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THE ELEMENTARY THEORY OF STATICAL STABILITY.

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THERE are some problems in elementary mathematics that the average student is apt to consider as matter for memorising without any great effort being either useful or advisable at understanding them. Experience in teaching elementary statics shows that the theory of stability is an example of such a problem. The average student often does not appreciate the principle of the method generally adopted to investigate the stability of the equilibrium of one body resting on another. He is not generally in a position to grasp the significance of an "infinitesimal displacement," or the meaning of "proceeding to the second order of small quantities" when the equilibrium is "neutral" to the first order. He is further puzzled by the fact that what appears to be neutral equilibrium becomes, when higher orders of small quantities are taken into account, stable in one case, unstable in another, whilst remaining neutral in a third. These difficulties can be overcome if we adopt a slight modification of the treatment given in the text-books. The object of this article is to explain the method used by the author for lecturing purposes, and not to obtain new results.

Let us take the elementary problem as ordinarily formulated. A body with a spherical base rests on a fixed body with a spherical top, sufficiently rough to prevent slipping when a displacement takes place; to investigate the nature of the equilibrium.

Let the figure represent a displacement of the top body. A is the original point of contact, C the centre of the lower sphere. B is the new point of contact, C' the new position of the centre of the top sphere, so that CBC' are collinear. $C'A'$ is the new position of that radius of the top sphere that was originally vertical, so that A' was originally at A . The centre of gravity of the top body is somewhere in the line $A'C'$, produced if necessary.

At B draw BN vertically to meet $C'A'$ in N . Following a notation suggested by Minchin, we shall call N the critical point for the displacement defined by the angle ACB ($=\theta$).

N will in general vary with θ . If the centre of gravity of the top body is below N , say at G , the gravity will bring the body back to its original position; if the centre of gravity is above N , say at G' , the gravity will

take the top body still further away from its original position ; if the centre of gravity is at N , neither effect will be produced.

Now suppose that when θ is zero, the critical point is at N_0 , and let θ be gradually increased from zero. The critical point, being dependent on θ , will either begin to rise, or to sink : as a limiting case it may happen to be functionally independent of θ and remain at N_0 . If N rises from N_0 , then when the centre of gravity is *below* or *at* N_0 , there will be a restoring couple in a disturbed position and the equilibrium is *stable*, but when the centre of gravity is *above* N_0 , the equilibrium is *unstable*, since for a small enough value of θ , the centre of gravity is above N . If N sinks from N_0 , then when the centre of gravity is *above* or *at* N_0 , it remains above N and the equilibrium is *unstable*, but when the centre of gravity is *below* N_0 , the equilibrium is *stable*, since for a small enough value of θ , the centre of gravity is below N . If N remains at N_0 , then when the centre of gravity is *below* N_0 the equilibrium is *stable*, if *above* N_0 the equilibrium is *unstable*, but if the centre of gravity is *at* N_0 the equilibrium is truly *neutral*.

To apply these results, let us call $A'N \equiv H$, the critical height for the angle θ . H can be easily calculated in terms of θ . Let the angle through which the top body has rotated be called ϕ , and let the radius of the top sphere be r and of the bottom sphere R ; then $r\phi = R\theta$, and, equating the horizontal distance of N from C' to that of B from C' , we get

$$r \sin \theta = (r - H) \sin (\theta + \phi),$$

i.e.

$$H = r \left\{ 1 - \frac{\sin \theta}{\sin \frac{R+r}{r} \theta} \right\} \dots \dots \dots (1)$$

We shall assume at present that neither body is flat, so that r and R are both finite. When θ is zero the critical height is

$$H_0 = \frac{Rr}{R+r} \dots \dots \dots (2)$$

N will rise from N_0 as θ increases from zero if H increases with θ ; N will sink if H decreases as θ increases from zero; N will remain at N_0 if H does not really involve θ .

The only case giving stationary N is when $\sin \frac{R+r}{r} \theta$ is a multiple of $\sin \theta$. For finite values of r, R this can only occur when $R = -2r$, and then $H = H_0 = 2r$. Thus, if the lower sphere is also concave upwards and its radius is double that of the top sphere, the equilibrium is truly neutral when the centre of gravity of the top body is at a height equal to the diameter of the top sphere; the equilibrium is stable if the centre of gravity is lower, unstable if higher.

