Presidential Address delivered

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XXXVII. *Presidential Address delivered by Prof. OLIVER LODGE, LL.D., D.Sc., F.R.S., at the Annual General Meeting on February 10th, 1899.*

PART I.—*Preliminary Portion.*

IN taking the Chair of this Society for one year, notwithstanding that I am a country member and therefore shall not be able always to be present, I am I daresay traversing the views of what is considered best for the Society by a minority of members. I have no knowledge that it is so, but it can hardly be otherwise. I hesitated myself for some time, but ultimately consented to be put in nomination by the Council; and I can promise that I shall try to be present as often as my duties permit.

I have been a member of the Society from its earliest infancy, for though not technically an "original member," nor one admitted in the first year, that is a mere accident of not having presented myself for admittance by the Chair; perhaps I had not paid my subscription, but I used to attend all the meetings, and my memory goes back to the preliminary discussions that preceded the birth of the Society, between Prof. Foster, Prof. Adams, Dr. Gladstone, Dr. Atkinson, and Dr. Guthrie, especially Dr. Guthrie, to whom the idea of the Society was, I believe, largely due.

In those days we used to meet at South Kensington on Saturday afternoons; and my experience of recent years is altogether too remote and detached to make me aware whether the change, from the hospitable laboratory at South Kensington to the no doubt equally hospitable but less *physically* inspiring quarters of Burlington House, has been an unmixed advantage. If it is an unmixed advantage, the Society has been fortunate.

Our death-roll for the past year contains the names of two original members whose figures were familiar at the

meetings in those early days : Henry Perigal and Latimer Clark. It also contains the names of Sir J. N. Douglass, so well known in connexion with lighthouse work ; of Dr. Obach, the skilled electrician and designer of Messrs. Siemens Bros. ; of the Rev. Bartholomew Price, known not only as a mathematician but also as a man of public spirit and enterprize in the University of Oxford, who had I believe more to do with the early financial success of the Clarendon Press than is generally known ; of Dr. J. E. Myers, and alas of Dr. John Hopkinson.

It is unfitting that this year's deliverance from the Chair of any Society of which John Hopkinson was a member should fail to contain some reference to his untimely loss ; yet there is nothing for me to say but what you already feel, and no words of mine are needed. I cannot say that I knew him in his youth, but I heard of him. He and I as self-taught students—at least, I was at that time self-taught, and he could hardly have found a better teacher than himself—once figured together in the so-called honours list of the South Kensington May examinations in Electricity and Magnetism or else in Sound Light and Heat, or in both. Needless to say, his name stood above mine. He passed, for fun, in nearly the whole range of South Kensington subjects ; I took a good many of them, but had nothing like either his comprehensive range or his educational advantages. And then he went and wrested the Senior Wranglership from that exceptionally brilliant mathematical genius, J. W. L. Glaisher. That was a surprising achievement for a Whitworth Scholar and practical Engineer. May I be allowed to congratulate Mrs. Hopkinson and her family not only on a peculiarly noble donation, but also on its special application to the welding together the names of Cambridge, of Engineering, and of Hopkinson.

It is remarkable, as Mr. Arthur Balfour said in a speech the other day (at the opening of the Battersea Institute, I think), it is remarkable how in modern times the pursuit of science tends in the direction of rapid and immediate application.

In Newton's time pure science was altogether aloof from

practice : I do not suppose that the gravitational theory of astronomy had any effect on His Majesty's coins at the Mint, nor did the prismatic sifting of white-light lead at that time to any chemical applications of spectrum analysis. Even as late as Faraday's day the induction of currents by motion and magnetism exercised no appreciable effect on commerce and industry.

Nowadays, Fourier's theorem is applied by Lord Kelvin to work an Atlantic cable ; Hopkinson's researches on the magnetic circuit (an idea that has revolutionized the practical treatment of magnetism) result at once in improvements in the dynamo ; and Hertz's experimental detection of Maxwell's waves seems likely to result before long in a new system of wireless telegraphy.

What is the cause of this keen interest in practical applications felt in varying degrees by nearly all modern physicists, felt most keenly and consistently I think, of those I know, by your past-President Prof. Ayrton ? Is it the example and inspiration of Lord Kelvin, or is it that the human race generally is beginning to be better educated than it used to be, to take more kindly to the result of scientific researches, to be more eager to learn about them and, if possible, to apply them ; partly, no doubt, because it has gradually discovered that by judicious treatment it can convert them into substantial commodity ?

Surely, on the whole, with some drawbacks, the result from the point of view of science is good. If science were really remote from all human use or interest, as some scoffers still endeavour to maintain it to be, would there not be a fear lest gradually the human race should get tired of it, should cease to encourage even by approval or applause those whose instinct or mission it might be to develop it ? So that gradually a blight might settle down, and advanced scientific knowledge become as extinct as a dead language or a fossil genus. A few scholars here and there would take an interest in it, as they did in the Middle Ages, and it might perhaps for a time be instilled into youth at the point of a cane, as the dead languages not so long ago used to be ; but unless a branch of learning enters in some way into the life and well-

being of a community, unless it has some real interest—a literary interest it may be, a commercial interest perhaps, a living human interest of some kind—there is fear that however artificially bolstered up it may be for a time, by those whose stock in trade it is and who know little else, it will ultimately be suffered to go by the board and become to all practical intents and purposes dead, buried, and extinct.

I said just now that I supposed the public as a whole were gradually getting better educated, but how miserably slow is the process and how lamentable the present result. I have no feud with orthodox secondary schoolmasters, indeed some of them are my very good friends, but is it not painful to be unable to speak to the average so-called educated Englishman (except a few here and there of course) on any scientific discovery without at once being met by a hopeless wall of ignorance.

Is it so in other countries? I hardly think so; I do not think it is so marked even among Scotchmen, at any rate not among Scotchmen who have had their Natural Philosophy year at the University. A year for training in the whole of science—it is not much. If scientific men were ignorant of letters in the same proportion they would hardly be able to read, except in words of one syllable, nor write, except with the laborious contortions of the village urchin; a year extended on the subjects of all the Arts would not carry one very far. And as to sums, how many men there are whose mathematical equipment is limited to the practice of compound addition and subtraction, and who shy if they see an algebraic symbol, even in a newspaper,—indeed, when they occur in newspapers they are often of a kind that deserve to be shyed at.

They learn languages, some people seem to enjoy learning foreign languages—though the multiplicity of them is an artificial arrangement which an international convention might alter if it chose,—but the exact and beautiful language in which a great part of science is and must be written is to the majority of men unknown. Are teachers wholly free from blame in respect of the dislike which many boys feel

for the most elementary mathematics, for problems which may be made exhilarating when properly presented? I believe they are not. If the teacher dislikes mathematics, the boys will hate it, and if they hate it when young they will love it but little better when they grow up and proceed to be teachers themselves. There is a law of geometrical progression or compound interest here, and it seems to have been at work for some time. I must not however be understood here to be speaking of really cultivated intelligences or authorities in any subject. A thorough training in any branch of knowledge may so stimulate and enlarge the faculties as to fit them for intelligent appreciation and criticism of many other branches. Some men of letters I know whose minds are copiously and essentially scientific, they have not the detailed knowledge of the professed worker in science—that would be impossible and unnecessary—but they appreciate and assimilate all the main doctrines of science, and their broad criticisms and suggestions are often of value to the special worker. Any educational strictures on which I have ventured have reference not to scholars but to average people; at the same time I believe many scholars feel their deficiency in mathematics, and I believe it to be the result of bad teaching.

This is not my address, it is a preliminary excursus, my address has reference to a physical problem which has interested me, and about which I thought it might be interesting if I said something. It is customary to review some topic in an address, not to adduce anything specially new, but to treat of or attempt to clarify something already known. Usually it is some experimental fact which is thus treated, sometimes it is a point of theory; it is the latter to which I wish to ask your attention.

As to experiments, if I quickly review some of those that strike me in the past year—there is further progress to be reported concerning the developments of Prof. Zeeman's great discovery, the most important being (1) the new manifestations of it, or of a closely allied phenomenon, discovered by Professor Righi, to which Prof. S. P. Thompson called the attention of Section A at Bristol last September; and

(2) some extension of the resolution of complex lines, discriminating between the effect as exhibited by different lines, a discrimination which seems likely to lead to an extension of the powers of spectrum analysis by the application of a magnetic field to the source of light, as studied with apparatus of splendid dispersive power by Prof. Preston, and with his own ingenious method by Prof. Michelson. There is a notable continuation of the remarkably powerful and beautiful researches into atomic properties carried on at Cambridge by Prof. J. J. Thomson ; about which a great deal more might be said. And then there is the prediction made in the columns of 'Nature' by Prof. G. F. FitzGerald, but not yet verified, that circularly polarized light sent through an absorbing medium would constitute a magnet.

Many have been the attempts, from Faraday downwards, to excite magnetization by means of light, to detect a real and not an imaginary "magnetization of a ray of light," but all the experimenters hitherto have failed to realise the necessity of an absorbent medium. The effect is bound to be small, and of course it may not be there after all, but I understand it is being looked for, and it has at any rate been on rational grounds predicted.

Dr. Barton has continued experiments on the damping of waves on wires, and has not yet reconciled a discrepancy between theory and experiment ; being able, as I understand him, to account for only $5\frac{1}{2}$ per cent. of the outstanding discrepancy. I have not myself looked into the matter, but I understand that a note has been communicated by Mr. Heaviside. I hope it will be published in full.

The subject of Terrestrial Magnetism, in the hands of Prof. Schuster on the theoretical side and of Prof. Rücker on the practical side, has recently entered on an enlarged existence and seems to be taking up a position almost of an independent science, so that when the multifurcation or explosion of Section A occurs, as sooner or later I fear it must, the fragment entitled Meteorology and Terrestrial Magnetism will perhaps not be one of the smallest.

The publication called 'Science Abstracts' has continued to be welcomed by workers all over the country, and the abstracts sent by Mr. Fournier d'Albe to the 'Electrician' are also useful and admirably done. I have no knowledge myself of the mode of control of the publication of our 'Science Abstracts,' but I confess to a personal preference for the older form, the one with a less startling cover, and with a list of authors' names, instead of unnecessary advertisements of well-known firms, on its back page. Undoubtedly the inauguration of these 'Abstracts' supplies a long-felt want, and even if one does not always find time to read them it is a comfort to feel that there they are, to be read in some less pressing season; and one need not envy the German worker his 'Beiblätter' as one used to.

