PERPETUANT SYZYGIES OF THE n-TH KIND

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CONTENTS.

A GREAT part of modern invariant theory deals with the question of what are known as "complete systems." Hilbert, in particular, has proved a series of theorems* of vast generality and power dealing with the formation and structure of such systems.

The present paper treats of "perpetuant types,"† and obtains for them a "syzygy chain" of the Hilbertian model. To the best of my belief, no previous example of a syzygy chain in invariant theory‡ has extended beyond the second link. The chain given here is of indefinite extent.

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* The most important of these are to be found in *Math. Annalen*, Bd. XXXVI. In what follows, whenever Hilbert is mentioned the reference is to this paper.

† Defined further on.

‡ Examples for a rational integral function of n variables have been given by Hilbert himself and by Schönflies (*Gött. Nach.*, 1891).

A brief sketch of the history of the subject may not be out of place.

George Boole is generally recognised as having laid the first foundations. In 1841 he showed that if the variables of any given form were subjected to any linear transformation, then the discriminant of the transformed expression would be the same as that of the original one, save for a factor containing only the constants of the substitution. This is the property of invariance.

The next step was to find other functions of the coefficients having this property. When Cayley first turned his attention to the subject, the discovery of each single invariant or covariant* was a distinct acquisition. All this was changed by his invention of the calculus of hyperdeterminants. By this he could evolve, from a given form, an infinite number of concomitants.

To Cayley also fell the further success of showing that in the case of the binary quantics of the four lowest orders, any of these concomitants can (if rational and integral) be expressed as a rational integral function of a certain finite number of them. This is what is meant by the statement that the concomitants of the forms in question possess a *complete system.*

As the result of his investigations on the binary quintic, Cayley concluded that for it there is no such complete system.

However, Gordan, using the symbolic notation† of Aronhold and Clebsch, was able to show that not only the binary quintic, but also the general binary quantic of any order, possesses a complete system. It is interesting to note that the erroneous conclusion as to the quintic was due to the non-recognition of the existence of a syzygy of the second kind.

In the memoir to which we have already referred and in later papers,‡ Hilbert greatly extended our knowledge of complete systems. His method is founded upon the following theorem :—

Let a homogeneous function of any number of variables be formed according to any definite laws ; then, although there may be an infinite number of functions F satisfying the conditions laid down, nevertheless a finite number $F_1 F_2 \ldots F_r$ can always be chosen so that any other F can be written in the form

$$F = A_1F_1+A_2F_2+\ldots+A_rF_r.$$

* A covariant only differs from an invariant in that it contains the variables as well as the coefficients. We shall use the term *concomitant* to include these as well as other functions (*e.g.*, "mixed concomitants") having the invariantive property.

† Which is really equivalent to Cayley's hyperdeterminants.

‡ *Gött. Nach.*, 1891 and 1892.

where the A's are homogeneous integral functions of the variables, but do not necessarily satisfy the conditions for the F's.

By means of this very general theorem he extended Gordan's theorem to quantics (and sets of quantics) of any order, in any number of variables, or even in several sets of variables. But this was not all. Having proved that a complete system can always be obtained, he next proceeded to study its internal structure. To follow his conclusions it will be convenient to recall a few definitions:—

The members of a complete system are called the *irreducible* concomitants. In terms of these all other concomitants can be expressed as rational integral functions.

A *syzygant* (of the *first* kind) is a rational integral function of the irreducible concomitants which, *when these last are replaced by their expressions as rational integral functions of the coefficients of the quantics*, is identically zero.* The equation expressing this fact is called a *syzygy*.

If a syzygant is obtainable from one of lower degree by multiplication throughout with a concomitant, it is called *compound.* A *reducible* syzygant is one which is compound or equivalent to the sum of two or more compound ones.

Hilbert has proved that, starting from a given form, the number of *irreducible* syzygants is finite, or, in other words, that *syzygants have a complete system.*

From syzygants of the first kind, we pass to those of higher kinds.

If $S_1 S_2 \dots S_r$ be irreducible syzygants of the first kind, and $P_1 P_2 \dots P_r$ certain concomitants or products of concomitants, it may happen that the expression

$$P_1 S_1 + P_2 S_2 + \dots + P_r S_r \equiv 0,$$

the expressions P, S being regarded as functions of the concomitants (which for the moment are treated as independent variables). If so, $P_1 S_1 + P_2 S_2 + \dots + P_r S_r$ is called a syzygant of the *second kind.* It will be seen that it is an identically vanishing sum of compound syzygants of the first kind. Syzygants of the second kind may, like those of the first, be reducible or irreducible. Hilbert has shown that the number of irreducible ones is finite.

