The Multiplication of Alephs.

By

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In order to prove that

$$\aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma,$$

where γ is any (finite or transfinite) ordinal number, we shall generalise Cantor's*) method of proving that

$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

This method is to consider an aggregate $\{(\mu, \nu)\}$ of couples of integers, where μ and ν are — independently of one another — any finite positive integers**), so that this aggregate is evidently of cardinal number $\aleph_0 \cdot \aleph_0$; to arrange this aggregate in a doubly-infinite series; and to rearrange, by 'diagonal enumeration'***), the double series as

*) „Ein Beitrag zur Mannigfaltigkeitslehre“. Journ. für Math. Bd. 84, 1878, pp. 242—258; cf. Math. Ann. Bd. 46, 1895, p. 494. Fundamentally the same method was used by Cantor („Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen“ (Journ. für Math. Bd. 77, 1874, pp. 258—263) in proving the enumerability of the aggregate of real algebraic numbers.

Cantor (see Bernstein, „Untersuchungen aus der Mengenlehre“, Diss. Göttingen, Halle a. S., 1901, p. 49; or, in the reprint, with corrections and additions, in Math. Ann. Bd. 61, 1905, pp. 150—151) has indicated that his theorem can be generalised into one on the multiplication of any Alephs, but he neither gave a proof nor mentioned the difficulty about the «multiplicative axiom» (which can be overcome, in this case).

**) The couples are *couples with sense*; that is to say, (μ, ν) is identical ($=$) with (μ', ν') only if $\mu = \mu'$ and $\nu = \nu'$, so that, if μ is not identical with ν , (μ, ν) is not with (ν, μ) , as it would be if (μ, ν) represented a *class* (without any particular ordering-relation). Here 'finite positive integer' means 'ordinal number less than ω '.

***) The meaning of this phrase is that the term $u_{\mu, \nu}$ in the double series $(u_{\mu, \nu})$ is the λ^{th} term counted, where

$$\lambda = \frac{1}{2}(\mu + \nu - 1)(\mu + \nu - 2) + \mu.$$

Cf. Jourdain, 'On Unique, Non-Repeating, Integer-Functions', Mess. of Math., May 1901.

a *simply* infinite series (the cardinal number of whose terms is $\aleph_0$), in such a way that each term of the one series occurs in the other.

This manner of enumeration can be characterised as follows. If (μ, ν) and (μ', ν') are any two different couples, and

$$\mu + \nu < \mu' + \nu',$$

or if

$$\mu + \nu = \mu' + \nu' \quad \text{and} \quad \mu < \mu',$$

then, in the simply-infinite series,

$$(\mu, \nu) < (\mu', \nu').$$

If

$$\mu + \nu > \mu' + \nu',$$

or if

$$\mu + \nu = \mu' + \nu' \quad \text{and} \quad \mu > \mu',$$

then

$$(\mu, \nu) > (\mu', \nu').$$

The only remaining case, that

$$\mu + \nu = \mu' + \nu' \quad \text{and} \quad \mu = \mu',$$

is only possible if $\nu = \nu'$, and thus if (μ, ν) and (μ', ν') are identical.

1.

Now this rule for ordering the couples (μ, ν) can be extended to the case of μ and ν being any, finite or transfinite, ordinal numbers, without any alteration. In fact, if $\alpha, \beta, \alpha',$ and β' are any ordinal numbers, we have only the possibilities

$$\alpha + \beta > \alpha' + \beta', \quad \alpha + \beta < \alpha' + \beta', \quad \text{and} \quad \alpha + \beta = \alpha' + \beta';$$

and, in the last case, if also $\alpha = \alpha'$, we must have $\beta = \beta'$.*)

Accordingly the aggregate $\{(\alpha, \beta)\}$, where α and β are, each independently of the other, any ordinal numbers contained in the number-classes preceding some definite one**) (that which begins with ω_γ) can

*) If β is transfinite, we *can* have

$$\alpha + \beta = \alpha' + \beta', \quad \beta = \beta', \quad \text{and yet} \quad \alpha' > \alpha;$$

let, for example, $\beta = \alpha = \omega$ and $\alpha' = \omega + 1$.