But there is one most important event that has occurred during the year, perhaps to us conjointly as physicists the most important of all, an event of which the science of Physics will feel the effects—I trust the wholly favourable effects—for centuries to come: I mean the decision of the Government to begin on a small scale the inauguration of a National Physical Laboratory. If its effects are ever in any respects unfavourable to true science, a heavy indictment will lie against its future governing body, and against its Superintendent or Physicist Royal.

I wish to congratulate Sir Douglas Galton, as well as to some extent myself, on the result, the speedy result, of our urging of the subject, within the present decade, on the British Association; and as a Physical Society I think we owe a debt of gratitude to the Committee appointed by the Treasury to examine into the question, whose members gave themselves an immense amount of trouble, travelling to Berlin and elsewhere to make themselves acquainted with foreign ideals, and altogether going into the subject with extreme thoroughness, scientific judgment, and public spirit. The chairman of the committee was Lord Rayleigh, but during his absence in India the acting chairman was constantly Professor Rücker; Mr. Chalmers represented the Treasury with urbane frugality; and to Prof. Rücker and

Mr. Chalmers, as well as to the other members of the committee, I am sure you will agree with me our thanks are justly due.

The record of their labours and of the voluminous evidence which they obtained, to which evidence indeed several of us here contributed, is embodied in a 'blue-book' of more than usual interest and instruction. I commend the perusal of it not only to professed physicists, but to all scientific teachers and their advanced students, that they may learn what accuracy is, and what it is practically wanted for.

It is not for me as your President to touch at any length upon outside or political topics, but briefly I venture to think that the year 1898 has been a year of no ordinary good fortune, at least to the Anglo-Saxon race. A year, and a Government, which has witnessed and assisted not only the beginning of a National Physical Laboratory, but such great outside events as the Imperial expansion and cooperation of America, (for America, and not the United States, it will now have to be called), the liberation of Crete, and the restoration of the civilization of Upper Egypt, is a year upon which we can look back with satisfaction, and is a Government which, irrespective of party, we may legitimately congratulate, and even, though that I admit is unusual, thank.

Would that we could add to its laurels the commencement on wise lines of a real and comprehensive, dignified and progressive, genuine University of London.

PART II.—*On Opacity.*

MY attention has recently been called to the subject of the transmission of electromagnetic waves by conducting dielectrics—in other words, to the opacity of imperfectly conducting material to light. The question arose when an attempt was being made to signal inductively through a stratum of earth or sea, how far the intervening layers of moderately conducting material were able to act as a screen; the question also arises in the transmission of Hertz waves through partial conductors, and again in the transparency of gold-leaf and other homogeneous substances to light.

The earliest treatment of such subjects is due of course to Clerk Maxwell thirty-four years ago, when, with unexampled genius, he laid down the fundamental laws for the propagation of electric waves in simple dielectrics, in crystalline media, and in conducting media. He also realised there was some strong analogy between the transmission of such waves through space and the transmission of pulses of current along a telegraph-wire. But naturally at that early date not every detail of the investigation was equally satisfactory and complete.

Since that time, and using Maxwell as a basis, several mathematicians have developed the theory further, and no one with more comprehensive thoroughness than Mr. Oliver Heaviside, who has gone into these matters with extraordinarily clear and far vision. I may take the opportunity of calling or recalling to the notice of the Society, as well as of myself, some of the simpler developments of Mr. Heaviside's theory and manner of unifying phenomena and processes at first sight apparently different; but first I will deal with the better-known aspects of the subject.

Maxwell deals with the relation between conductivity and opacity in his Art. 798 and on practically to the end of that famous chapter xx. ('Electromagnetic Theory of Light'). He discriminates, though not very explicitly or obtrusively, between the two extreme cases, (1) when inductive capacity

or electric inductivity is the dominant feature of the medium—when, for instance, it is a slightly conducting dielectric, and (2) the other extreme case, when conductivity is the predominant feature.

The equation for the second case, that of predominant conductivity, is

$$\frac{d^2 F}{dx^2} = \frac{4\pi\mu}{\sigma} \frac{dF}{dt}, \quad (1)$$

F being practically any vector representing the amplitude of the disturbance; for since we need not trouble ourselves with geometrical considerations such as the oblique incidence of waves on a boundary &c., we are at liberty to write the ∇ merely as d/dx , taking the beam parallel and the incidence normal.

No examples are given by Maxwell of the solution of this equation, because it is obviously analogous to the ordinary heat diffusion fully treated by Fourier.

Suffice it for us to say that, taking F at the origin as represented by a simple harmonic disturbance $F_0 = e^{ipt}$, the solution of equation (1)

$$\frac{d^2 F}{dx^2} = \frac{4\pi\mu ip}{\sigma} F (1')$$

is $F = F_0 e^{-Qx} = e^{-Qx + ipt},$

where $Q = \sqrt{\left(\frac{4\pi\mu ip}{\sigma}\right)} = \sqrt{\frac{2\pi\mu p}{\sigma}} \cdot (1+i);$

wherefore

$$F = e^{-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}} x} \cos \left(pt - \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}} x \right), \quad . . . (2)$$

an equation which exhibits no true elastic wave propagation at a definite velocity, but a trailing and distorted progress, with every harmonic constituent going at a different pace, and dying out at a different rate; in other words, the *diffusion* so well known in the case of the variable stage of heat-conduction through a slab.

In such conduction the gain of heat by any element whose heat capacity is $cpdx$ is proportional to the difference of the

temperature gradient at its fore and aft surfaces, so that

$$c\rho \, dx \, \frac{d\theta}{dt} = d \cdot k \, \frac{d\theta}{dx},$$

or, what is the same thing,

$$\frac{d^2\theta}{dx^2} = \frac{c\rho}{k} \frac{d\theta}{dt},$$

the same as the equation (1) above; wherefore the constant $c\rho/k$, the *reciprocal* of the thermometric conductivity, takes the place of $4\pi\mu/\sigma$, that is, of electric conductivity; otherwise the heat solution is the same as (2). The 4π has come in from an unfortunate convention, but it is remarkable that the conductivity term is inverted. The reason of the inversion of this constant is that, whereas the *substance* conveys the heat waves, and by its conductivity aids their advance, the *æther* conveys the electric waves, and the substance only screens and opposes, reflects, or dissipates them.

This is the case applied to sea-water and low frequency by Mr. Whitehead in a paper which he gave to this Society in June 1897, being prompted thereto by the difficulty which Mr. Evershed and the Post Office had found in some trials of induction signalling at the Goodwin Sands between a coil round a ship at the surface and another coil submerged at a depth of 10 or 12 fathoms. It was suspected that the conductivity of the water mopped up a considerable proportion of the induced currents, and Mr. Whitehead's calculation tended, or was held to tend, to support that conclusion.

To the discussion Mr. Heaviside communicated what was apparently, as reported, a brief statement; but I learn that in reality it was a carefully written note of three pages, of which recently he has been good enough to lend me a copy. In that note he calls attention to a theory of the whole subject which in 1887 he had worked out and printed in his collected 'Electrical Papers,' but which has very likely been overlooked. It seems to me a pity that a note by Mr. Heaviside should have been so abridged in the reported discussion as to be practically useless; and I am permitted to quote it here as an appendix (p. 413).

Meanwhile, taking the diffusion case as applicable to seawater with moderately low acoustic frequency, we see that the

induction effect decreases geometrically with the thickness of the oceanic layer, and that the logarithmic decrement of the amplitude of the oscillation is $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$, where σ is the specific resistance of sea-water and $p/2\pi$ is the frequency.

Mr. Evershed has measured σ and found it 2×10^{10} c.g.s., that is to say $2 \times 10^{10} \mu$ square centim. per second; so putting in this value and taking a frequency of 16 per second, the amplitude is reduced to 1/eth of what its value would have been at the same distance in a perfect insulator, by a depth

$$\sqrt{\frac{\sigma}{2\pi\mu p}} = \sqrt{\left(\frac{2 \times 10^{10}\mu}{2\pi\mu \times 2\pi \times 16}\right)} = \sqrt{\frac{10^{10}}{320}} = \frac{10^5}{18} \text{ centim.} \\ = 55 \text{ metres.}$$

Four or five times this thickness of intervening sea would reduce the result at the 16 frequency to insignificance (each 55-metre-layer reducing the energy to $\frac{1}{e}$ of what entered it); but if the frequency were, say, 400 per second instead of 16 it would be five times more damped, and the damping thickness (the depth reducing the amplitude in the ratio $e:1$) would in that case be only eleven metres.

It is clear that in a sea 10 fathoms (or say 20 metres) deep the failure to inductively operate a "call" responding to a frequency of 16 per second was *not* due to the screening effect of sea-water*.

Maxwell, however, is more interested in the propagation of actual light, that is to say, in waves whose frequency is about 5×10^{14} per second; and for that he evidently does not consider that the simple diffusion theory is suitable. It certainly is not applicable to light passing through so feeble a conductor as salt water. He attends mainly therefore to the other and more interesting case, where electric inductive-capacity predominates over the damping effect of conductivity, and where true waves therefore advance with an

* I learn that the ship supporting the secondary cable was of metal, and that the primary or submerged cable was sheathed in uninsulated metal, viz. in iron, which would no doubt be practically short-circuited by the sea-water. Opacity of the medium is in that case a superfluous explanation of the failure, since a closed secondary existed close to both sending and receiving circuit.

approximately definite velocity

$$v = \frac{1}{\sqrt{\mu K}};$$

though it is to be noted that the slight sorting out of waves of different frequency, called dispersion, is an approximation to the case of pure diffusion where the speed is as the square root of the frequency, and is accompanied, moreover, as it ought to be, by a certain amount of differential or selective absorption.