* The simplest example (due to Cayley) is that for the binary cubic. See Elliott's *Algebra of Quantics*, p. 110.

We go on to syzygants of the third kind. Let $S_1^{(2)} S_2^{(2)} \dots S_r^{(2)}$ be irreducible syzygants of the second kind, and $P_1 P_2 \dots P_r$ be certain concomitants (or products of them) such that $P_1 S_1^{(2)} + P_2 S_2^{(2)} + \dots + P_r S_r^{(2)}$, when the $S^{(2)}$'s are replaced by their equivalent sums of compound syzygants of the first kind, is, *solely in virtue of this replacement*,* identically zero; then $P_1 S_1^{(2)} + P_2 S_2^{(2)} + \dots + P_r S_r^{(2)}$ is called a syzygant of the third kind.

In the same way a syzygant of the n-th kind is defined in the terms of those of the $(n-1)$-th kind and concomitant products.†

Hilbert has succeeded in proving that the syzygants of each species possess a complete system.

The object of this paper is to illustrate these conclusions. The particular syzygies selected for consideration are those connecting "perpetuant types."‡

In a paper by Young and Wood,§ the syzygies of the first kind connecting perpetuant types have been considered as far as degree 9. These authors differ fundamentally from all previous writers‖ on the same subject in that their methods are based upon the symbolical notation, and in particular upon Grace's perpetuant type theorem.¶ This theorem of Grace's, when taken in conjunction with a paper of Wood's,** gives in a neat form the symbolical expressions for the perpetuant types. Young and Wood's paper deals with the syzygies of the first kind connecting these, and in the present paper I deal with the syzygies of the second, third, and finally n-th kind.

A general method†† for the formation of these is developed and a some-

* *I.e.*, the syzygants of the first kind are *not* to be replaced, in their turn, by their equivalent sums of concomitant products.

† It will be seen that the definition of the third and higher kinds is slightly different from that of the second, which again differs from that of the first.

‡ A *perpetuant* is an *irreducible* seminvariant of a set of binary quantics of infinite order, and a perpetuant *type* is one linear in the coefficients of each quantic concerned. In case it should be thought that the problem proposed for consideration is an unduly specialised one, it may be remarked that theorems concerning perpetuants seem to be generally capable of extension to ordinary covariants (*i.e.*, those of forms of finite order). For instance, Young has so extended Grace's perpetuant type theorem and also the results of the paper (referred to below) on perpetuant syzygies of the first kind. See *Proc. London Math. Soc.*, Ser. 2, Vols. 1 and 3.

§ *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 221–265.

‖ A full list of previous writings is given in Young and Wood's paper. They all deal exclusively with the *first* kind.

¶ *Proc. London Math. Soc.*, Vol. xxxv.

** *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

†† Explained in Section VI.

what curious theorem is arrived at, differentiating those of kind n, and of degrees less than $2n+2$, from those of the same kind and higher degrees.

This general method, which we may call *symbolical multiplication*, is an extension of the well known method of deriving syzygies of the first kind from the Jacobian identity.

The results obtained are as follows:—

No perpetuant syzygy of the n-th kind can exist for degrees $< (n+2)$.

For degree $(n+2)$ one, and only one, perpetuant syzygy of the n-th kind exists. From this, by *symbolical multiplication*, many other perpetuant syzygies of the n-th kind of all degrees $> (n+2)$ may be derived.

Let these, together with the syzygy from which they spring, be called *primary perpetuant syzygies of the n-th kind.*

Then the principal theorem obtained is that *all perpetuant syzygies of the n-th kind of degree* $< (2n+2)$ *are primary*. We have thus obtained those members of the complete system that belong to the first n degrees for which perpetuant syzygies of the n-th kind exist at all. For the first kind this merely gives the fact that the Jacobian identity is the only syzygy of degree 3.

It is somewhat unexpected that the range of simplicity should increase as the "kind" of the syzygy increases.

After an examination in detail of these primary syzygies, introducing a principle of arrangement of symbolical letters analogous to that in Grace's perpetuant type theorem, and after finding the generating function corresponding to each case, a proof is given that *for degree* $(2n+2)$ *new syzygies that are not primary do actually exist.*

Among the minor results, it may be of interest to notice that perpetuant syzygies of the first kind, which, in the notation of Young and Wood, *reduce* perpetuant products having a factor C_1, are always compound.* This agrees with their results as far as they go.