**) This limitation is added, since 'the series where α and β are *any* ordinal numbers' is not, in spite of appearances, a series at all (cf. Russell, Proc. Lond. Soc. (2), vol. IV, 1906, p. 33). It is necessary to conclude this from the self-contradictory results to which the supposition that there *is* a series of all ordinal (or cardinal) numbers leads. It is Russell's merit to have stated explicitly the analogous fact (*ibid.*, pp. 32—36) that certain logically determinate propositional functions do not determine classes at all. In this way, his own (cardinal) contradiction is solved in a way that agrees with my somewhat obscurely expressed solution of Burali-Forti's (ordinal) contradiction.

be put, by the above method, in the form of a simply-ordered series. We will next prove that this series is *well-ordered* (§ 2), and of type ω_γ^* (§ 3).

2.

To prove that the rule of order described brings about a well-ordering, we must show how to find, in any given part P of the aggregate $\{(\alpha, \beta)\}$ an element (α_0, β_0) which is the lowest in rank of all the members of P , when P is arranged in the way described.

Each member (α, β) of P determines one, and only one, ordinal number $\alpha + \beta$.**) Among the members of the aggregate $\{\alpha + \beta\}$, where the couples (α, β) are members of P , there is a least ordinal number κ ; for any aggregate of ordinal numbers has a least (in order of magnitude). This κ determines an aggregate u , of one or more members, of couples (α, β) of P such that $\alpha + \beta = \kappa$. In u there is one and only one couple (α_0, β_0) such that α_0 is the least (in order of magnitude) ordinal number of all the first terms α of the couples which are the members of u .***)

This couple (α_0, β_0) is the first of P when P is arranged in the above order; for if there were a member (α_1, β_1) of P such that

$$(\alpha_1, \beta_1) < (\alpha_0, \beta_0),$$

we should have, either

$$\alpha_1 + \beta_1 < \alpha_0 + \beta_0, \quad \text{or} \quad \alpha_1 + \beta_1 = \alpha_0 + \beta_0 \quad \text{and} \quad \alpha_1 < \alpha_0,$$

and these are excluded by the above selections of $\alpha_0 + \beta_0$ as the least in $\{\alpha + \beta\}$, and α_0 as the least of all the α 's in u .

It is to be noticed that, in the case of this aggregate $\{(\alpha, \beta)\}$, we have indicated how, in any part P of it, a specialised element (α_0, β_0) can be found, and thus that the 'multiplicative axiom'†) can be proved. If u is any aggregate, and we can prove the multiplicative axiom for u , Zermelo††) has shown that u can be well-ordered, and the converse is evidently true.

*) Following Russell ('The Principles of Mathematics', vol. I, Cambridge, 1903 p. 322; Jourdain, Phil. Mag., March 1904, p. 295), ω_γ denotes the first ordinal number of the $(2 + \gamma)^{\text{th}}$ numberclass; so that, as will follow from our main theorem (see §§ 3 and 4), ω_γ is the least ordinal number belonging to the cardinal number $\aleph_\gamma$, and every ordinal number less than ω_γ belongs to a smaller Aleph.

**) One ordinal-number ξ may determine *many* couples (γ, δ) of P for which $\gamma + \delta = \xi$; thus, if $\xi = \omega + \omega$, two corresponding couples may be (ω, ω) and $(\omega + 1, \omega)$.

***) There cannot be *two* couples in u with the same α_0 , say (α_0, β_0) and (α_0, β_0') ; for $\alpha_0 + \beta_0$ is never equal to $\alpha_0 + \beta_0'$ unless $\beta_0 = \beta_0'$.

†) Cf. my paper in the Quart. Journ. of Math. 1907, pp. 356, 360—366.

††) „Beweis, daß jede Menge wohlgeordnet werden kann“, Math. Ann. Bd. 59, 1904, pp. 514—516.

3.