To treat the case of waves in a conductor, the same damping term as before has to be added to the ordinary wave equation, and so we have

$$\frac{d^2 F}{dx^2} = \mu K \frac{d^2 F}{dt^2} + \frac{4\pi\mu}{\sigma} \frac{dF}{dt}. \quad (3)$$

Taking $F_0 = e^{ipt}$ again, it may be written

$$\frac{d^2 F}{dx^2} = \left(-\mu K p^2 + \frac{4\pi\mu ip}{\sigma} \right) F, \quad (3')$$

the same form as equation (1') ; so the solution is again

$$F = e^{-qx + ipt},$$

with Q^2 equal to the coefficient of F in (3'). Maxwell, however, does not happen to extract the square root of this quantity, but, assuming the answer to be of the form (for a simply harmonic disturbance) [modifying his letters, vol. ii. § 798]

$$e^{-rx} \cos(pt - qx),$$

he differentiates and equates coefficients, thus getting

$$q^2 - r^2 = \mu K p^2, \quad 2rq = \frac{4\pi\mu p}{\sigma},$$

as the conditions enabling it to satisfy the differential equation. This of course gives for the logarithmic decrement, or coefficient of absorption,

$$r = \frac{2\pi\mu}{\sigma} \cdot \frac{p}{q},$$

p/q being precisely the velocity of propagation of the train of waves. Though not exactly equal to $1/\sqrt{\mu K}$, the true velocity of wave propagation, except as a first approximation, in an absorbing medium, yet practically this velocity p/q or λ/T

is independent of the frequency except in strongly absorbent substances where there are dispersional complications; and so the damping is, in simple cases, practically independent of the frequency too.

With this simple velocity in mind Maxwell proceeds to apply his theory numerically to gold-leaf, calculating its theoretical transparency, and finding, as every one knows, that it comes out discordant with experiment, being out of all comparison * smaller than what experiment gives.

But then it is somewhat surprising to find gold treated as a substance in which conductivity does not predominate over specific inductive capacity.

The differential equation is quite general and applies to any substance, and since the solution given is a true solution, it too must apply to any substance when properly interpreted; but writing it in the form just given does not suggest the full and complete solution. It seems to apply only to slightly damped waves, and indeed, Maxwell seems to consider it desirable to rewrite the original equation with omission of K , for the purpose of dealing with good conductors.

By a slip, however, he treats gold for the moment as if it belonged to the category of poor conductors, and as if absorption in a thickness such as gold-leaf could be treated as a moderate damping of otherwise progressive waves.

The slip was naturally due to a consideration of the extreme frequency of light vibrations; but attention to the more complete expression for the solution of the same differential equation, given in 1887 by Mr. Heaviside and quoted in the note to this Society above referred to, puts the matter in a proper position. Referring to his 'Electrical Papers,' vol. ii. p. 422, he writes down the general value of the coefficient of absorption as follows (translating into our notation)

$$r = \frac{p}{v\sqrt{2}} \left\{ \left[1 + \left(\frac{4\pi}{\sigma p K} \right)^2 \right]^{\frac{1}{2}} - 1 \right\}^{\frac{1}{2}}$$

without regard to whether the conductivity of the medium is large or small; where v is the undamped or true velocity of wave propagation in the medium $(\mu K)^{-\frac{1}{2}}$.

* The fraction representing the calculated transmission by a film half a wave thick has two thousand digits in its denominator; see below, top of p. 370.

Of course Maxwell could have got this expression in an instant by extracting the square root of the quantity Q , the coefficient of F in equation (3') written above. I do not suppose that there is anything of the slightest interest from the mathematician's point of view, the interest lies in the physical application; but as this is not a mathematical Society it is permissible, and I believe proper, to indicate steps for the working out of the general solution of equation (3) by extracting the square root of the complex quantity Q .

The equation is

$$\frac{d^2 F}{dx^2} + \left(\mu K p^2 - \frac{4\pi\mu ip}{\sigma} \right) F = 0,$$

and the solution is

$$F = e^{-Qx + ipx},$$

where

$$Q = \sqrt{-\mu K p^2 + \frac{4\pi\mu ip}{\sigma}} = \alpha + i\beta \text{ say.}$$

Squaring we get, just as Maxwell did,

$$\alpha^2 - \beta^2 = -\mu K p^2, \quad 2\alpha\beta = +\frac{4\pi\mu p}{\sigma}.$$

Squaring again and adding

$$(\alpha^2 + \beta^2)^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2 = \mu^2 K^2 p^4 + \frac{16\pi^2 \mu^2 p^2}{\sigma^2}$$

$$\therefore \alpha^2 + \beta^2 = \mu K p^2 \left\{ 1 + \left(\frac{4\pi}{K p \sigma} \right)^2 \right\}^{\frac{1}{2}},$$

wherefore

$$2\beta^2 = \mu K p^2 \left\{ \sqrt{\left(1 + \left(\frac{4\pi}{K p \sigma} \right)^2} + 1 \right), \dots \right. \quad (4)$$

and $2\alpha^2$ = the same with the last sign negative,

$$\text{or} \quad \alpha = p \sqrt{\left(\frac{1}{2} \mu K \right)} \left\{ \left[1 + \left(\frac{4\pi}{K p \sigma} \right)^2 \right]^{\frac{1}{2}} - 1 \right\}^{\frac{1}{2}}, \dots \quad (5)$$

which is the logarithmic decrement of the oscillation per unit of distance, or the reciprocal of the thickness which reduces the amplitude in the ratio $1 : e$ (or the energy to $\frac{1}{2}$) of the value it would have at the same place without damping.

Using these values for α and β , the radiation-vector in general, after passing through any thickness x of any medium

whose magnetic permeability and other properties are *constant*, is

$$F = F_0 e^{-\alpha x} \cos (pt - \beta x), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the speed of advance of the wave-train being p/β .

Now not only the numerical value but the form of this damping constant α depends on the magnitude of the numerical quantity $\frac{4\pi}{\sigma p K}$, which may be called the critical number*, and may also be written

$$\frac{4\pi\mu_0 v_0^2}{p\sigma K/K_0}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where K , the absolute specific inductive capacity of the medium, is replaced by its relative value in terms of K_0 for vacuum, and by $\frac{1}{\sqrt{K_0\mu_0}}$ = the velocity of light *in vacuo* = v_0 .

Now for all ordinary frequencies and good conductors this critical number is large; and in that case it will be found that

$$\alpha = p \sqrt{\frac{\mu K}{2}} \cdot \sqrt{\frac{4\pi}{\sigma p K}} = \sqrt{\frac{2\pi\mu p}{\sigma}},$$

and that β is identically the same. This represents the simple diffusion case, and leads to equation (2).

On the other hand, for luminous frequency and bad conductors, the critical quantity is small, and in that case

$$\begin{aligned} \alpha &= p \sqrt{(\tfrac{1}{2} \mu K)} \left\{ 1 + \tfrac{1}{2} \left(\frac{4\pi}{\sigma p K} \right)^2 - 1 \right\}^{\frac{1}{2}} \\ &= \tfrac{1}{2} p \sqrt{(\mu K)} \cdot \frac{4\pi}{\sigma p K} = \frac{2\pi\mu v}{\sigma}, \end{aligned}$$

while

$$\beta = p \sqrt{\mu K} = \frac{p}{v},$$

giving the solution

$$F = F_0 e^{-\frac{2\pi\mu v}{\sigma} x} \cos p \left(t - \frac{x}{v} \right). \quad . \quad . \quad . \quad (8)$$

* An instructive mode of writing α and β in general is given in (11'') or (12'') below, where the above critical number is called $\tan \epsilon$:—

$$\begin{aligned} \alpha v \sqrt{\cos \epsilon} &= p \sin \tfrac{1}{2} \epsilon, \\ \beta v \sqrt{\cos \epsilon} &= p \cos \tfrac{1}{2} \epsilon. \end{aligned}$$

This expresses the transmission of light through imperfect insulators, and is the case specially applied by Maxwell to calculations of opacity. Its form serves likewise for telegraphic signals or Hertz waves transmitted by a highly-conducting aerial wire; the damping, if any, is independent of frequency and there is true undistorted wave-propagation at velocity $v=1/\sqrt{LS}$; the constants belonging to unit length of the wire. The current (or potential) at any time and place is

$$C = C_0 e^{-\frac{Rx}{2Lv}} \cos p(t - x \sqrt{LS}). \quad . \quad . \quad (9)$$

The other extreme case, that of diffusion, represented by equation (2), is analogous to the well-known transmission of slow signals by Atlantic cables, that is by long cables where resistance and capacity are predominant, giving the so-called KR law (only that I will write it RS),

$$C = C_0 e^{-\sqrt{\frac{1}{2}pRS}x} \cos \{pt - \sqrt{\frac{1}{2}pRS}x\}; \quad . \quad . \quad (10)$$

wherefore the damping distance in a cable is

$$x_0 = \sqrt{\left(\frac{2}{pRS}\right)}.$$

Thus, in comparing the cable case with the penetration of waves into a conductor and with the case of thermal conduction, the following quantities correspond :

$$\frac{1}{2}pRS, \quad \frac{4\pi\mu p}{2\sigma}, \quad \frac{pc\rho}{2k}.$$

$c\rho$ is the heat-capacity per unit volume, S is the electric capacity per unit length; k is the thermal conductivity per unit volume, $1/R$ is the electric conductance per unit length. So these agree exactly; but in the middle case, that of waves entering a conductor, there is a notable inversion, representing a real physical fact. $4\pi\mu$ may be called the density and may be compared with ρ or with $1/S$, that is with elasticity $\div v^2$; but σ is the *resistance* per unit volume instead of the conductance. The reason of course is that whereas good conductivity helps the cable-signals or the heat along, it by no means helps the waves into the conductor. Conductivity aids their slipping *along* the boundary of a conductor, but it retards their passing *across* the boundary and entering a

conductor. As regards waves entering a conductor, the effect of conductivity is a screening effect, not a transmitting effect, and it is the bad conductor which alone has a chance of being a transparent medium.

It may be convenient to telegraphists, accustomed to think in terms of the "KR-law" and comparing equations (2) and (10), to note that the quantity $4\pi\mu\sigma$ —that is, practically, the specific conductivity in electromagnetic measure (multiplied by a meaningless 4π because of an unfortunate initial convention)—takes the place of KR (*i. e.* of RS), but that otherwise the damping-out of the waves as they enter a good conductor is exactly like the damping-out of the signals as they progress through a cable; or again as electrification travels along a cotton thread, or as a temperature pulse makes its way through a slab; and yet another case, though it is different in many respects, yet has some similarities, viz. the ultimate distance the melting-point of wax travels along a bar in Ingenhousz's conductivity apparatus,—the same law of inverse square of distance for effective reach of signal holding in each case.

Now it is pointed out by Mr. Heaviside in several places in his writings that, whereas the transmission of high-frequency waves by a nearly transparent substance corresponds by analogy to the conveyance of Hertz waves along aerial wires (or along cables for that matter, if sufficiently conducting), and whereas the absorption of low-frequency waves by a conducting substance corresponds, also by analogy, to the diffusion of pulses along a telegraph-cable whose self-induction is neglected—its resistance and capacity being prominent,—the intermediate case of waves of moderate frequency in a conductor of intermediate opacity corresponds to the more general cable case where self-induction becomes important and where leakage also must be taken into account; because it is leakage conductance that is the conductance of the dielectric concerned in plane waves. This last is therefore a real, and not only an analogic, correspondence.

