



LXIII. On the variable elements of a disturbed planet, and the equations of its motion on the plane of the orbit

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LXIII. *On the Variable Elements of a Disturbed Planet, and the Equations of its Motion on the Plane of the Orbit.* By the Rev. BRICE BRONWIN*.

THE following method of determining the elements of a disturbed planet, based upon a theorem of Lagrange, leads to a useful result. R being the perturbing function, the well-known equations of the planet's motion are

$$\left. \begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0, \\ \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0. \end{aligned} \right\} \dots (A.)$$

Let x, y, z be determined from

$$\left(\frac{d^2x}{dt^2} \right) + \frac{\mu x}{r^3} = 0, \quad \left(\frac{d^2y}{dt^2} \right) + \frac{\mu y}{r^3} = 0, \quad \left(\frac{d^2z}{dt^2} \right) + \frac{\mu z}{r^3} = 0. \dots (a.)$$

These values of x, y, z will satisfy (a.), however the elements may vary, since t alone varies in $\left(\frac{d^2x}{dt^2} \right)$, &c. They may therefore be made to satisfy (A.). But we have six quantities to determine and only three equations: we must then assume three conditions; let them be

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0, \quad \dots \dots (b.)$$

δ denoting the variation of the elements only. Then

$$\frac{dx}{dt} = \left(\frac{dx}{dt} \right), \text{ \&c.}, \quad \frac{d^2x}{dt^2} = \left(\frac{d^2x}{dt^2} \right) + \frac{\delta dx}{dt^2}, \text{ \&c.}$$

Put these values in (A.) and subtract (a.), there result

$$\frac{\delta dx}{dt^2} + \frac{dR}{dx} = 0, \quad \frac{\delta dy}{dt^2} + \frac{dR}{dy} = 0, \quad \frac{\delta dz}{dt^2} + \frac{dR}{dz} = 0, \dots (c.)$$

where d refers to the variation of t only, δ that of the elements.

If $\Delta x, \Delta y, \Delta z$ denote variations indeterminate, and independent one of another, we easily form from (b.) and (c.) the following, which is equivalent to (b.) and (c.), and contains the whole solution of the problem:

$$\left. \begin{aligned} \Delta x \frac{\delta dx}{dt} - \delta x \frac{\Delta dx}{dt} + \Delta y \frac{\delta dy}{dt} - \delta y \frac{\Delta dy}{dt} \\ + \Delta z \frac{\delta dz}{dt} - \delta z \frac{\Delta dz}{dt} + \Delta R dt = 0. \end{aligned} \right\} \dots (B.)$$

Leaving out $\Delta R dt$, this is independent of t . For since

$$\frac{d \Delta x}{dt} = \frac{\Delta dx}{dt}, \quad \frac{d \delta x}{dt} = \frac{\delta dx}{dt}, \text{ \&c.},$$

* Communicated by the Author.

the whole vanishes when we differentiate $\Delta x, \delta x, \Delta y, \delta y, \&c.$ for t . Also

$$\frac{d\delta dx}{dt^2} = \frac{\delta d^2x}{dt^2} = -\mu \delta \left(\frac{x}{r^3} \right) \text{ by (a.)} = 0 \text{ by (b.)};$$

and the same of $\frac{d\delta dy}{dt^2}, \frac{d\delta dz}{dt^2}$. And lastly, $\frac{d\Delta dx}{dt^2}, \&c.$ are multiplied by $\delta x = 0, \&c.$ The variation relative to t being nothing, it is independent of t .

The quantities $x, \frac{dx}{dt}, y, \frac{dy}{dt}, \&c.$ are composed of terms of the form $A \sin(in t + m), B \cos(in t + m)$; their variations therefore will contain terms of the form $A t \sin(in t + m), B t \cos(in t + m)$, multiplied by δn . And by substitution (B.) will be of the form

$$M + Nt + Pt^2 + \Delta R dt = 0. \quad . \quad . \quad (d.)$$

But we may make t anything or nothing without altering the value of the three first terms of (d.); therefore $N = 0, P = 0$. Hence if we make $t = 0$ where it appears in the coefficients, or leave out those terms, the final result will not be affected. And hence if we use $\int n dt$ instead of nt for the mean longitude and consider n a function of the time, the result will be the same; for the variations $\delta x, \&c.$ will only differ from their former values by wanting the terms containing t in their coefficients.