We generate*) the Alephs corresponding to the ordinal numbers by defining an $\aleph_\gamma$ for any γ , but we do not assume that all these Alephs are different, that is to say, that $\aleph_\xi < \aleph_\gamma$ if, and only if, $\xi < \gamma$. This proposition is equivalent to our theorem on the multiplication of Alephs. In fact, if α is some ordinal number, and $\xi < \alpha$ implies that

$$\aleph_\xi \cdot \aleph_\xi = \aleph_\xi,$$

then, and only then,

$$\aleph_\alpha > \aleph_\xi,$$

and $\aleph_\alpha$ is the next greater to all these $\aleph_\xi$'s.**)

Let us now start from the aggregate $\{(\alpha, \beta)\}$, where $\alpha < \omega_\gamma$, $\beta < \omega_\gamma$, arranged, by the above method, in a well-ordered series S , and assume that

$$(1) \quad \xi < \omega_\gamma \text{ implies that } \bar{\xi} \cdot \bar{\xi} = \bar{\xi},$$

where $\bar{\xi}$ is the cardinal number associated with the ordinal number ξ ***) Then $\aleph_\gamma$ is the next Aleph greater than all the $\bar{\xi}$'s, and we have only to prove that

$$\aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma. \dagger)$$

Let S' be any given segment of S and σ' the type of S' ; we prove that $\bar{\sigma}' < \aleph_\gamma$. Let (α', β') be the lowest couple in S not in S' , then

$$\alpha' + \beta' < \omega_\gamma.$$

For

$$\overline{\alpha' + \beta'} = \bar{\alpha}' + \bar{\beta}' \leq 2 \cdot \bar{\xi} \leq \bar{\xi} \cdot \bar{\xi},$$

where $\bar{\xi}$ is the larger of the two, α' , β' ; and (1) gives

$$\bar{\xi} \cdot \bar{\xi} = \bar{\xi} < \aleph_\gamma;$$

hence

$$\overline{\alpha' + \beta'} < \aleph_\gamma.$$

All the couples in S' are formed with numbers α , β , such that $\alpha + \beta \leq \alpha' + \beta'$. Put $\alpha' + \beta' = \lambda$, then,

*) See Russell, op. cit., p. 322.

**) See Jourdain, Phil. Mag., Jan. 1904, p. 74, and March 1904, pp. 297, 300.

***) That is to say, the cardinal number of any aggregate which can be arranged in type ξ . Thus $\bar{\omega}_\gamma = \aleph_\gamma$.

†) If we prove this, we shall have proved the theorem on multiplication for all Alephs by the generalised theorem of complete induction: We know that the theorem holds for the least Aleph $\aleph_0$; and, if it holds for all Alephs less than $\aleph_\gamma$, it holds for $\aleph_\gamma$; then it holds for all Alephs.

$$\overline{\sigma'} \leq \bar{\lambda} \cdot \bar{\lambda},$$

since all the couples (α, β) of S' are such that

$$\alpha < \lambda, \quad \beta < \lambda,$$

and, since also, by (1)*),

$$\bar{\lambda} \cdot \bar{\lambda} = \bar{\lambda} < \aleph_\gamma,$$

we have

$$(2) \quad \overline{\sigma'} < \aleph_\gamma.$$

Now (2) implies that

$$(3) \quad \bar{\sigma} \leq \aleph_\gamma^{**}),$$

since, if not, there would be a segment of S of cardinal number $\aleph_\gamma$.

We have

$$(4) \quad \bar{\sigma} = \aleph_\gamma \cdot \aleph_\gamma;$$

and since we also evidently have

$$(5) \quad \aleph_\gamma \cdot \aleph_\gamma \geq \aleph_\gamma,$$

we conclude, from (3), (4), and (5), that

$$(6) \quad \aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma.$$

Also we can conclude that S is of type ω_γ .

4.