Writing R_1 S_1 L_1 Q_1 for the resistance, the capacity ("permittance"), the inductance, and the leakage-conductance ("leakance") respectively, per unit length, the general equations to cable-signalling are given in Mr. Heaviside's

‘ Electromagnetic Theory ’ thus :—

$$S_1 \frac{dV}{dt} + Q_1 V + \frac{dC}{dx} = 0, \quad L_1 \frac{dC}{dt} + R_1 C + \frac{dV}{dx} = 0 ;$$

or for a simple harmonic disturbance,

$$\begin{aligned} \frac{d^2 V}{dx^2} &= (R_1 + ipL_1)(Q_1 + ipS_1)V \quad . \quad . \quad . \quad (11) \\ &= (\alpha + i\beta)^2 V, \end{aligned}$$

whose solution therefore is

$$V = V_0 e^{-\alpha x} \cos (pt - \beta x) * . \quad . \quad . \quad (11')$$

There are several interesting special cases :—

The old cable theory of Lord Kelvin is obtained by omitting both Q and L ; thus getting equation (2).

The transmission of Hertz waves along a perfectly-conducting insulated wire is obtained by omitting Q and R ; the speed of such transmission being $1/\sqrt{(L_1 S_1)}$. Resistance in the wire brings it to the form (9), where the damping depends on the ratio of the capacity constant RS to the self-induction constant L/S ; because the index $R/2Lx$ equals half the square root of this ratio ; but it must be remembered that R has the throttled value due to merely superficial penetration. The case is approximated to in telephony sometimes.

A remarkable case of undistorted (though attenuated) transmission through a cable (discovered by Mr. Heaviside, but not yet practically applied) is obtained by taking

$$R/L = Q/S = r ;$$

the solution being then

$$V = e^{-\frac{rx}{v}} f\left(t - \frac{x}{v}\right),$$

due to $f(t)$ at $x=0$. All frequencies are thus treated alike,

* I don't know whether the following simple general expression for α and β has been recorded by anyone : writing $R/pL = \tan \epsilon$ and $Q/pS = \tan \epsilon'$,

$$\alpha \text{ or } \beta = \frac{p}{v} \cdot \frac{\sin \text{ or } \cos \frac{1}{2}(\epsilon + \epsilon')}{(\cos \epsilon \cos \epsilon')^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (11'')$$

which is shorter than (12).

and a true velocity of transmission makes its reappearance. This is what he calls his "distortionless circuit," which may yet play an important part in practice.

And lastly, the two cases which for brevity may be treated together, the case of perfect insulation, $Q=0$, on the one hand, and the case of perfect wire conduction, $R=0$, on the other. For either of these cases the general expression

$$\alpha^2 \text{ or } \beta^2 = \frac{1}{2} p^2 L_1 S_1 \left[\left\{ 1 + \left(\frac{R}{pL} \right)^2 \right\}^{\frac{1}{2}} \left\{ 1 + \left(\frac{Q}{pS} \right)^2 \right\}^{\frac{1}{2}} \pm \left\{ \frac{RQ}{p^2 LS} - 1 \right\} \right] \quad (12)$$

becomes exactly of the form (4) or (5) reckoned above for the general screening-effect, or opacity, of conducting media in space.

For the number which takes the place of the quantity there called the critical number, namely either R/pL or Q/pS , the other being zero, we may write $\tan \epsilon$; in which case the above is

$$\alpha^2 \text{ or } \beta^2 = \frac{1}{2} p^2 L_1 S_1 (\sec \epsilon \mp 1); \quad . \quad . \quad . \quad (12')$$

or, rewriting in a sufficiently obvious manner, with $2\pi/\lambda$ for p/v if we choose,

$$\alpha = \frac{p \sin \frac{1}{2} \epsilon}{v(\cos \epsilon)^{\frac{1}{2}}}, \quad \beta = \frac{p \cos \frac{1}{2} \epsilon}{v(\cos \epsilon)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (12'')$$

Instead of attending to special cases, if we attend to the general cable equation (11) as it stands, we see that it is more general than the corresponding equation (3) to waves in space, because it contains the extra possibility R of wire resistance, which does not exist in free space.

Mr. Heaviside, however, prefers to unify the whole by the introduction of a hypothetical and as yet undiscovered dissipation-possibility in space, or in material bodies occupying space, which he calls magnetic conductance, and which, though supposed to be non-existent, may perhaps conceivably represent the reciprocal of some kind of hysteresis, either the electric or the magnetic variety. Calling this g , (gH^2 is to be the dissipation term corresponding with RC^2), the equation to waves in space becomes

$$\nabla^2 F = (g + ip\mu) \left(\frac{4\pi}{\sigma} + ipK \right) F, \quad . \quad . \quad . \quad (13)$$

just like the general cable case. And a curious kind of transparency, attenuation without distortion, would belong to a medium in which both conductivities coexisted in such proportion that $g : \mu = 4\pi k : K$; for g would destroy H just as k destroys E .

In the cable, F may be either current or potential, and $LSv^2 = 1$. In space, F may be either electric or magnetic intensity, and $\mu Kv^2 = 1$; but observe that g takes the place not of Q but of R , while it is $4\pi/\sigma$ that takes the place of Q . Resistance in the wire and electric conductivity in space do not produce similar effects. If there is any analogue in space to wire resistance it is magnetic not electric conductivity.

The important thing is of course that the wire does not convey the energy but dissipates it, so that the dissipation by wire-resistance and the dissipation by space-hysteresis to that extent correspond. The screening effect of space-conductivity involves the very same dielectric property as that which causes leakage or imperfect insulation of the cable core.

Returning to the imaginary magnetic conductivity, let us trace what its effects would be if it existed, and try to grasp it. Its effect would be to kill out the magnetism of permanent magnets in time, and generally to waste away the energy of a static magnetic field, just as resistance in wires wastes the energy of an unmaintained current and so kills out the magnetism of *its* field. I spoke above as if it were conceivable that such magnetic conductivity could actually in some degree exist, likening it to a kind of hysteresis; but hysteresis—the enclosure of a loop between a to and fro path—is a phenomenon essentially associated with *fluctuations*, and cannot exist in a steady field with everything stationary. Admitted: but then the molecules are not stationary, and the behaviour of molecules in the Zeeman and Righi phenomena, or still more strikingly in the gratuitous radiations discovered by Edmond Becquerel, and more widely recognized by others, especially by Monsieur et Madame Curie, (not really gratuitous but effected probably by conversion into high-pitched radiation of energy supplied from low-pitched sources),—the way molecules of absorbent substances behave, seems to render

possible, or at least conceivable, something like a minute magnetic conductivity in radiative or absorptive substances. Mr. Heaviside, however, never introduced it as a physical fact for which there was any experimental evidence, but as a physical possibility and especially as a mathematical auxiliary and unifier of treatment, and that is all that we need here consider it to be ; but we may trace in rather more detail its effect if it did exist.

Suppose the magnetism of a magnet decayed, what would happen to its lines of force ? They would gradually shrink into smaller loops and ultimately into molecular ones. The generation of a magnetic field is always the opening out of previously existing molecular magnetic loops ; there is no such thing as the creation of a magnetic field, except in the sense of moving it into a fresh place or expanding it over a wider region *. So also the destruction of a magnetic field merely means the shrinkage of its lines of force (or lines of induction, I am not here discriminating between them). Now consider an electric current in a wire :—a cylindrical magnetic field surrounds it, and if the current gradually decreases in strength the magnetic energy gradually sinks into the wire as its lines slowly collapse. But observe that the *electric* energy of the field remains unchanged by this process : if the wire were electrostatically charged it would remain charged, its average potential can remain constant. Let the wire for instance be perfectly conducting, then the current needs no maintenance, the potential might be uniform (though in general there would be waves running to and fro), and both the electric and magnetic fields continue for ever, unless there is some dissipative property in space.

Two kinds of dissipative property may be imagined in matter filling space : first, and most ordinary, an electric conductivity or simple leakage, the result of which will be to equalize the potential throughout space and destroy the electric field, without necessarily affecting the magnetic field, and so without stopping the steady circulation of the current manifested by that field. The other dissipative property in space that could be imagined would be magnetic conductivity ; the result of which would be to shrink all the circular lines of

* This may be disagreed with.

magnetic force slowly upon the wire, thus destroying the magnetic field, and with it (by the circuital relation) the current; but leaving the electrostatic potential and the electric field unchanged. And this imaginary effect of the medium in surrounding space is exactly the real effect caused by what is called electric resistance in the wire*.

Now for a simply progressive undistorted wave, *i. e.* one with no character of diffusion about it, but all frequencies travelling at the same quite definite speed $1/\sqrt{\mu K}$, it is essential that the electric and magnetic energies shall be equal. If both are weakened *in the same proportion*, the wave-energy is diminished, and the pulse is said to be "attenuated," but it continues otherwise uninjured and arrives "undistorted," that is, with all its features intact and at the same speed as before, but on a reduced scale in point of size.

This is the case of Mr. Heaviside's "distortionless circuit" spoken of above, and its practical realization in cables, though it would not at once mean Atlantic telephony, would mean greatly improved signalling, and probably telephony through shorter cables. In a cable the length of the Atlantic the attenuation would be excessive, unless the absence of distortion were secured by increasing rather the wire-conductance than the dielectric leakage; but, unless excessive, simple attenuation does no serious harm. Articulation depends on the features of the wave, and the preservation of the features demands, by Fourier's analysis, the transmission of every frequency at the same rate.

But now suppose any cause diminishes one of the two fields without diminishing the other: for instance, let the electric field be weakened by leakage alone, or let the magnetic field be weakened by wire-resistance alone, then what happens? The preservation of E and the diminution of H , to take the latter—the ordinary—case, may be regarded as a superposition on the advancing wave of a gradually growing reverse field of intensity δH ; and, by the relation $E = \mu v H$,

* There is this difference, that in the real case the heat of dissipation appears locally in the wire, whereas in the imaginary case it appears throughout the magnetically conducting medium; but I apprehend that in the imaginary case the lines would still shrink, by reason of molecular loops being pinched off them.

this reversed field, for whatever it is worth, must mean a gradually growing wave travelling in the reverse direction.