Let i be the inclination, θ the longitude of the node on the fixed plane, ϑ the same longitude on the plane of the orbit, having a fixed origin on it, in which case it is easy to see that $d\vartheta = \cos i d\theta$. Also let ξ and η be the rectangular co-ordinates on the plane of the orbit, the axis of ξ passing through the origin of ϑ . Transforming by known methods from the system (x, y, z) to that in which x lies in the line of the nodes, then to another in which y lies in the plane of the orbit, and lastly to the system (ξ, η) , we find

$$x = A\xi + B\eta, \quad y = C\xi + D\eta, \quad z = E\xi + F\eta, \quad . \quad (1.)$$

$$\begin{aligned} A &= \cos \theta \cos \vartheta + \cos i \sin \theta \sin \vartheta, & B &= \cos \theta \sin \vartheta - \cos i \sin \theta \cos \vartheta, \\ C &= \sin \theta \cos \vartheta - \cos i \cos \theta \sin \vartheta, & D &= \sin \theta \sin \vartheta + \cos i \cos \theta \cos \vartheta, \\ E &= -\sin i \sin \vartheta, & F &= \sin i \cos \vartheta. \end{aligned}$$

If we differentiate these last, making $d\vartheta = \cos i d\theta$, and compare the values of $dA, dE; dB, dF; dC, dE; dD, dF$; we shall see that

$$\left. \begin{aligned} dA &= \tan i \sin \theta dE, & dB &= \tan i \sin \theta dF, \\ dC &= -\tan i \cos \theta dE, & dD &= -\tan i \cos \theta dF. \end{aligned} \right\} \quad . \quad (2.)$$

And if in $x^2 + y^2 + z^2 = \xi^2 + \eta^2 = r^2$ we substitute for x, y , and z their values from (1.), it will become identical. Equalizing therefore the coefficients of ξ^2 , $\xi\eta$ and η^2 to nothing in the result, we have

$$A^2 + C^2 + E^2 = 1, \quad B^2 + D^2 + F^2 = 1, \quad AB + CD + EF = 0. \quad (3.)$$

These may be proved by putting for A, B, C , &c. their values given above.

Differentiate the two first of (3.), and

$$A \, dA + C \, dC + E \, dE = 0, \quad B \, dB + D \, dD + F \, dF = 0.$$

These by (2.) will give

$$(\tan i \sin \theta A - \tan i \cos \theta C + E) \, dE = 0,$$

$$(\tan i \sin \theta B - \tan i \cos \theta D + F) \, dF = 0.$$

Divide the first of these by dE , then multiply by dF ; divide the second by dF , then multiply by dE , the results by (2.) will be

$$\left. \begin{aligned} AdB + CdD + EdF &= 0, & BdA + DdC + FdE &= 0, \\ AdA + CdC + EdE &= 0, & BdB + DdD + FdF &= 0, \end{aligned} \right\} \quad (4.)$$

the two last being added from above.

Since

$$\frac{dx}{dt} = A \frac{d\xi}{dt} + B \frac{d\eta}{dt},$$

d denoting the variation of t only. Therefore

$$\frac{\delta dx}{dt} = A \frac{\delta d\xi}{dt} + B \frac{\delta d\eta}{dt} + \delta A \frac{d\xi}{dt} + \delta B \frac{d\eta}{dt}.$$

$$\text{And} \quad \Delta x = A \Delta \xi + B \Delta \eta + \xi \Delta A + \eta \Delta B.$$

Multiply the two last; there results

$$\begin{aligned} \Delta x \frac{\delta dx}{dt} &= A^2 \Delta \xi \frac{\delta d\xi}{dt} + A B \Delta \xi \frac{\delta d\eta}{dt} + A \delta A \Delta \xi \frac{d\xi}{dt} \\ &\quad + A \delta B \Delta \xi \frac{d\eta}{dt} + A B \Delta \eta \frac{\delta d\xi}{dt} + B^2 \Delta \eta \frac{\delta d\eta}{dt} + B \delta A \Delta \eta \frac{d\xi}{dt} \\ &\quad + B \delta B \Delta \eta \frac{d\eta}{dt} + A \Delta A \xi \frac{\delta d\xi}{dt} + B \Delta A \xi \frac{\delta d\eta}{dt} + \Delta A \delta A \xi \frac{d\xi}{dt} \\ &\quad + \Delta A \delta B \xi \frac{d\eta}{dt} + A \Delta B \eta \frac{\delta d\xi}{dt} + B \Delta B \eta \frac{\delta d\eta}{dt} + \Delta B \delta A \eta \frac{d\xi}{dt} \\ &\quad + \Delta B \delta B \eta \frac{d\eta}{dt}. \end{aligned}$$

