SOLUTION OF PROBLEMS IN GEOMETRY

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It is maintained by many that one of the chief factors in the solution of problems in geometry is the ability to visualize clearly and vividly the figure that will satisfy the conditions of the problem. To discover the extent to which visualization enters into such solutions, to determine the precise manner in which visualization functions, and to ascertain what other differences in behavior are found between those who excel in geometry and those who do poorly in it, were the purposes that animated the present investigation.

As a part of another study the Stockard-Bell tests in plane geometry were given to one hundred and forty-two freshmen students in the department of education of the University of Texas. The scores of these students ranged from 5 to 46 out of a possible maximum of 70. Assuming that these tests afford a reasonably accurate indication of attainments in geometry, the range of scores was wide enough to justify the conclusion that those having the higher scores might well be called "good" geometry students, while those at the bottom of the list would qualify as "poor" geometry students. Our intention was to take the highest five and the lowest five for intensive study, but for various reasons it was found impossible to secure for the experimental study those who stood at the very bottom of the list. The students ultimately available were the following.

Good group			Poor group		
Student	Age	Score	Student	Age	Score
GU...	17.0	46	PD..	21.0	20
GV....	17.0	38	PA..	20.0	11
GY.....	17.0	45	PB.....	18.0	14
GZ.....	18.0	28	PC..	19.0	12
GX.....	19.0	34	PE.....	19.0	13
GW.....	19.0	30			
	—	—		—	—
Average.....	17.8	37		19.4	14

It will be seen that while the average score of the good group is almost three times that of the poor group, the average age of the latter is almost two years higher than that of the former. This is quite in accord with other studies on elementary and high school pupils, and supports the generalization that in a given group classified by ordinary school progress the duller pupils are apt to be the older and the brighter pupils the younger. All the students participating in the experiment were girls.

THE TESTS

The material selected for the experimental work consisted of four exercises in plane geometry. Exercises I and III presented locus problems, Exercise II is the famous Binet test of the results of a wedge-shaped cut on the edge of a folded paper, and Exercise IV was one involving a simple geometrical construction. Exercise II was included because it appeals to what is traditionally called visual imagery, and because it was found by Kelley¹ to be more highly correlated with geometrical ability than any other test used in his analysis. The other three exercises were designed to afford room for original reasoning, and at the same time to enable the experimenter to observe the behavior of the subjects and to judge of their method of attack.

It soon became apparent that these exercises were too difficult for even the best students to perform independently. Since, however, the object of the study was not the mere performance of the tasks, but rather the analysis of the student's methods of attack, the difficulty of the exercises helped rather than hindered the attainment of the desired goal. If the problems had been such that the subjects could grasp their solution at a flash, the experimenter would have had to rely entirely on the introspective reports of inexperienced observers. Such introspection is apt to be quite untrustworthy. But where the subject has to halt and think carefully, it not only renders him more amenable to the questions of the experimenter, but also enables the latter to make a better comparison of the attitudes, habits of attention, persistence, certainty, etc., of the good and the poor groups. A list of suggestions was prepared for each exercise, to be given only when it had become apparent that the

¹Truman Lee Kelley: Educational Guidance: An Experimental Study in the Analysis and Prediction of Ability in High School Pupils. *Teachers College Contributions to Education*, No. 71, 1914, p. 22.

student could go no further without help. The suggestions were arranged in such a way as to indicate by their efficacy something of the nature of the processes going on in the mind of the subject. They consisted of three groups, the first appealing to pure reason, the second to concrete imagery, and the third to immediate sensory experience or objective experiment. In this way the number of suggestions given furnished a significant index to the mental reactions of the subject.