The ordinary wave is now no longer left alone and uninjured, it has superposed upon itself a more or less strong reflected wave, a reflected wave which constantly increases in intensity as the distance along the cable, or the penetration of the wave into a conducting medium, increases; all the elementary reflected waves get mixed up by re-reflexion in the rear, constituting what Mr. Heaviside calls a diffusive "tail"; and this accumulation of reflected waves it is which constitutes what is known as "distortion" in cables, and what is known as "opacity" inside conducting dielectrics.

There is another kind of opacity, a kind due to heterogeneity, not connected with conductivity but due merely to a change in the constants K and μ ,—properly a kind of translucency, a scattering but not a dissipation of energy,—like the opacity of foam or ground glass.

This kind of opacity is an affair of boundaries and not of the medium itself, but after all, as we now see, it has features by no means altogether dissimilar to the truer kind of opacity. Conducting opacity is due to reflexion, translucent opacity is due to reflexion,—to irregular reflexion as it is called, but of course there is nothing irregular about the reflexion, it is only the distribution of boundaries which is complicated, the reflexion is as simple as ever;—except, indeed, to some extent when the *size* of the scattering particles has to be taken into account and the blue of the sky emerges. But my point is that this kind of opacity also is after all of the reflexion kind, and the gradual destruction of the advancing wave—whether it be by dust in the air or, as Lord Rayleigh now suggests, perhaps by the discrete molecules themselves, by the same molecular property as causes refraction and dispersion—must result in a minute distortion and a mode of wave propagation not wholly different from cable-signalling or from the transmission of light through conductors. So that the red of the sunset sky and the green of gold-leaf may not be after all very different; nor is the arrival-curve of a telegraph signal a wholly distinct phenomenon.

There is a third kind of opacity, that of lampblack, where the molecules appear to take up the energy direct, converting

it into their own motion, that is into heat, and where there appears to be little or nothing of the nature of reflexion. I am not prepared to discuss that kind at present.

It is interesting to note that in the most resisting and capacious cable that ever was made, where all the features of every wave arrive as obliterated as if one were trying to signal by heat-pulses through a slab, that even there the *head* of every wave travels undistorted, with the velocity of light, and suffers nothing but attenuation; for the superposed reversed field is only called out by the arrival of the direct pulse, and never absolutely reaches the strength of the direct field. The attenuation may be excessive, but the signal is there in its right time if only we have a sensitive enough instrument to detect it; though it would be practically useless as a signal in so extreme a case, being practically *all* tail.

Nothing at all reaches the distant end till the light-speed-time has elapsed; and the light-speed-time in a cable depends on the μ and K of its insulating sheath, depends, if that is not simply cylindrical, on the product of its self-inductance and capacity per unit length; but at the expiration of the light-speed time the head of the signalling pulse arrives, and neither wire-resistance nor insulation-leakage, no, nor magnetic-conductivity, can do anything either to retard it or to injure its sharpness: they can only enfeeble its strength, but they can do that very effectually.

The transmitter of the pulse is self-induction in conjunction with capacity: the chief practical enfeebler of the pulse is wire-resistance in conjunction with capacity; and before Atlantic telephony is possible (unless a really distortionless cable is forthcoming) the copper core of an ordinary cable will have to be made much larger. Nothing more is wanted in order that telephony to America may be achieved. There may be practical difficulties connected with the mechanical stiffness of a stout core and the worrying of its guttapercha sheath, and these difficulties may have to be lessened by aiming at distortionless conditions—it is well known also that for high frequencies a stout core must be composed of insulated strands unless it is hollow—but when such telephony is accomplished, I hope it will be recollected that the full and complete principles of it and of a great deal else connected with tele-

graphy have been elaborately and thoroughly laid down by Mr. Heaviside.

There is a paragraph in Maxwell, concerning the way a current rises in a conductor and affects the surrounding space, which is by no means satisfactory : it is Art. 804. He takes the current as starting all along the wire, setting up a sheath of opposition induced currents in the surrounding imperfectly insulating dielectric, which gradually diffuse outwards and die away, leaving at last the full inductive effect of the core-current to be felt at a distance. Thus there is supposed to be a diffusion of energy outwards from the wire, which he likens to the diffusion of heat.

But, as Mr. Heaviside has shown, the true phenomenon is the transmission of a wave in the space surrounding the wire—a plane wave if the wire is perfectly conducting, a slightly coned wave if it resists,—a wave-front perpendicular to the wire and travelling along it,—*a sort of beam of dark light with the wire as its core.*

Telegraphic signalling and optical signalling are similar ; but whereas the beam of the heliograph is abandoned to space and must go straight except for reflexion and refraction, the telegraphic beam can follow the sinuosities of the wire and be guided to its destination.

If the medium conducts slightly it will be dissipated *in situ* ; but if the wire conducts imperfectly, a minute trickle of energy is constantly directed inwards radially towards the wire core, there to be dissipated as heat. Parallel to the wire flows the main energy stream, but there is a small amount of tangential grazing and inward flow. The initial phenomenon does not occur in the wire, gradually to spread outwards, but it occurs in the surrounding medium, and a fraction of it gradually converges inwards. The advancing waves are not cylindrical but plane waves, and though the diffusing waves are cylindrical they advance inwards, not outwards.

I will quote from a letter of Mr. Heaviside's:—"The easiest way to make people understand is, perhaps, to start with a conducting dielectric with plane waves in it without wires, [thus getting] one kind of attenuation and distortion. Then introduce wires of *no* resistance ; there is no difference except

in the way the lines of force distribute [enabling the wires to guide the plane waves]. Then introduce magnetic conductivity in the medium, [thereby getting] the other kind of attenuation and distortion. Transfer it to the wires, making it electrical resistance. Then abolish the first electric conductivity, and you have the usual electric telegraph."

OPACITY OF GOLD-LEAF.

Now returning to the general solution (5) let us apply it to calculate the opacity of gold-leaf to light.

Take $\sigma = 2000 \mu$ square centim. per sec.,
 $p = 2\pi \times 5 \times 10^{14}$ per sec. ;

then the critical quantity $4\pi/p\sigma K$ or (7) is

$$\frac{2 \times 9 \times 10^{20}}{5 \times 10^{14} \times 2000 K/K_0} = \frac{1800}{K/K_0}.$$

This number is probably considerably bigger than unity (unless, indeed, the specific inductive capacity K, K_0 of gold is immensely large, which may indeed be the case—refractive index 40, for instance,—only it becomes rather difficult to define) ; so that, approximately,

$$\alpha = \sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)} = \sqrt{\frac{40 \times 5 \times 10^{14}}{2000}} = \sqrt{10^{13}} = 3 \times 10^6 ;$$

or the damping distance is

$$\frac{1}{3} \times 10^{-5} \text{ centim.} = \frac{1}{3} \text{ microcentimetre,}$$

whereas the wave-length in air is

$$6 \times 10^{-5} \text{ centim.} = 60 \text{ microcentimetres.}$$

The damping distance is therefore getting nearer to the right order of magnitude, but the opacity is still excessive.

A common thickness for gold-leaf is stated to be half a wave length of light ; that is to say, 90 times the damping distance. Hence the amplitude of the light which gets through a half-wave thickness of gold is e^{-90} of that which enters ; and that is sheer opacity.

[Maxwell's calculation in Art. 798, carried out numerically, makes the damping

$$e^{-\frac{2\pi\mu p}{\sigma}x}, = \exp. (-10^8x) \text{ for gold,}$$

see equation (8) above; or, for a thickness of half a wave-length, 10^{-1000} , which is billions of billions of billions (indeed a number with 960 digits) times greater opacity than what we have here calculated, and is certainly wrong.]

It must, however, be granted, I think, that the green light that emerges from gold-leaf is not properly transmitted; it is light re-emitted by the gold *. The incident light, say the red, is all stopped by a thickness less than half a wave-length. The green light may conceivably be due to atoms vibrating fairly in concordance, and not calling out the conducting opacity of the metal. If the calculated opacity, notwithstanding this, is still too great, it is no use assuming a higher conductivity at higher frequency, for that would act the wrong way. What must be assumed is either some special molecular dispersion theory, or else greater specific resistance for oscillations of the frequency which get through; nor must the imaginative suggestion made immediately below equation (13) be altogether lost sight of.

There is, however, the possibility mentioned above that the relative specific inductive capacity of gold, K/K_0 , if a meaning can be attached to it, may be very large, perhaps (though very improbably, see Drude, *Wied. Ann.* vol. xxxix. p. 481) comparable with 1800. Suppose for a moment that it is equal to 1800; then the value of the critical quantity (7) is 1 and the value of α is

$$p \sqrt{\frac{1}{2} \mu K \times .4} = \frac{p}{v} \sqrt{\left(\frac{K}{5K_0} \right)} = \frac{2\pi \times 5 \times 10^{14}}{3 \times 10^{10}} \sqrt{360} \\ = 19 \times 10^5,$$

which reduces the calculated opacity considerably, though still not enough.

In general, calling $K/K_0 = c$, and writing the critical number $\frac{4\pi\mu v^2}{\sigma p c}$ as h/c , we have

$$\frac{2v^2}{p^2} \alpha^2 = \frac{1}{2} \alpha^2 / \beta^2 = \lambda^2 \alpha^2 / 2\pi^2 = \sqrt{(h^2 + c^2)} - c;$$

* This would be fluorescence, of course; and Dr. Larmor argues in favour of a simple ordinary exponential coefficient of absorption even in metals. See *Phil. Trans.* 1894, p. 738, § 27.

so that $a\lambda/2\pi$ ranges from $\sqrt{\frac{1}{2}}h$ when h/c is big, to $\frac{1}{2}h/\sqrt{c}$ when h/c is small.

Writing the critical number h/c as $\tan \epsilon$, the general value of α is given by

$$a\lambda = \pi \sqrt{2c(\sec \epsilon - 1)}. \quad (14)$$

This is the ratio of the wave-length in air to the damping distance in the material in general; meaning by "the damping distance" the thickness which reduces the amplitude in the ratio $e:1$. (14) represents expression (5); compare with (12').

Theory of a Film.

So far nothing has been said about the limitation of the medium in space, or the effect of a boundary, but quite recently Mr. Heaviside has called my attention to a special theory, a sort of Fresnel-like theory, which he has given for infinitely thin films of finite conductance; it is of remarkable simplicity, and may give results more in accordance with experiment than the theory of the universal opaque medium without boundary, hitherto treated: a medium in which really the source is immersed.