In this result change A into C , B into D , and we have the expression of $\Delta y \frac{\delta dy}{dt}$. Again, change A into E , B into F ,

and we have that of $\Delta z \frac{\delta d z}{d t}$. Whence, if we have respect to (3.) and (4.), we find

$$\begin{aligned} \Delta x \frac{\delta d x}{d t} + \Delta y \frac{\delta d y}{d t} + \Delta z \frac{\delta d z}{d t} &= \Delta \xi \frac{\delta d \xi}{d t} + \Delta \eta \frac{\delta d \eta}{d t} \\ &+ (\Delta A \delta A + \Delta C \delta C + \Delta E \delta E) \xi \frac{d \xi}{d t} + (\Delta A \delta B + \Delta C \delta D + \Delta E \delta F) \xi \frac{d \eta}{d t} \\ &+ (\Delta B \delta A + \Delta D \delta C + \Delta F \delta E) \eta \frac{d \xi}{d t} + (\Delta B \delta B + \Delta D \delta D + \Delta F \delta F) \eta \frac{d \eta}{d t}. \end{aligned}$$

If in this we make Δ and δ change places, we shall have the expression of

$$\delta x \frac{\Delta d x}{d t} + \delta y \frac{\Delta d y}{d t} + \delta z \frac{\Delta d z}{d t}.$$

But by (2.),

$$\left. \begin{aligned} \Delta A \delta B + \Delta C \delta D + \Delta E \delta F &= \frac{1}{\cos^2 i} \Delta E \delta F \\ \Delta B \delta A + \Delta D \delta C + \Delta F \delta E &= \frac{1}{\cos^2 i} \Delta F \delta E \end{aligned} \right\} . \quad (5.)$$

Whence by substitution, making $\frac{\xi d \eta - \eta d \xi}{d t} = h$, and having regard to (5.), (B.) becomes

$$\left. \begin{aligned} \Delta \xi \frac{\delta d \xi}{d t} - \delta \xi \frac{\Delta d \xi}{d t} + \Delta \eta \frac{\delta d \eta}{d t} - \delta \eta \frac{\Delta d \eta}{d t} \\ + \frac{h}{\cos^2 i} (\Delta E \delta F - \Delta F \delta E) + \Delta R d t &= 0. \end{aligned} \right\} . \quad (C.)$$

This last divides into two. Make $p = -E$, $q = F$, and we have

$$\frac{h}{\cos^2 i} (\Delta q \delta p - \Delta p \delta q) + \Delta R d t = 0. \quad (D.)$$

Or by a further and obvious transformation,

$$h \sin i (\Delta i \delta \theta - \Delta \theta \delta i) + \Delta R d t = 0. \quad (E.)$$

And we shall also have for the determination of the co-ordinates or elements on the plane of the orbit,

$$\Delta \xi \frac{\delta d \xi}{d t} - \delta \xi \frac{\Delta d \xi}{d t} + \Delta \eta \frac{\delta d \eta}{d t} - \delta \eta \frac{\Delta d \eta}{d t} + \Delta R d t = 0. \quad (F.)$$

This, from the nature of the characteristic Δ , is equivalent to

$$\frac{\delta d \xi}{d t} + \frac{d R}{d \xi} = 0, \quad \frac{\delta d \eta}{d t} + \frac{d R}{d \eta} = 0, \quad \delta \xi = 0, \quad \delta \eta = 0. \quad (e.)$$

These will give the four elements, a , e , π and ϵ . But with any elements whatever, constant or variable, the values of ξ

and η will satisfy

$$\left(\frac{d^2\xi}{dt^2}\right) + \frac{\mu\xi}{r^3} = 0, \quad \left(\frac{d^2\eta}{dt^2}\right) + \frac{\mu\eta}{r^3} = 0. \quad \dots (f.)$$

Therefore from (e.) and (f.) by addition, there results

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} + \frac{dR}{d\xi} = 0, \quad \frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} + \frac{dR}{d\eta} = 0, \quad \dots (G.)$$

which are the same as if the plane of the orbit were fixed, the result intimated at the beginning of this paper.

To effect a further transformation of (F.), let v be the longitude on the orbit. Then $\xi = r \cos v$, $\eta = r \sin v$;

$$\frac{d\xi}{dt} = \cos v \frac{dr}{dt} - r \sin v \frac{dv}{dt}, \quad \frac{d\eta}{dt} = \sin v \frac{dr}{dt} + r \cos v \frac{dv}{dt};$$

$$\frac{\delta d\xi}{dt} = \cos v \frac{\delta dr}{dt} - r \sin v \frac{\delta dv}{dt} - \sin v \delta v \frac{dr}{dt} - \sin v \delta r \frac{dv}{dt} - r \cos v \delta v \frac{dv}{dt};$$

$$\frac{\delta d\eta}{dt} = \sin v \frac{\delta dr}{dt} + r \cos v \frac{\delta dv}{dt} + \cos v \delta v \frac{dr}{dt} + \cos v \delta r \frac{dv}{dt} - r \sin v \delta v \frac{dv}{dt}.$$

$$\text{Also } \Delta\xi = \cos v \Delta r - r \sin v \Delta v, \quad \Delta\eta = \sin v \Delta r + r \cos v \Delta v.$$