THE DIRECTIONS

Exercise I

"In this exercise, which consists of a problem in plane geometry, and which I shall presently read, you are to solve the problem mentally, without using paper and pencil and without moving your hands. The problem will be read through twice, then you will be allowed two minutes in which to study it. You are not expected to solve it in two minutes: it will not be easy, but neither will it be excessively difficult. At the end of two minutes, you will be asked certain questions which you will answer as clearly and concisely as you can without moving your hands; *i.e.*, if you speak of a vertical line, do not move your hand up and down, or if you speak of a horizontal line, do not move the hand back and forth. Work as rapidly as you can, but do not hurry: think first, then speak. You must not use pencil and paper in any problem except when specifically directed. Do not ask any questions during your two-minute study period; in general, do not ask questions on any problem except when you are told to do so. Do not grow impatient if you desire to ask a question, as you will be given ample opportunity in due time. If you misunderstand anything, or forget something, do not ask questions; try to think it out by yourself, try to remember it without help, then whether you have succeeded or not, answer all questions as correctly and clearly as you can. We want to know what you think about the problem and how you think it, whether you solve it or not. Here is the problem. I shall read it through twice, and after two minutes, I shall ask you some questions which you will answer without moving your hands:"

"A straight line AB moves so that it remains parallel to a given straight line XY, and touches at one end a given circumference. Find the locus of the other end."

These instructions in their entirety were read to each subject that took problem I. The purpose of the instructions is sufficiently obvious without explanation; the main thing to notice is that they were uniform, as were also the questions and suggestions following the presentation of the problem. All the suggestions were given in the order above indicated until the subject succeeded in solving the problem.

QUESTIONS AND SUGGESTIONS

1. Repeat the problem just as it was read to you.
2. What is the locus of the other end?
3. How did you arrive at this conclusion?
4. Have you a picture in your mind of the figure as it would appear on paper?
5. Are the lines in your picture clear, can you see them plainly?
6. Describe your figure just as you see it in your mind.
7. Move your line AB so that the end touching the circumference is about 90 degrees from the starting point. Move it an equal amount twice more in the same manner.
8. Connect the points where the other end of the line has been.
9. How many sides has the figure thus formed? Is it regular?
10. What difference would there have been in the figure had you moved the line only 45 degrees at a time? and seven times?
11. Therefore, what limit would the shape of your figure approach as you lessened the distance you move AB each time? What kind of a line does the other end of AB generate as you move it under the given conditions?
12. Draw your figure on paper, using compass, ruler, and protractor.
(The examiner should here repeat the questions above beginning with 7 until the subject succeeds.)
13. (If the subject still fails to respond correctly, the examiner must go through an objective demonstration; but when this has been required, the student's reaction has been counted as a complete failure to solve the problem.)

We have said that the suggestions were arranged in groups. While the arrangement may seem at first sight arbitrary, it is not entirely so. Suggestions (7) to (11) inclusive plainly appeal to pure reason primarily. Hence, if a subject solves the problem with the aid of the 11th suggestion, he may be said to have succeeded with the use of pure reason alone. But if the 12th is required, it has been counted as due to a concrete suggestion. And if nothing but the 13th has availed, the solution has been accounted a failure, and the most effective suggestion as objective experiment.

Exercise II

"In this exercise you will use this square piece of paper. I have here another piece of paper, of exactly the same shape and size. I

am going to fold it a certain way, which you will notice carefully, and then I am going to cut out a piece while it is yet folded. Your problem is to find the exact appearance of my piece of paper when I unfold it. You will indicate your conclusions by drawing on your piece of paper the shape and position of the cuts as you think they will appear on my paper. Do not ask any questions until told to do so. Keep your hands still while the paper is being cut, and do not raise your paper from the table in front of you, nor in any way move it while solving the problem. Here is your problem:"

The experimenter places the square piece of paper with its sides vertical in full view of the subject. He folds it along the diagonal from the right upper to the lower left hand corner (as the experimenter faces the paper), so that the vertex of the lower right coincides with the vertex of the upper left hand corner. He folds along the other diagonal so that the lower left vertex coincides with the upper right vertex, and the left side coincides with the upper base; he folds along the vertical median so that the top base will be folded back over itself to the left, and the two diagonals will coincide throughout; now, beginning at the edge formed by the fold along the vertical median he cuts with the scissors in a line parallel to the bases about one and one-half inches, and from the end of this cut he cuts back to the median along a line parallel to the diagonals (the excised portion will thus represent a right isosceles); as soon as the cut is made he exposes it for three seconds, and lets the subject study it for two minutes.