Let a film, not so thick as gold-leaf, but as thin as the black spot of a soap-bubble, be interposed perpendicularly between source and receiver. I will quote from 'Electrical Papers,' vol. ii. p. 385:—"Let a plane wave $E_1 = \mu v H_1$ moving in a nonconducting dielectric strike flush an exceedingly thin sheet of metal [so thin as to escape the need for attending to internal reflexions, or the double boundary, or the behaviour inside]; let $E_2 = \mu v H_2$ be the transmitted wave out in the dielectric on the other side, and $E_3 = -\mu v H_3$ be the reflected wave*.

* *General Principles.*—It may be convenient to explain here the principles on which Mr. Heaviside arrives at his remarkably neat expression for a wave-front in an insulating medium,

$$\mathbf{E} = \mu v \mathbf{H},$$

or as it may be more fully and vectorially written,

$$\nabla(\nu \mathbf{E}) = \mu v \mathbf{H},$$

where $\mathbf{E}$ is a vector representing the electric intensity (proportional to the electric displacement), $\mathbf{H}$ is the magnetic intensity, and ν is unit normal to the wave-front. $\mathbf{E}$ and $\mathbf{H}$ are perpendicular vectors in the

“At the sheet we have

$$\begin{aligned} E_1 + E_3 &= E_2 \\ H_1 + H_3 &= H_2 + 4\pi k z E_2, \end{aligned}$$

k being the conductivity of the sheet of thickness z . Therefore

$$\frac{E_2}{E_1} = \frac{H_2}{H_1} = \frac{E_1 + E_3}{E_1} = \frac{1}{1 + 2\pi\mu k z v}, \quad \dots \quad (15)$$

same plane, *i.e.* in the same phase, and $\mathbf{E} \mathbf{H} v$ are all at right angles to each other.

The general electromagnetic equations in an insulating medium are perhaps sufficiently well known to be, on Mr. Heaviside's system,

$$\text{curl } \mathbf{H} = K \dot{\mathbf{E}} \quad \text{and} \quad -\text{curl } \mathbf{E} = \mu \dot{\mathbf{H}},$$

where “curl” is the vector part of the operator ∇ , and where Maxwell's vector-potential and other complexities have been dispensed with.

[In case these equations are not familiar to students I interpolate a parenthetical explanation which may be utilised or skipped at pleasure.

The orthodox definition of Maxwell's name “curl” is that $\mathbf{b}$ is called the curl of $\mathbf{a}$ when the surface-integral of $\mathbf{b}$ through an area is equal to the line-integral of $\mathbf{a}$ round its boundary, $\mathbf{a}$ being a vector or a component of a vector agreeing everywhere with the boundary in direction, and $\mathbf{b}$ being a vector or component of vector everywhere normal to the area. Thus it is an operator appropriate to a pair of looped or interlocked circuits, such as the electric and the magnetic circuits always are. The first of the above fundamental equations represents the fact of electromagnetism, specially as caused by displacement currents in an insulator, the second represents the fact of magneto-electricity, Faraday's magneto-electric induction, in any medium. Taking the second first, it states the fundamental law that the induced EMF in a boundary equals the rate of change in the lines of force passing through it; since the EMF or step of potential all round a contour is the line-integral of the electric intensity $\mathbf{E}$ round it, so that

$$\text{EMF} = \int_{\text{cycle}} \mathbf{E} d\mathbf{s} = - \frac{dN}{dt} = - \iint \mathbf{B} d\mathbf{S} = - \iint \mu \dot{\mathbf{H}} d\mathbf{S};$$

wherefore $-\mu \dot{\mathbf{H}}$ equals the curl of $\mathbf{E}$. (The statement of this second circuital law is entirely due to Mr. Heaviside; it is now largely adopted and greatly simplifies Maxwell's treatment, abolishing the need for vector potential.) [See, however, Appendix III. p. 386. *May* 1899.]

The first of the above two fundamental equations, on the other hand, depends on the fact that a current round a contour excites lines of magnetic force through the area bounded by it, and states the law that the total magnetomotive force, or line-integral of the magnetic intensity round the boundary, is equal to 4π times the total current through it; the total current being the “ampere-turns” of the practical Engineer.

H is reflected positively and E negatively. A perfectly conducting barrier is a perfect reflector; it doubles the magnetic force and destroys the electric force on the side containing the incident wave, and transmits nothing."

[I must here interpolate a remark to the effect that though it can hardly be doubted that the above boundary conditions (tangential continuity of both E and H) are correct, yet in general we cannot avoid some form of æther-theory

Expressing this law in terms of current density c , we write

$$\text{MMF} = \int_{\text{cycle}} H ds = 4\pi C = \iint 4\pi c dS;$$

so always current-density represents the curl of the magnetic field due to it, or $\text{curl } H = 4\pi c$.

Now in a conductor $c = kE$, but in an insulator $c = \dot{D}$, the rate of change of displacement or Maxwell's "displacement-current"; and the displacement itself is proportional to the intensity of the electric field, $D = \frac{K}{4\pi} E$; hence the value of current density in general is

$$c = kE + \frac{K}{4\pi} \dot{E},$$

whence in general

$$\text{curl } H = 4\pi kE + K\dot{E} = (4\pi k + Kp)E,$$

and in an insulator the conductivity k is nothing.

The connexion between "curl" so defined and $V\nabla$ is explained as follows. The operator ∇ applied to a vector R whose components are X Y Z gives

$$\left(i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz}\right) (iX + jY + kZ),$$

which, worked out, yields two parts

$$S\nabla R \text{ or } -\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right),$$

also called convergence, and

$$V\nabla R \text{ or } i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right)$$

or say $i\xi + j\eta + k\zeta$, where $\xi \eta \zeta$ are the components of a spin-like vector ω . Now a theorem of Sir George Stokes shows that the normal component of ω integrated over any area is equal to the tangential component of R integrated all round its boundary; hence $V\nabla$ and curl are the same thing.

Whenever ω is zero it follows that R has no circulation but is the derivative of an ordinary single-valued potential function, whose $dV = Xdx + Ydy + Zdz$. In electromagnetism this condition is by no means

when we have to lay down continuity conditions, and, according to the particular kind of æther-theory adopted so will the boundary conditions differ. My present object is to awaken a more general interest in the subject and to represent Mr. Heaviside's treatment of a simple case; but it must be understood that the continuity conditions appropriate to oblique incidence have been treated by other great mathematical physicists, notably by Drude, J. J. Thomson, and

satisfied. $\dot{E}$ and H or $\dot{H}$ and E are both full of circulation, and their circuits are interlaced. Fluctuation in E by giving rise to current causes H ; fluctuation in H causes induced E .]

Now differentiating only in a direction normal to a plane wave advancing along x , the operator ∇ becomes simply id/dx when applied to any vector in the wave-front, the scalar part of ∇ being nothing.

So the second of the above fundamental equations can be written

$$-i \frac{dE}{dx} = \mu \frac{dH}{dt},$$

or

$$\frac{dE}{dH} = \mu v;$$

so, ignoring any superposed constant fields of no radiation interest, E and H are vectors in the same phase at right angles to each other, and their tensors are given by $E = \mu v H$.

Similarly of course the other equation furnishes $H = KvE$; thus giving the ordinary $K\mu v^2 = 1$, and likewise the fact that the electric and magnetic energies per unit volume are equal, $\frac{1}{2}KE^2 = \frac{1}{2}\mu H^2$.

A wave travelling in the opposite direction will be indicated by $E = -\mu v H$; hence, as is well known, if either the electric or the magnetic disturbance is reversed in sign the direction of advance is reversed too.

(The readiest way to justify the equation $E = \mu v H$, *a posteriori*, is to assume the two well-known facts obtained above, viz. that the electric and magnetic energies are equal in a true advancing wave, and that $v = 1/\sqrt{\mu\kappa}$; then it follows at once.)

Treatment of an insulating boundary.—At the boundary of a different medium without conductivity the tangential continuity of E and of H across the boundary gives us the equations

$$E_1 + E_3 = E_2$$

$$H_1 + H_3 = H_2,$$

where the suffix 1 refers to incident, the suffix 2 to transmitted, and the suffix 3 to reflected waves.

$H_1 + H_3$ may be replaced by $\mu v(E_1 - E_3)$, since the reflected wave is

Larmor, also by Lord Rayleigh, and it would greatly enlarge the scope of this Address if I were to try to discuss the difficult and sometimes controversial questions which arise. I must be content to refer readers interested to the writings of the Physicists quoted—especially I may refer to J. J. Thomson's 'Recent Researches,' Arts. 352 to 409, and to Larmor, Phil. Trans, 1895, vol. 186, Art. 30, and other places.]

Now apply this to an example. Take k for gold, as we have done before, to be $1/2000 \mu$ seconds per square centim.

reversed; so we shall have, for the second of the continuity equations,

$$E_1 - E_3 = \frac{\mu_2 v_2}{\mu_1 v_1} E_2 = nm E_2;$$

n being the index of refraction, and m the relative inductivity. Hence, adding and subtracting,

$$\frac{E_2}{E_1} = \frac{2}{1+nm},$$

and

$$\frac{E_3}{E_1} = -\frac{1-nm}{1+nm};$$

well-known optical expressions for the transmitted and reflected amplitudes at perpendicular incidence, except that the possible magnetic property of a transparent medium is usually overlooked.

Treatment of a conducting boundary.—But now, if the medium on the other side of the boundary is a conductor instead of a dielectric, a term in one of the general equations must be modified; and, instead of curl $H = KpE$, we shall have, as the fundamental equation inside the medium,

$$-\frac{dH}{dx} = 4\pi kE;$$

or more generally $(4\pi k + Kp)E$.

So, on the far side of a thin slice of thickness z , the magnetic intensity H_2 is not equal to the intensity $H_1 + H_3$ on the near side, but is less by

$$dH = 4\pi kE dx = 4\pi kE_2 z = 4\pi k\mu r z H_2;$$

and this explains the second of the continuity equations immediately following in the text.

In a quite general case, where all the possibilities of conductivity and capacity &c. are introduced at once, the ratio of E/H is not μv or $(\mu/K)^{\frac{1}{2}}$, but is $(g + \mu p)^{\frac{1}{2}}(4\pi k + Kp)^{-\frac{1}{2}}$ for waves in a general material medium, (g may always be put zero), or $(R + pL)^{\frac{1}{2}}(Q + pS)^{-\frac{1}{2}}$ for waves guided by a resisting wire through a leaky dielectric.