The theorem (6) is chiefly important in proving that there actually is a series of different Alephs. We can define cardinal numbers of various segments of the ordinal number-series, but we cannot prove that they are different, — in other words, if ω_ω (the first of the $(2 + \omega)^{\text{th}}$ class) is the least ordinal number which is greater than all the ordinal numbers γ which form the extension of a certain aggregate u (for example, u may consist of the numbers less than ω_ω), that

$$\aleph_\omega > \aleph_\gamma,$$

and that $\aleph_\omega$ is the *least* such cardinal number — without a previous proof of (6).***)

Also, in the generalisation of König's†) theorem, that $2^{\aleph_0}$ is not

*) Notice that we do not assume an equality like (1) for *finite* cardinal numbers (for which it is not true, in general), but only for certain Alephs.

**) The use of the relation $\leq$ between two cardinal numbers always indicates that the Schröder-Bernstein theorem (see my remarks in the Quart. Journ. of Math., 1907, pp. 355—356) has been applied.

***) I pointed out this in the Phil. Mag., Jan. 1904, p. 74. See also a note in § 3 above.

†) „Zum Kontinuum-Problem“, Verhandl. des dritten internat. Math.-Kongr., 1905, pp. 144—147.

an Aleph of the form $\aleph_{\gamma+\omega}$, where γ is any ordinal number, to the proof that $2^{\aleph_\xi}$ is not an Aleph of the form $\aleph_{\gamma+\zeta}$, where γ is any ordinal number and ζ any Limes-number of the $(2+\xi)^{\text{th}}$ class, (6) is again required.

5.

The proof that I designed in 1903*) of the theorem that

$$(7) \quad \aleph_1 \cdot \aleph_1 = \aleph_1$$

and its generalisation to (6), is only sketched, and, though having at the bottom the same thought as that used here, has a difficulty of detail**), from which the present proof is free. Also that former proof assumes the principle of arbitrary selection, but this principle can be avoided by an easily-given specification of the order in which the couples (α, β) are to be enumerated. In fact, the essence of the proof given in this paper is such a specification.

Thus my proof was not subject to the criticism which A. E. Harward***) has brought against it; — the question is quite different from that which Harward raised†) *à propos* of my mistaken argument††) on the Schröder-Bernstein theorem; but this Harward did not seem to perceive. However, the proof of (7) given by Harward†††) as a modification of my own is quite satisfactory, except that it does not emphasise the difficulty (as to the multiplicative axiom) that it overcomes. His method has the further advantage of proving that the number

$$\xi = \sum_{\xi < \beta} \xi + \alpha$$

takes every value less than ω_γ *once and once only*, when α and β vary *independently* of one another through all values less than ω_γ ; — a generalisation of the theorem quoted in the last note of the introduction above.

*) Phil. Mag., March 1904, pp. 298—301.

**) Namely, a series of type ω_1 of numbers κ such that both

$$\alpha + \beta < \kappa \quad \text{and} \quad \gamma < \kappa$$

where γ is the suffix of general term of a simple series $\{a_n\}$ to which the double series $\{u_{\alpha,\beta}\}$ is to be put in correspondence and the terms a_γ of simple series have the same cardinal number as those $u_{\alpha,\beta}$ of the double series. This difficulty was pointed out to me by Prof. Zermelo.

***) 'On the Transfinite Numbers', Phil. Mag., October 1905, pp. 439—460, especially pp. 457—459.

†) Ibid., pp. 455—457.

††) Ibid., Jan. 1904, pp. 71—72.

†††) Loc. cit., pp. 458—459.

Finally, Gerhard Hessenberg*) proved, by a wholly different method, the theorem that, quite generally,

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta,$$

which is easily proved to be equivalent to the theorem expressed by (6).

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Dec. 27th, 1907, and Jan. 28th, 1908.**)

*) „Grundbegriffe der Mengenlehre“, Göttingen 1906, pp. 106—109. On p. 109, Hessenberg stated that Zermelo had also communicated a proof of (6) to him which, since it is essentially different from his own, will be published elsewhere.

**) I must here refer gratefully to the trouble which Prof. Zermelo has taken in repeatedly criticising weak points in my proofs and suggesting improvements.