QUESTIONS AND SUGGESTIONS

1. "Without folding your paper, explain in detail how you have obtained your result. Tell me exactly how I folded and cut my paper."
2. "You have made a slight oversight. I am going to tell you exactly how I folded and cut the paper, and see if you cannot detect your error." (The experimenter will here give the stimulus verbally as written out above.)
3. (If the subject still misses the point, the experimenter will give the verbal stimulus again, accompanied by actual folding of the paper, but without cutting it.)
4. (If the subject still fails, let him go through the motions himself, up to the actual cutting, but without unfolding the cut paper.)

Exercise III

"You will treat this problem exactly as you did the first one."
(Read the instructions of problem I). "Here is your problem:"

"A straight rod AB moves so that its ends constantly touch

two other straight rods which are perpendicular to each other. Find the locus of its middle point."

QUESTIONS AND SUGGESTIONS

1. Repeat the problem just as it was read to you.
2. What is the locus of its middle point?
3. Have you a picture of the figure in your mind?
4. Describe this picture. Are the lines clear?
5. The median upon the hypotenuse of a right triangle is equal to one-half the hypotenuse.
6. In this problem we have a right triangle whose legs are variable but whose hypotenuse = K . Therefore the median = $K/2$.
7. Of how many movements is AB capable under the given conditions?
8. Move AB as far as it will go one way and mark its middle point and move it to its limit the other way and again mark its mid-point. What is the distance from the right angle vertex to each of these points?
9. Mark similarly several intermediate positions of the mid-point.
10. How far is each of these from O ?
11. Does not each point with O determine the median of a right triangle whose hypotenuse is K ? From (5) above what is the relative length of these medians?
12. Draw the figure. (The experimenter will now give the suggestions over again from (5).)
13. If the subject fails, give an objective demonstration, showing the relation to a ladder, etc.

Exercise IV

GIVE THE SAME DIRECTIONS AS IN EXERCISE I

Through a given point P between the sides of a given angle ABC , draw a line terminated by the sides of the angle and bisected at P .

QUESTIONS AND SUGGESTIONS

1. Repeat the problem as it was read to you.
2. What is your solution?
3. Have you a picture in your mind of the figure?
4. Describe this picture.
5. The line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of it.
6. Suppose you have found the line. Then this line with the sides of the angle ABC makes a triangle, and its middle point is P .
7. Through P pass a line parallel to the lower side of the triangle.
8. How does this line divide the third side of the triangle?
9. What is the relation of the distance from this intersection to the vertex at B to the length of the whole intersected side?
10. Draw the figure carefully. (The experimenter will now repeat the suggestions above beginning with (6), until the subject succeeds.)

THE RESULTS

Some of the results obtained in this study require special explanation; and the method of their determination must be kept clearly in mind if they are to be correctly interpreted. In the first place, the term Memory indicates immediate recall, and was graded arbitrarily as follows: if the subject was able to answer question (1) at the first trial in a satisfactory manner, he was graded 100; satisfactory here meaning simply a correct statement of the problem, not necessarily verbatim. If the subject failed on the first trial, the experimenter read the problem again twice through just as the first time, allowed two minutes, then asked again question (1); if the student failed on this second trial, he was graded 75. If the subject required three trials, he was graded 50; if four, 25; if more than four, 0.

Under the term Behavior are included stability of imagery, facility of verbal expression, and inhibition of superfluous motor accompaniments. These are best illustrated by examples: GX, in the good group, in Exercise I, presented a striking case of very stable and clear mental images. When question (1) was put after the presentation of the problem, she gave a verbatim recall. When question (6) was put, GX fixed her vision on a certain part of the wall of the laboratory, and described her figure in faultless English just as though she were looking at a printed figure, and used letters without once confusing or misplacing them. It is to be noticed also that the clarity of GX's figure in no wise depended upon the external point of fixation. GX was asked to look at various objects and in every case her figures were as stable and clear as if she were looking at them on the blackboard. Such stability was graded 100. The grading of Verbal Expression for GX was equally easy. Her English was faultless and logical. Her verbal description of the figure was so clear and exact that any one could have reconstructed the figure from the first hearing. Her motor activities were well under control. The experimenter could not detect the slightest tendency to move any muscle except those of speech, and then only when speaking effectively. GX never left a word half spoken, nor used one incorrectly; but always thought out every word before hand.