The addition of dielectric capacity to conductivity in a film is therefore simple enough and results in an equation quoted in the text below.

and $v = 3 \times 10^{10}$ centim. per sec., for v is the velocity in the dielectric not in the conductor; then take a film whose thickness z is one twenty-fifth of a wave-length of the incident light; and the ratio of the transmitted to the incident amplitude comes out

$$1/2\pi\mu kvz = \frac{1000}{\pi vz} = \frac{1}{200}.$$

Some measurements made by W. Wien at Berlin in 1888 (Wied. Ann. vol. xxxv.), with a bunsen-burner as source of radiation, give as the actual proportion of the transmitted to the long-wave incident light, for gold whose thickness is

Simple treatment of the E.M. theory of light.—It is tempting to show how rapidly the two fundamental electromagnetic equations, in Mr. Heaviside's form, lead to the electromagnetic theory of light, if we attend specially to the direction normal to the plane of the two perpendicular vectors E and H , to the direction along say x , so that $\nabla = i d/dx$ and $\nabla^2 = -d^2/dx^2$.

In an insulating medium the equations are

$$\text{curl } H = K \dot{E} \quad \text{and} \quad -\text{curl } E = \mu \dot{H};$$

now $\text{curl} = \nabla \nabla = \nabla$, since $S \nabla = 0$ in this case, so

$$\nabla^2 H = K \nabla \dot{E} = K \text{curl } \dot{E} = -K \mu \ddot{H};$$

or, in ordinary form,

$$\frac{d^2 H}{dx^2} = K \mu \frac{d^2 H}{dt^2},$$

and there are the waves.

If this is not rigorous, there is no difficulty in finding it done properly in other places. I believe it to be desirable to realize things simply as well.

In a conducting medium the fundamental equations are, one of them,

$$\text{curl } H = K \dot{E} + 4\pi k E = (Kp + 4\pi k) E,$$

while the other remains unchanged; unless we like to introduce the non-existent auxiliary g , which would make it

$$-\nabla \nabla E = (v + \mu p) H,$$

and would cover wires too.

$$\text{So} \quad -\nabla^2 H = (4\pi k + Kp)(g + \mu p) H,$$

the general wave equation. In all these equations p stands for d/dt ; but, for the special case of simply harmonic disturbance of frequency $p/2\pi$, of course ip can be substituted.

10⁻⁵ centim., '0033 or 1/300; while for gold one quarter as thick the proportion was 0.4 (see Appendix II. page 385).

He tried also two intermediate thicknesses, and though approximately the opacity increases with the square of the thickness, it really seems to increase more rapidly: as no doubt it ought, as the boundaries separate. However, for a thickness $\lambda/25$ I suppose we may assume that about 1/3rd of the light would be transmitted, whereas the film-theory gives $(1/200)^2$; so even now a metal calculates out too opaque though it is rather less hopelessly discrepant than it used to be. The result, we see, for the infinitely thin film, is independent of the frequency.

Specific inductive capacity has not been taken into account in the metal, but if it is it does not improve matters. It does not make much difference, unless very large, but what difference it does make is in the direction of increasing opacity. In a letter to me Mr. Heaviside gives for the opacity of a film of highly conducting dielectric

$$\frac{E_1}{E_2} = \left\{ (1 + 2\pi\mu kvz)^2 + (\frac{1}{2}mczp/v)^2 \right\}, \dots (16)$$

where I have replaced his $\frac{1}{2}\mu v z K_p$ last term by an expression with the merely relative numbers K/K_0 and μ/μ_0 , called c and m respectively, thus making it easier to realise the magnitude of the term, or to calculate it numerically.

Theory of a Slab.

An ordinary piece of gold-leaf, however, cannot properly be treated as an infinitely thin film; it must be treated as a slab, and reflexions at its boundaries must be attended to. Take a slab between $x=0$ and $x=l$. The equations to be satisfied inside it are the simplified forms of the general fundamental ones

$$-\frac{dE}{dx} = \mu p H, \quad -\frac{dH}{dx} = 4\pi k E,$$

k being $1/\sigma$, and K being ignored; while outside, at $x=l$, the condition $E = \mu v H$ has to be satisfied, in order that a wave may emerge.

The following solutions do all this if $q^2 = 4\pi\mu kp$:—

$$E = Ae^{qx} \left(1 + \frac{qv+p}{qv-p} e^{2q(l-x)} \right),$$

$$\mu v H = -\frac{Aqv}{p} e^{qx} \left(1 - \frac{qv+p}{qv-p} e^{2q(l-x)} \right).$$

Conditions for the continuity of both E and H at $x=0$ suffice to determine A , namely if $E_1 H_1$ is the incident and $E_3 H_3$ the reflected wave on the entering side, while $E_0 H_0$ are the values just inside, obtained by putting $x=0$ in the above,

$$E_1 + E_3 = E_0,$$

$$E_1 - E_3 = \mu v (H_1 + H_3) = \mu v H_0.$$

Adding, we get a value for A in terms of the incident light E_1 ,

$$2E_1 p(qv-p) = A\{(qv+p)^2 e^{2ql} - (qv-p)^2\}.$$

whence we can write E anywhere in the slab,

$$\frac{E}{E_1} = \frac{2p(qv-p)}{(qv+p)^2 e^{2ql} - (qv-p)^2} \left\{ e^{qx} + \frac{qv+p}{qv-p} e^{2ql} e^{-qx} \right\}.$$

Put $x=l$, and call the emergent light E_2 ; then

$$\frac{E_2}{E_1} = \frac{4p q v}{(qv+p)^2 e^{2ql} - (qv-p)^2 e^{-ql}} = \rho, \text{ say,} \quad (17)$$

and this constitutes the measure of the opacity of a slab, ρ^2 being the proportion of incident light transmitted.

It is not a simple expression, because of course p signifies the operator d/dt , and though it becomes simply ip for a simply harmonic disturbance, yet that leaves q complex. However, Mr. Heaviside has worked out a complete expression for ρ^2 , which is too long to quote (he will no doubt be publishing the whole thing himself before long), but for slabs of considerable opacity, in which therefore *multiple* reflexions may be neglected, the only important term is

$$\rho = \frac{4\sqrt{2} e^{-\alpha l} p/\alpha v}{1 + (1 + p/\alpha v)^2}, \quad \dots \dots (18)$$

with

$$\alpha = \sqrt{(2\pi\mu p/\sigma)} = 3 \times 10^6$$

for light in gold ; and

$$\frac{\rho}{\alpha v} = \frac{2\pi}{\alpha \lambda} = \frac{2\pi}{30 \times 5} = \frac{1}{25} \text{ about.}$$

So the effect of attending to reflexion at the walls of the slab is to still further diminish the amplitude that gets through, below the $e^{-\alpha l}$ appropriate to the unbounded medium, in the ratio of $\frac{2\sqrt{2}}{25}$, or about a ninth.

Effect of each Boundary.

It is interesting to apply Mr. Heaviside's theory to a study of what happens at the first boundary alone, independent of subsequent damping.

Inside the metal, by the two fundamental equations, we have

$$E_0 = \left(\frac{\mu p}{4\pi k} \right)^{\frac{1}{2}} H_0,$$

and by continuity across the boundary

$$\begin{aligned} E_1 + E_3 &= E_0, \\ E_1 - E_3 &= \mu v H_0 = \mu v E_0 \left(\frac{4\pi k}{\mu p} \right)^{\frac{1}{2}} = \frac{qv}{p} E_0, \end{aligned}$$

where still $q^2 = 4\pi \mu k p$.

Therefore, for the transmitted amplitude

$$\frac{E_0}{E_1} = \frac{2p}{p + qv},$$

and for the reflected

$$\frac{E_3}{E_1} = \frac{p - qv}{p + qv},$$

or rationalising and writing amplitudes only, and understanding by p no longer d/dt in general, but only 2π times the frequency,

$$\frac{E_0}{E_1} = \frac{2p}{\sqrt{((p + \alpha v)^2 + \alpha^2 v^2)}} = \rho_1 \text{ say. . (19)}$$

Any thickness of metal multiplies this by the factor $e^{-\alpha x}$, and then comes the second boundary, which, according to what has been done above, has a comparatively small but peculiar effect ; for it ought to change the amplitude from ρ_1

into ρ , that is to give an emergent amplitude

$$\frac{4\sqrt{2} \cdot p/\alpha v}{1 + (1 + p/\alpha v)^2} E_1 e^{-\alpha l},$$

instead of the above incident on the second boundary

$$\frac{2p/\alpha v}{\sqrt{1 + (1 + p/\alpha v)^2}} E_1 e^{-\alpha l}; \quad . \quad . \quad (20)$$

that is for the case of light in gold, for which $p/\alpha v$ is small, to change $2/\sqrt{2}$ into $2\sqrt{2}$, in other words, to double it.

The effect of the first boundary alone, ρ_1 , is $\frac{2\sqrt{2}\pi}{\alpha\lambda}$, or say $1/18$, and this is a greater reduction effect than that reckoned above for the two boundaries together.

Thus the obstructive effect of the two boundaries together comes out less than that of the first boundary alone—an apparently paradoxical result. About one-eighteenth of the light-amplitude gets through the first boundary, but about one-ninth gets through the whole slab (ignoring the geometrically progressive decrease due to the thickness, that is ignoring $e^{-\alpha l}$, and attending to the effect of the boundaries alone; which, however, cannot physically be done). At first sight this was a preposterous and ludicrous result. The second or outgoing boundary ejects from the medium nearly double the amplitude falling upon it from inside the conductor! But on writing this, in substance, to Mr. Heaviside he sent all the needful answer by next post. "The incident disturbance inside is not the whole disturbance inside."

That explains the whole paradox—there is the reflected beam to be considered too. At the *entering* boundary the incident and reflected amplitudes are in opposite phase, and nearly equal, and their algebraic sum, which is transmitted, is small. At the *emerging* boundary the incident and reflected amplitudes are in the same phase, and nearly equal, and their algebraic sum, which is transmitted, is large—is nearly double either of them. But it is a curious action:—either more light is pushed out from the limiting boundary of a conductor than reaches it inside, or else, I suppose, the velocity of light inside the metal must be greater than it is outside, a result not contradicted by Kundt's refraction experiments, and suggested by most optical

theories. It is worth writing out the slab theory a little more fully, to make sure there is no mistake, though the whole truth of the behaviour of bodies to light can hardly be reached without a comprehensive molecular dispersion theory. I do not think Mr. Heaviside has published his slab theory anywhere yet. A slab theory is worked out by Prof. J. J. Thomson in Proc. Roy. Soc. vol. xlv., but it has partly for its object the discrimination between Maxwell's and other rival theories, so it is not very simple. Lord Kelvin's Baltimore lectures probably contain a treatment of the matter. All that I am doing, or think it necessary to do in an Address, is to put in palatable form matter already to a few leaders likely to be more or less known: in some cases perhaps both known and objected to.

