case they may or may not discover the general rule. Then each boy repeats the above exercise, choosing his own parabola, and verifying the result, as was done in the case of the straight line, and we, in this case also, have now a number of independent results showing that in curves of the form $y = k \cdot x^2$ the value of $\frac{\delta y}{\delta x}$ is not constant.

(3) The general case is taken. Here, as with the straight line, points on the curve $\{(x - \delta x_1), (y - \delta y_1)\}$ and $\{(x + \delta x_1), (y + \delta y_2)\}$ are taken on either side of the point (x, y) , and from the equations

$$(1) \quad y - \delta y_1 = k(x - \delta x_1)^2,$$

$$(2) \quad y + \delta y_2 = k(x + \delta x_1)^2,$$

we obtain
$$\frac{\delta y_1 + \delta y_2}{2 \cdot \delta x_1} = 2kx \quad \text{or} \quad \frac{\delta y}{\delta x} = 2kx ;$$

in which
$$\delta y = \delta y_1 + \delta y_2 \quad \text{and} \quad \delta x = 2 \cdot \delta x_1 ;$$

\therefore the value of $\frac{\delta y}{\delta x}$ depends upon the value of x , and is not constant.

The boys then work out a table of coordinates, setting it out in the same way as that given above. Then they (1) plot y against x and obtain a parabola ; (2) Plot $\frac{\delta y}{\delta x}$ against x and obtain a straight line ; (3) Given the curve (1) they obtain the derived straight line by graphic differentiation, without any reference to the table of coordinates.

A good many hints and some help will be needed here at first ; dividers should be used to take off the increments of y (take $\delta x = 1$ in the first case).

The straight line thus obtained by graphic differentiation should be compared with that obtained by plotting $\frac{\delta y}{\delta x}$ against x .

When enough exercises of this type have been worked, and the equations of the derived straight lines written down, then the reverse exercises should be worked, viz., given $\frac{\delta y}{\delta x} = 2kx$, find the relation between y and x .

(1) Coordinate values of $\frac{\delta y}{\delta x}$ and x are given in tabular form, and the corresponding values of δy , y and δx are obtained.

$$\frac{\delta y}{\delta x} = 2x.$$

δx	x	y	δy	$\frac{\delta y}{\delta x}$
1	$\left\{ \begin{array}{l} 2 \\ 2.5 \\ 3 \end{array} \right.$	$\left\{ \begin{array}{l} 4 \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	5
1	$\left\{ \begin{array}{l} 3.5 \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	7
1	$\left\{ \begin{array}{l} 4 \\ 4.5 \\ 5 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	9

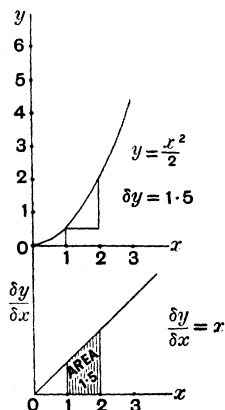
Note.—The boys will probably discover that unless one value of y is given they can commence equally well with any value, but I do not go into that question at this point, but just give them a value of y to start with.

(2) Given the straight line $\frac{\delta y}{\delta x} = 2kx$, obtain the integral curve by graphic integration. Very little explanation is needed here; the boys soon find out how to reverse the successive steps of the differentiating process.

Dividers are generally used here, and also the method of using a strip of paper and marking the values of $\frac{\delta y}{\delta x}$ in succession along it, and thus automatically adding them together, is used. Plenty of exercises should be worked, so that the method becomes quite familiar, and the various steps of the process should be illustrated numerically and followed out on a table of coordinates.

The next step is to point out the relation between these two curves, viz., that an area contained by the derived curve is given by the length of an ordinate of the integral curve.

This is made clear by actual illustration from a diagram and counting up of squares and numerical calculations.



It is then shown to be true generally, thus :

$$\text{Area} = \frac{\delta y}{\delta x} \times \delta x = \delta y,$$

and δy is the difference between the ordinates at x_1 and x_2 .