On the other hand let us review the case of PD, in the poor group, in Exercise III. PD had images, but they had the serious fault of being entirely wrong. It took three presentations to enable her to recall unaided the words in which the problem was presented. A

remarkable thing was that each time PD was entirely confident that she was right, and was much surprised when told that she did not have the correct idea at all. When it came to the description of her figure, the experimenter was unable, even with the most exaggerated patience, to arrive at any definite conclusion as to just what the figure was. PD was profuse in confessing that she did not know herself just what she was trying to describe. The experimenter then described the proper figure and drilled her on it so that she could describe it independently with a certain degree of facility and accuracy. But no sooner was the work of solving begun than PD twisted her figure into a hopeless tangle. This lack of stability was given the grade 0. PD's English was very faulty and difficult to understand. Her verbal expressions consisted chiefly of timorous interjections, exclamations, and verbless and unfinished sentences. A mark of 25 seemed to represent fairly her proficiency in English expression. A striking thing about PD was her strong, uncontrollable motor tendencies. Contrary to directions, she immediately seized a pencil and began drawing lines as soon as the experimenter had presented a problem. This she continued to do in spite of repeated protests on the part of the experimenter.

Table I shows some rather interesting things. We see that the good group bests the poor group in memory by a score of 94 to 56, which is sufficient alone, if the scores are to be interpreted absolutely, to explain the difference in geometrical ability between the two groups. But it is in the second item that we find the most significant indications. According to the results here expressed, the so-called visual tendency appears to be only moderately more pronounced in the good group. The difference in ability to use good English is far greater, and, in the opinion of the authors, is intimately associated causally, with the lack of stability and lack of clarity in the visual manifestations. Still more significant is the difference in scores for motor accompaniments. All but one of the good group were able to inhibit undesirable motor activities quite satisfactorily, while not one of the poor group showed the ability to do so. In the case of the good student GU we have to do with a high degree of automatic motor activity that scarcely reaches the threshold of consciousness. She was always scribbling, and frequently had no idea of what she had written. This motor activity was accompanied by a distinctly inferior type of verbal expression, and was characterized by a considerable number of incorrect solutions.

TABLE I

Subject	Exercise	Memory	Behavior			Effective suggestions			Number incorrect solutions	Final correct solution
			Stability of imaginary	Verbal expression	Motor accompaniments	Pure reason	Concrete imagery	Objective experiments		
I. Good group										
GU.....	I	100	100	50	100	...	100	...	2	1
	II	100	100	75	100	...	100	...	1	1
	III	100	100	75	100	...	...	100	2	1
	IV	100	100	75	50	...	100	...	2	1
GV.....	I	100	100	100	0	100	...	...	0	1
	II	100	100	100	0	100	...	...	1	1
	III	75	75	100	0	100	...	...	1	1
	IV									
GW.....	I	100	100	100	0	100	...	...	1	1
	II	100	0	100	0	...	100	...	1	1
	III	100	100	100	0	...	100	...	2	1
	IV									
GX.....	I	100	100	100	0	...	100	...	1	1
	II	100	100	100	0	100	...	...	0	1
	III	100	100	100	0	...	100	...	2	1
	IV	100	100	100	0	...	100	...	0	1
GY.....	I	100	100	100	0	...	100	...	1	1
	II	100	50	75	100	...	100	...	3	1
	III	25	100	100	0	...	100	...	2	1
	IV	100	100	80	0	...	100	...	0	1
GZ.....	I	100	100	100	0	...	...	100	0	1
	II	100	100	100	0	100	...	...	0	1
	III	100	100	100	0	...	100	...	2	1
	IV	75	100	100	0	...	100	...	1	1
Average.....		94	92	92	20	27	64	10	1.2	1
II Poor group										
PA.....	I	50	100	50	100	...	...	100	2	0
	II	50	100	50	100	...	...	100	3	0
	III	50	100	50	50	...	...	100	2	0
	IV	0	100	0	100	...	...	100	5	0
PB.....	I	100	100	100	100	...	...	100	3	0
	II	50	100	100	100	...	...	100	3	0
	III	100	100	100	100	...	...	100	3	0
	IV	100	100	75	100	...	...	100	4	0
PC.....	I	0	100	0	100	...	...	100	5	0
	II	100	100	25	100	...	...	100	1	1
	III	100	100	75	100	...	...	100	2	1
	IV	100	100	50	100	...	...	100	2	0
PD.....	I	50	100	0	100	...	...	100	2	0
	II	25	0	25	100	...	...	100	2	0
	III	25	0	25	100	...	...	100	0	0
	IV	75	75	0	100	...	...	100	2	0
PE.....	I	0	80	0	100	...	...	100	4	0
	II	25	0	25	100	...	...	100	2	1
	III									
	IV									
Average.....	..	56	75	42	97	...	...	100	2.6	17