To investigate the general case we use the following facts :

$$\frac{\sin n\theta}{\sin \theta}, \quad n > 1, \quad \text{decreases as } \theta \text{ increases from zero, } \dots \dots \dots (3)$$

$$\frac{\sin n\theta}{\sin \theta}, \quad n < 1, \quad \text{increases as } \theta \text{ increases from zero. } \dots \dots \dots (4)$$

The student can readily verify these statements either by differentiation if he is sufficiently advanced in his pure mathematics, or by means of plotting the function $\sin n\theta / \sin \theta$ for different values of n by means of a table of sines.

We now consider the three cases that can occur with finite radii. If the top sphere is concave upwards and the bottom sphere is concave downwards, then $(R+r)/r$ is greater than unity, and therefore by (3) H decreases as θ increases from zero, i.e. N sinks from N_0 . Thus in this case the equilibrium is unstable if the centre of gravity of the top body is above or at the height $Rr/(R+r)$, stable if the centre of gravity is lower.

If both spheres are concave upwards, we use $-R$ instead of R for the bottom sphere, and we must write

$$\left. \begin{aligned} H &= r \left\{ 1 + \frac{\sin \theta}{\sin \frac{R-r}{r} \theta} \right\}, \\ H_0 &= \frac{Rr}{R-r}. \end{aligned} \right\} \dots\dots\dots (5)$$

R is of course greater than r . Here we get three sub-cases.

If $R > 2r$, $\sin \frac{R-r}{r} \theta / \sin \theta$ decreases as θ increases from zero, and therefore by (5) H increases, so that N rises from N_0 . Thus, if the centre of gravity of the top body is below or at the height $Rr/(R-r)$, the equilibrium is stable; if the centre of gravity is higher the equilibrium is unstable.

If $R = 2r$, $H = H_0 = 2r$, and we get truly neutral equilibrium when the centre of gravity of the top body is at the height $2r$, stable when below, unstable when above.

If $R < 2r$, $\sin \frac{R-r}{r} \theta / \sin \theta$ increases as θ increases from zero, by (4), and therefore by (5) H decreases as θ increases, so that N sinks from N_0 . Thus, if the centre of gravity of the top body is above or at the height $Rr/(R-r)$ the equilibrium is unstable; if lower the equilibrium is stable.

When both spheres are concave downwards, we use $-r$ instead of r for the top sphere, and we get

$$\left. \begin{aligned} H &= r \left\{ -1 + \frac{\sin \theta}{\sin \frac{r-R}{r} \theta} \right\}, \\ H_0 &= Rr/(r-R). \end{aligned} \right\} \dots\dots\dots (6)$$

r is greater than R , but $(r-R)/r$ is always less than unity. Hence $\sin \frac{r-R}{r} \theta / \sin \theta$ increases as θ increases from zero, and H decreases, so that N sinks from N_0 . The equilibrium is therefore unstable if the centre of gravity of the top body is above or at the height $Rr/(r-R)$, stable if lower.

The cases when either r or R is infinite, *i.e.* when either the top or the bottom body is flat, need special investigation, but they are very simple. When R is infinite, then N is clearly at the centre of the top sphere and invariable. The equilibrium is truly neutral if the centre of gravity of the top body is at the centre of its sphere, stable if lower, unstable if higher.

When r is infinite, we get at once

$$\left. \begin{aligned} H &= R\theta / \tan \theta, \\ H_0 &= R. \end{aligned} \right\} \dots\dots\dots (7)$$

The student will readily verify that $\tan \theta / \theta$ increases as θ increases from zero, so that N sinks from N_0 . The equilibrium is therefore unstable if the centre of gravity is above or at the height R , stable if lower.

In this manner the difficulty experienced by the student can be avoided, and at the same time we eliminate the use of the expansions for $\sin \theta$, $\sin \frac{R+r}{r} \theta$, etc., in terms of θ , expansions that come comparatively late in pure mathematics.