$\therefore \delta y$ represents the area of a portion contained by the curve, the x -axis and the ordinates at x_1 and x_2 .

And this is true for each successive δy , whence it follows that the whole of a larger area is given by the sum of a series of δy 's.

It is then shown, both from the diagram and from a tabular form, that the sum of any number of terms in the δy column can be obtained (without actual addition) by reference to the y column, *i.e.* that the difference between two terms in the y column gives the sum of the intervening terms in the δy column.

δx	x	y	$\frac{\delta y}{\delta x}$
	4	16	
1	4.5	...	9
	5	25	
1	5.5	...	11
	6	36	
1	6.5	...	13
	7	49	
1	7.5	...	15
	8	64	
1	8.5	...	17
	9	81	
			65

i.e. the sum of the terms shown in the δy column 9...17 is given by the difference between the 81 and the 16 of the y column.

It is well to spend time upon these points until they are clearly understood, a good number of examples being worked and explained. It is then pointed out that it is not necessary to graph the integral curve in order to get the result, but that its equation may be written down and the lengths of the required ordinates calculated.

Thus, given that $\frac{\delta y}{\delta x} = 2kx$, \therefore equation of the integral curve is $y = kx^2$.

When

$$x=1, \quad \text{then } y_1 = k,$$

$$x=2, \quad y_2 = 4k;$$

$$\therefore \text{area} = y_2 - y_1 = 3k,$$

i.e. the area contained by the line $\frac{\delta y}{\delta x} = 2kx$, the x -axis and ordinates at $x=2$ and $x=1$.

A number of examples should now be worked, and the work should, I think, be set out in the form indicated above, and every now and again the actual curve should be drawn and the length of the line read off. In other cases the diagram to scale may be omitted and a small sketch of the figure drawn.

Then later on the usual method of setting out the integral sign may be shown, *e.g.* $\text{area} = \int_a^b y \delta x$. But it is very essential to emphasise that the line on the integral curve represents the area on the derived curve between the same ordinates.

Concurrently with the working of these exercises is a convenient time to deal with the cubic curve. Taking the type $y = kx^3$, I have begun with the curve $y = \frac{x^3}{6}$ by drawing up a table of coordinate values of x , y , δx , δy , and $\frac{\delta y}{\delta x}$.

δx	δx_1	x	y	$\frac{\delta y_1}{\delta y_2}$	δy	$\frac{\delta y}{\delta x}$
1	·5	·5	$\frac{1}{8}$	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{1}{2}$
	·5	1	$\frac{1}{6}$	$\frac{1}{6}$		
1	·5	1·5	$\frac{9}{16}$	$\frac{3}{8}$	$2\frac{1}{4}$	$2\frac{1}{4}$
	·5	2	$1\frac{1}{3}$	$1\frac{5}{8}$		
		2·5	$2\frac{25}{8}$			

In this table the alternative values of δy are distinguished as δy_1 and δy_2 respectively, the two values taken together being called δy , and denoting the increment of y about its mean position.

The exercises that follow are :

- (1) The curve $y = \frac{x^3}{6}$ is graphed by plotting y against x .
- (2) $\frac{\delta y}{\delta x}$ is plotted against x and verified graphically, or the previous curve is differentiated.
- (3) This curve is compared with the curve $y = \frac{x^2}{2}$, which has been drawn previously, testing it either by tracing paper or by measurement, and noting that it is the same curve, but is shifted slightly in position.

It is, in fact, raised $\frac{1}{24}$ ", when $\delta x = 1$.

It is then shown that if the interval δx were taken larger than 1, then the derived curve would be raised still further ; and again, if δx be smaller than 1 the shifting of its position would be reduced.

We then proceed to investigate this by taking the general case

Let $y = kx^3$;
then $y - \delta y_1 = k(x - \delta x_1)^3$,
and $y + \delta y_2 = k(x + \delta x_1)^3$;
 $\therefore \frac{\delta y_1 + \delta y_2}{2\delta x} = k\{3x^2 + (\delta x_1)^2\}$
or $\frac{\delta y}{\delta x} = k\left\{3x^2 + \frac{(\delta x)^2}{4}\right\}$.