On the whole it seems evident that the good students were able to inhibit their motor activities much better than the poor ones. Is this due to the greater motility of the poor students, and the lack of it among the good ones? From other observations it seems likely that there is little difference between the two groups in general motor tendencies, but rather that the good students remembered the directions and consciously suppressed tendencies to movement, while the poor group forgot all about them.

The results for the rubric Effective Suggestions seems to support this idea, especially with reference to the matter of breadth of attention and strength of memory. The poor group were not helped by the Pure Reason Suggestions nor by the Concrete Imagery, because they could not remember what was said. Probably this fault of memory was largely due to a lack of concentration on the matter in hand, to an inability to fasten the attention on the essential things, and therefore the inability to make fundamental logical associations. Without these, memory does not go very far and amounts to little as a factor in reasoning. But a weak and faulty attention works both ways; for in addition to making logical memory very problematical, it makes rote-memory useless. Suppose the subject has memorized a certain block of *logical* data in a more or less *rote*-fashion, as all the poor group memorized every thing they did memorize; now if the subject cannot fasten his attention on each essential in order long enough to examine accurately its full meaning and relations to the other elements of the problem, there is hardly a chance that he will 'bridge the gap' correctly. Hence we see that the poor group offered a multitude of incorrect solutions: these 'solutions' were only vague ideas which perhaps occurred to the good group as often and as incorrectly as to the poor group. The difference was that the good students had developed such a neural organization — had such a group of associations already formed that they immediately rejected these irrelevant ideas and sought for something more closely related to the conditions of the problem. This neural organization, this readiness to lay hold on the significant features of a situation and reject the irrelevant ones, seems to depend upon native tendencies that are far-reaching, and that have been built into life habits of accurately noting and recalling details, of care and precision in the use of language, and of conscious control of motor activities. Unless these life habits are already well formed, the prospect of developing a good geometer is not a very favorable

one. To what extent these life habits are developed by the study of formal geometry would be an interesting problem for experimental investigation.

The most striking difference revealed by the table is that in every case the good group reached a final correct solution, while in only three cases out of eighteen was this true of the poor group. Thus, so far as these exercises give any indication, the Stockard-Bell tests enabled us to select two groups, the one of which was distinctly superior, the other distinctly inferior, in geometrical ability.

CONCLUSIONS

1. Excellence in memory (for immediate recall) is very highly correlated with geometrical ability.

2. There was less observable difference in the quantity and quality of visual imagery than in any of the other elements compared; but the difference was in favor of the good group.

3. Excellence in verbal expression is very significantly related to ability in geometry.

4. Motor manifestations are exceedingly prominent in the poor group, and almost entirely absent in the good group. This may be due either to differences in motor tendencies, or differences in ability to control equally strong tendencies.

5. The poor group depend almost wholly on objective stimuli, and were most helped by objective suggestions.

6. The better memory of the good group may depend much on comprehension, but it seems closely related to more highly organized habits of neutral control.

7. The results of this study, in so far as they apply, show the efficacy of the Stockard-Bell Tests.