The optical fractions of Sir George Stokes, commonly written b c e f , are defined, as everyone knows, as follows. A ray falling upon a denser body with incident amplitude 1 yields a reflected amplitude b and a transmitted c . A ray falling upon the boundary of a rare body with incident amplitude 1 has an internally reflected amplitude e and an emergent f . General principles of reversibility show that $b + e = 0$, and that $b^2 + cf = 1$ in a transparent medium.

Now in our present case we are attending to perpendicular incidence only, and we are treating of a conducting slab; indeed, we propose to consider the obstructive power of the material of the slab so great that we need not suppose that any appreciable fraction of light reflected at the second surface returns to complicate matters at the first surface. This limitation by no means holds in Mr. Heaviside's complete theory, of course, but I am taking a simple case.

The characteristic number which governs the phenomenon is $\frac{p}{av}$ or $\frac{2\pi}{a\lambda}$, a number which for light and gold we reckoned as being about $\frac{1}{27}$, that is decidedly smaller than unity, a being $\sqrt{(2\pi\mu kp)}$ or $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$. The characteristic number p/av

we will for brevity write as h , and we will express amplitudes for perpendicular incidence only, as follows :—

Incident amplitude 1,

$$\text{externally reflected } b = - \left\{ \frac{1 + (1-h)^2}{1 + (1+h)^2} \right\}^{\frac{1}{2}}$$

$$\text{entering } c = \frac{2h}{\{1 + (1+h)^2\}^{\frac{1}{2}}},$$

Incident again 1,

$$\text{internally reflected } e = \left\{ \frac{1 + (1-h)^2}{1 + (1+h)^2} \right\}^{\frac{1}{2}},$$

$$\text{emergent } f = \frac{2\sqrt{2}}{\{1 + (1+h)^2\}^{\frac{1}{2}}}.$$

(It must be remembered that e and f refer to the second boundary *alone*, in accordance with the above diagram.)

Thus the amplitude transmitted by the whole slab, or rather by both surfaces together, ignoring the opacity of its material for a moment, is

$$\text{transmitted } cf = \frac{4\sqrt{2} \cdot h}{1 + (1+h)^2}.$$

To replace in this the effect of the opaque material, of thickness l , we have only to multiply by the appropriate exponential damper, so that the amplitude ultimately transmitted by the slab is

$$\frac{4\sqrt{2} \cdot p/\alpha v}{1 + (1 + p/\alpha v)^2} e^{-\alpha l}$$

times the amplitude originally incident on its front face.

This agrees with the expression (18) specifically obtained above for this case, but, once more I repeat, multiple reflexions have for simplicity been here ignored, and the medium has been taken as highly conducting or very opaque.

But even so the result is interesting, especially the result for f . To emphasize matters, we may take the extreme case when the medium is so opaque that h is nearly zero; then b is nearly -1 , c is nearly 0, being $h\sqrt{2}$, e is the same as b except for sign, and f is nearly 2.

An opaque slab transmits $8h^2e^{-2\alpha l}$ of the incident light energy; its first boundary transmits only $2h^2$. The second

or emergent boundary doubles the amplitude. Taken in connexion with the facts of selective absorption and the timing of molecules to vibrations of certain frequency, I think that this fact can hardly be without influence on the green transparency of gold-leaf.

APPENDIX I.

MR. HEAVISIDE'S *Note on Electrical Waves in Sea-Water.*

[Contributed to a discussion at the Physical Society in June 1897 : see Mr. Whitehead's paper, *Phil. Mag.* August 1897.]

"To find the attenuation suffered by electrical waves through the conductance of sea-water, the first thing is to ascertain whether, at the frequency proposed, the conductance is paramount, or the permittance, or whether both must be counted.

"It is not necessary to investigate the problem for any particular form of circuit from which the waves proceed. The attenuating factor for plane waves, due to Maxwell, is sufficient. If its validity be questioned for circuits in general, then it is enough to take the case of a simply-periodic point source in a conducting dielectric ('*Electrical Papers*,' vol. ii. p. 422, § 29). The attenuating constant is the same, viz. (equation (199) *loc cit.*):—

$$n_1 = \frac{n}{v} \sqrt{\frac{1}{2}} \left[\left\{ 1 + \left(\frac{4\pi k}{cn} \right)^2 \right\}^{\frac{1}{2}} - 1 \right]^{\frac{1}{2}},$$

where $n/2\pi$ is the frequency, k the conductivity, c the permittivity, and $v = (\mu c)^{-\frac{1}{2}}$, μ being the inductivity.

"The attenuator is then $e^{-n_1 r}$ at distance r from the source, as in plane waves, disregarding variations due to natural spreading. It is thus proved for any circuit of moderate size compared with the wave-length, from which simply periodic waves spread.

"The formula must be used in general, with the best values of k and c procurable. But with long waves it is pretty certain that the conductance is sufficient to make $4\pi k/cn$ large. Say with common-salt-solution $k = (30^{11})^{-1}$, then

$$\frac{4\pi k}{cn} = \frac{2k\mu v^2}{f}$$

if f is the frequency. This is large unless f is large, whether we

assume the specific c/c_0 to have the very large value 80 or the smaller value effectively concerned with light waves. We then reduce n_1 to

$$n_1 = (2n\mu k\pi)^{\frac{1}{2}} = 2\pi(\mu k f)^{\frac{1}{2}},$$

as in a pure conductor.

"This is practically true perhaps even with Hertzian waves, of which the attenuation has been measured in common-salt-solution by P. Zeeman. If then $k^{-1} = 30^{11}$ [and if the frequency is 300 per second] we get $n_1 =$ about $\frac{1}{50000}$.

"Therefore 50 metres is the distance in which the attenuation due to conductivity is in the ratio 2.718 to 1, and there is no reason why the conductivity of sea-water should interfere, if the value is like that assumed above.

"These formulæ and results were communicated by me to Prof. Ayrton at the beginning of last year, he having enquired regarding the matter, on behalf of Mr. Evershed I believe.

"The doubtful point was the conductivity. I had no data, but took the above k from a paper which had just reached me from Mr. Zeeman. Now Mr. Whitehead uses $k^{-1} = 20^{10}$, which is no less than 15 times as great. I presume there is good authority for this datum *. None is given. Using it we obtain $n_1 = \frac{1}{18175}$.

"Thus 50 metres is reduced to 13.16 metres. But a considerably greater conductivity is required before it can be accepted that the statements which have appeared in the press, that the failure of the experiments endeavouring to establish telegraphic communication with a light-ship from the sea-bottom was due to the conductance of the sea, are correct. It seems unlikely theoretically, and Mr. Stevenson has contradicted it (in 'Nature') from the practical point of view. So far as I know, no account has been published of these experiments, therefore there is no means of finding the cause of the failure."

* Dr. J. L. Howard has recently set a student to determine the resistivity of the sea-water used by Professor Herdman, density 1.019 gr. per c.c., and he finds it to be 3×10^{10} c.g.s. at 15° C.—O. J. L., March 1899.

Added, May 1899.—An old student of mine, Prof. W. M. Thornton, of the Durham College, Newcastle, tells me that the resistance of sea-water collected at Tynemouth pier at flood-tide is 2.39×10^{10} c.g.s. at 15° C.

APPENDIX II.

Experiments on the Transparency of Metals.

The experiments of W. Wien on the transparency of metals, by means of a bolometer arranged to receive the radiation from a bunsen burner transmitted through different films, resulted in the following numbers for the proportion of radiation transmitted.

Metal.	Thickness in 10^{-7} centim.	Proportion transmitted.		Proportion reflected.
		Bunsen burner luminous.	Bunsen burner non-luminous.	
Platinum	20	·32	·37	·13
Iron & Platinum	40+20	·10	·14	·45
Gold 1 ..	56	·040	·041	·63
Gold 2	100	·0035	·0036	·80
Gold 3	24	·41	·41	·05
Gold 4	35	·20	·20	·19
Silver 1 (blue) ...	36	·058	·046	·78
Silver 2 (grey) ...	39·5	·058	·055	·60
Silver 3 (grey) ...	29	·25	·42	·40
Silver 4 (blue) ...	59·7	·0022	·0019	·95
Silver 5 (grey) ...	27·3	·31	·43	·24

The thickness is in millionths of a millimetre, *i. e.* is in terms of the milli-mikrom* called by microscopists $\mu\mu$.

The films were on glass, and the absorption of the glass was allowed for by control experiments.

* In spelling the word "mikrom" thus, I am following Lord Kelvin, who wishes to introduce two units—one, the ordinary mikrometre (the mikrom) or $\frac{1}{1000}$ of a millimetre, called by microscopists μ ; the other, a unit of time to be called a "michron," being the time required for light to traverse the above small distance. This suggestion of Lord Kelvin's commends itself to me; at any rate I take the opportunity of calling the attention of Physicists to it.

It is to be understood that of the whole incident light the proportion reflected is first subtracted, and the residue is then called 1 in order to reckon the fraction transmitted of that which enters the metal, it being understood that the residue which is not transmitted (say $\cdot 68$ or $\cdot 63$ in the case of platinum) is absorbed. It may be that more and better work has been done on the opacity of metals than this: at any rate there seems to me room for it. I do not quote these figures with a strong feeling of confidence in their accuracy. They are to be found in Wied. *Ann.* vol. xxxv. p. 57.

APPENDIX III. (*Added later.*)

The parenthetical statement near the bottom of page 372, concerning the first clear statement of the two circuital laws by Mr. Heaviside, represented subjectively the truth for me and for some others; but it should have been more carefully guarded if it was to be taken as representing objective truth: for I am referred to Maxwell, *Philosophical Transactions*, 1868 ("Collected Papers," vol. ii. p. 138), where the statement is very clear, and also to Rayleigh, *Philosophical Magazine*, August 1881, equations 8 and 9. I regret my inadequate acquaintance with scientific history.