We can use the same method in the more general problem of the two-dimensional stability for any two surfaces, the top one movable but resting on the lower one which is fixed, there being sufficient friction to prevent slipping. We shall suppose that in equilibrium the centre of gravity of the top body is in the vertical common normal at the point of contact.

We can use the same figure as before, the curves being any curves, not necessarily circles. Let it represent a two-dimensional displacement. A is the original point of contact, B the new point of contact, AC is the vertical at C , $C'BC$ is the common normal at B meeting AC in C , and $A'C'$ in C' , where A' is the point of the top body that was originally in contact with A and $A'C'$ the normal to the top body at A' . Draw BN vertically through B to meet $A'C'$ in N . Then N is the critical point for the displacement represented by AB , and will vary as AB increases from zero. Let N_0 be the critical point for AB zero. Then, if N rises from N_0 as AB increases from zero, the equilibrium is stable when the centre of gravity of the top body is below or at N_0 , unstable if higher. If N sinks, the equilibrium is unstable when the centre of gravity is above or at N_0 , stable if lower. If N remains permanently at N_0 , the equilibrium is stable when the centre of gravity is below N_0 , neutral when at N_0 , unstable when above N_0 .

To apply this principle, let $A'N \equiv H$. Then

$$H = y_2 + x_2 \cot(\theta + \phi), \dots\dots\dots(8)$$

where θ is the angle ACB , ϕ is the angle $A'C'B$; x_1, y_1 are the coordinates of B , taken on the lower curve, referred to an origin at A , a y -axis vertically downwards and an x -axis horizontally to the right; x_2, y_2 are the coordinates of B , taken on the upper curve, referred to an origin at A' , a y -axis along $A'C'$ and an x -axis along the tangent at A' to the right. (This is very similar to Minchin's treatment.) Putting

$$\left. \begin{aligned} \tan \phi &= \frac{dy_2}{dx_2}, \quad \tan \theta = \frac{dy_1}{dx_1}, \\ H &= y_2 + x_2 \frac{\frac{dx_1}{ds} \frac{dx_2}{ds} - \frac{dy_1}{ds} \frac{dy_2}{ds}}{\frac{dx_1}{ds} \frac{dy_2}{ds} + \frac{dy_1}{ds} \frac{dx_2}{ds}}, \end{aligned} \right\} \dots\dots\dots(9)$$

we get

s being the distance AB or $A'B$ measured along either curve.

Using the expansions

$$\left. \begin{aligned} x_1 &= s - \frac{s^3}{6\rho_1^2} + \frac{s^4}{8\rho_1^3} \frac{d\rho_1}{ds} + \dots, \\ y_1 &= \frac{s^2}{2\rho_1} - \frac{s^3}{6\rho_1^2} \frac{d\rho_1}{ds} + \dots, \end{aligned} \right\} \dots\dots\dots(10)$$

where ρ_1 is the radius of curvature for the lower curve, with similar equations for the upper curve in terms of its radius of curvature ρ_2 , we find

$$H = \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} + C_1 s + C_2 s^2 + \dots, \dots\dots\dots(11)$$

where C_1, C_2 are numbers depending on ρ_1, ρ_2 and their variations at A and A' .

Hence

$$H_0 = \frac{\rho_1 \rho_2}{\rho_1 + \rho_2}, \dots\dots\dots(12)$$

N will rise or sink from N_0 according as dH/ds is positive or negative near the original point of contact, *i.e.* according as C_m is positive or negative, where C_m is the first coefficient in $H - H_0$ which does not vanish.

Thus the equilibrium is stable for the centre of gravity of the top body at the height $\rho_1 \rho_2 / (\rho_1 + \rho_2)$ if C_m is positive, unstable if C_m is negative and truly neutral if no such coefficient exists at all, so that $H \equiv H_0$. In all cases the equilibrium is stable when the centre of gravity is lower than $\rho_1 \rho_2 / (\rho_1 + \rho_2)$, unstable if higher.

In this way the whole problem of statical stability can be discussed in an elementary manner without the introduction of anything more difficult than the increase or decrease of a certain function of a given variable. It is also evident that by plotting the values of H given by (1), (5), (6), (7), (9) the problem of a *finite* displacement admits of easy discussion. S. BRODETSKY.