This is verified numerically from the previous table, taking

$$k = \frac{1}{6} \text{ and } \delta x_1 = \cdot 5,$$

and it is shown that in this case

$$k\{3x^2 + (\delta x_1)^2\} = \frac{x^2}{2} + \frac{1}{24}.$$

Several cases are worked out, such as when $\delta x = 2, \frac{1}{2}, 1$, etc., and in this way it is shown that while in practical work it is not possible to eliminate entirely the error, still we can see that as δx becomes smaller and smaller the result approximates to $3kx^2$, and that if we assumed that the effect due to the size of δx were entirely eliminated, then the result would be $3kx^2$. We can therefore compensate for this error, and so obtain the true result.

In graphic work this can be easily done by drawing a line Ox_1 parallel to the x -axis at a distance $k(\delta x_1)^2$, and taking this instead of the true axis. In this way we get $y = 3kx^2$ as the correct derived curve of $y = kx^3$.

Exercises follow to illustrate the above facts, in which the practice is adopted of writing $\frac{dy}{dx}$ when it is assumed that the error due to the finite size of δx has been eliminated.

The two curves, $y = kx^3$ and $\frac{dy}{dx} = 3kx^2$, are arranged on similar axes, and it is shown that an area on the derived curve is represented by the difference of the two corresponding ordinates on the integral curve, and this result is verified by counting squares, and by Simpson's rule.

Plenty of exercises follow on differentiation, integration, and finding areas.

Being now able to find the area of the figure contained by the straight line $y = a + bx$, the x -axis, and the ordinates at x_1 and x_2 , we proceed to find the volumes of solids of revolution. We assume that the figure rotates about the x -axis, and that then the strip $\frac{\delta y}{\delta x} \times \delta x$ or $y \delta x$ becomes a narrow disc, of volume $\pi \cdot y^2 \delta x$; and the sum of all these discs into which the space between x_1 and x_2 is divided is the volume of the frustum of a cone, and may be represented as $\pi \int_{x_1}^{x_2} y^2 dx$, in which $y = a + bx$.

Next, if in the expression $V = \pi \int_{x_1}^{x_2} y^2 dx$ we write 1 instead of π , we obtain the volume of a square pyramid.

Also $\cdot 433 \int_{x_1}^{x_2} y^2 dx$ will give the volume of a pyramid, with an equilateral triangle for base, and in this way the volumes of the regular pyramids are worked out.

Not infrequently the curves are graphed, and it is well to make the boys realise that the various cross sections of the pyramids are represented respectively by the ordinates of parabolas, while in the cubic curves, which are the integrals of the parabolas, the ordinates represent the volumes of the pyramids.

W. KNOWLES.

(To be continued.)

REVIEWS.

(1) **Map Projections.** By A. R. HINKS, M.A. 5s. (Cambridge University Press.)

(2) **Maps and Survey.** By A. R. HINKS, M.A. 6s. (Cambridge University Press.)

(3) **The Text-Book of Topographical and Geographical Surveying.** By Col. C. F. CLOSE, C.M.G., R.E. 3s. 6d. (H.M. Stationery Office.)

(4) **Map Projections.** By J. L. CRAIG, F.R.S.E., Survey Department, Cairo.

(5) **The Theory of Map Projections.** By J. L. CRAIG, F.R.S.E., Survey Department, Cairo.

(6) **Maps and Map Making.** By E. A. REEVES, F.R.G.S. 8s. (Royal Geographical Society.)

While the above treatises are mainly designed to meet the wants of those engaged in various stages of training for survey work, we believe that they will be consulted and appreciated by a much wider public, and that they will be studied by many readers of the *Mathematical Gazette*. The keener interest now taken in Geography as a school subject, and the widened outlook of the teacher of Mathematics may well encourage this hope. We have heard much of the fusion of Mathematics and Science, and it must be admitted that the practical steps taken by the Committee of the Association to promote it have added greatly to the interest of the Annual