With these values we shall find

$$\begin{aligned} \Delta\xi \frac{\delta d\xi}{dt} + \Delta\eta \frac{\delta d\eta}{dt} &= \Delta r \frac{\delta dr}{dt} - r \Delta r \delta v \frac{dv}{dt} \\ &\quad + r^2 \Delta v \frac{\delta dv}{dt} + r \Delta v \delta v \frac{dr}{dt} + r \Delta v \delta r \frac{dv}{dt}. \end{aligned}$$

Making Δ and δ change places, we have the expression of

$$\delta\xi \frac{\Delta d\xi}{dt} + \delta\eta \frac{\Delta d\eta}{dt}.$$

By these values, observing that $\frac{dv}{dt} = \frac{h}{r^2}$, we shall find (F.) transformed into

$$\left. \begin{aligned} \Delta r \frac{\delta dr}{dt} - \delta r \frac{\Delta dr}{dt} + r^2 \left(\Delta v \frac{\delta dv}{dt} - \delta v \frac{\Delta dv}{dt} \right) \\ + \frac{2h}{r} (\Delta v \delta r - \Delta r \delta v) + \Delta R dt = 0. \end{aligned} \right\} \dots (H.)$$

We shall suppose t to have such a value that v may be equal to π , the longitude of the apse. Then

$$\frac{dr}{dt} = \frac{\mu e}{h} \sin(v - \pi), \quad \frac{\delta dr}{dt} = \frac{\mu e}{h} (\delta v - \delta \pi).$$

Also since

$$\frac{dv}{dt} = \frac{h}{r^2}, \quad \frac{\delta dv}{dt} = \frac{1}{r^2} \delta h - \frac{2h}{r^3} \delta r.$$

Putting these values in (H.), it becomes

$$\left. \begin{aligned} \frac{\mu e}{h} (\Delta r \delta v - \Delta v \delta r) + \frac{\mu e}{h} (\Delta \pi \delta r - \Delta r \delta \pi) \\ + \Delta v \delta h - \Delta h \delta v + \Delta R dt = 0. \end{aligned} \right\} \quad . \quad (g.)$$

If u be the eccentric anomaly,

$$r = a(1 - e \cos u), \quad \delta r = (1 - e) \delta a - a \delta e,$$

making $u = 0$ after taking the variations. Also

$$\tan \frac{v - \pi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}, \quad \delta v = \sqrt{\frac{1+e}{1-e}} \delta u + \delta \pi,$$

$$u - e \sin u = nt + \varepsilon - \pi, \quad (1-e) \delta u = -\frac{3nt}{2a} \delta a + \delta \varepsilon - \delta \pi = \delta \varepsilon - \delta \pi$$

by making $t = 0$, as it has been proved we may do. Therefore by elimination,

$$\delta v = \frac{(1-e^2)^{\frac{1}{2}}}{(1-e)^2} (\delta \varepsilon - \delta \pi) + \delta \pi.$$

With these values of δr , δv , (g.) becomes, making

$$k = 1 - \sqrt{1-e^2},$$

$$\left. \begin{aligned} \frac{n a k}{2} (\Delta a \delta \pi - \Delta \pi \delta a) + \frac{n a}{2} (\Delta \varepsilon \delta a - \Delta a \delta \varepsilon) \\ + \frac{\mu a e}{h} (\Delta e \delta \pi - \Delta \pi \delta e) + \Delta R dt = 0. \end{aligned} \right\} \quad . \quad (I.)$$

By making the coefficients of Δa , Δe , &c. separately equal to nothing, the equations (D.), (E.) and (I.) will give all the elements.

If e' be the eccentricity, π' the longitude of the apse, and ε' the epocha on the fixed plane, such as would be introduced by the integration of the differential equations relative to that plane, I find by a comparison of the values of the co-ordinates on the plane of the orbit and the fixed plane,

$$e \cos (\vartheta - \pi) = e' \cos (\theta - \pi'), \quad e \sin (\vartheta - \pi) = e' \cos i \sin (\theta - \pi');$$

and also $\varepsilon - \varepsilon' = \vartheta - \theta$ true to quantities of the order e^2 inclusive. Perhaps this last, like the two former, is strictly true. It is evident therefore that e' and π' at least differ from e and π , and that the same values cannot be used for either set indiscriminately without the hazard of error. I have not room to discuss this subject at length, but I wish to invite attention to it. The main object of this paper is the formulæ (G.), which I have found by a different method elsewhere.

Gunthwaite Hall, Penistone, Yorkshire,
September 30, 1844.