A corresponding theorem as to perpetuant syzygies of the n-th kind is also proved.†

Beside the above, I have obtained the linearly independent set of perpetuant syzygies of the second kind of degree 6, and some results for perpetuant syzygies of the second kind of degree 7, and perpetuant syzygies of the third kind of degree 8. As these are somewhat lengthy, I have not included them in this present paper.

* See Lemma I.

† See Lemma III.

I. *General Methods of Finding Canonical Sets of Perpetuant Syzygies of the n-th Kind.*

The general procedure for finding the *canonical system* of perpetuant syzygies of the n-th kind of degree δ (*i.e.*, the set of linearly independent ones in terms of which all other perpetuant syzygies of the n-th kind and degree δ may be linearly expressed), when we know those for the perpetuant syzygies of the $(n-1)$-th kind of degrees less than δ, divides itself into three parts:

(1) Deducing, as explained below, from our knowledge of perpetuant syzygies of the $(n-1)$-th kind of degree $<\delta$, what must be the generating function for linearly independent* perpetuant syzygies of the n-th kind of degree δ.

(2) Actually finding perpetuant syzygies of the n-th kind. In most cases the irreducible ones can all be found by *symbolical multiplication*.

(3) Finding how many of the perpetuant syzygies of the n-th kind (both irreducible and compound) that we now have got are linearly independent. If the generating function for these is equal to the L.G.F. formed by (1), we know that we have obtained all the linearly independent perpetuant syzygies of the n-th kind.

The method used for proving a set of perpetuant syzygies of the n-th kind to be linearly independent is as follows:—

All compound perpetuant syzygies of the $(n-1)$-th kind of the degree in question are arranged in accordance with a fixed sequence.† Now a perpetuant syzygy of the n-th kind can be looked upon as a linear relation between compound perpetuant syzygies of the $(n-1)$-th kind. In any particular perpetuant syzygy of the n-th kind, take the compound perpetuant syzygy of the $(n-1)$-th kind that precedes, according to our arrangement, all the others involved in the perpetuant syzygy of the n-th kind under consideration.

We shall say that this compound perpetuant syzygy of the $(n-1)$-th kind is *resolved* by the perpetuant syzygy of the n-th kind.

If we have m perpetuant syzygies of the n-th kind, say A, B, C, ... of the same degree, and each resolves a different compound perpetuant syzygy of the $(n-1)$-th kind, say a, b, c, ..., respectively, *they must all be*

* We shall, in future, use L.G.F. to denote these five words.

† For the details of which see Section II.

linearly independent; for if, of $a, b, c, \ldots$, a stand first of all the m, b second, c third, and so on, according to the fixed arrangement, then a cannot occur in any of $B, C, D, \ldots$; therefore there is certainly no linear relation connecting A with $B, C, D, \ldots$.

Similarly there is none connecting B with $C, D, E, \ldots$, none connecting C with $D, E, F, \ldots$, and so on; therefore $A, B, C, \ldots$ are all linearly independent.

We speak of a compound perpetuant syzygy of the $(n-1)$-th kind for which a perpetuant syzygy of the n-th kind can be found, to resolve it, as *resolvable*. If no such perpetuant syzygy of the n-th kind can be found, the perpetuant syzygy of the $(n-1)$-th kind is called *irresolvable*.*

To explain (1),

the L.G.F. for perpetuant syzygies of the n-th kind

= generating function for *all* compound perpetuant syzygies of the $(n-1)$-th kind

—generating function for canonical perpetuant syzygies of the $(n-1)$-th kind.

In consequence of our device for proving linear independence, there is a one-to-one correspondence between a *doubly compound* (*i.e.*, with at least two perpetuant factors) perpetuant syzygy of the $(n-2)$-th kind that is resolvable at all by a *compound* perpetuant syzygy of the $(n-1)$-th kind, and the *canonical* compound perpetuant syzygy of the $(n-1)$-th kind that resolves it. The other compound perpetuant syzygies of the $(n-1)$-th kind that resolve it are not canonical. By taking all doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ, we bring in, once and once only, all compound perpetuant syzygies of the $(n-1)$-th kind of that degree.

Therefore, if from the generating function for all *compound*† perpetuant syzygies of the $(n-1)$-th kind resolving a certain form of doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ, we subtract the generating function for all doubly compound perpetuant syzygies of the $(n-2)$-th kind of that form that *are resolvable* by *any* compound

* In the notation of Young and Wood for perpetuant products, *irreducible* is the word used. I was forced to use another word, as an *irreducible perpetuant syzygy of the $(n-1)$-th kind* has another meaning (see introduction).

† It is because we are only dealing with *compound* perpetuant syzygies of the $(n-1)$-th kind of degree δ that we need only know the canonical set of perpetuant syzygies of the $(n-1)$-th kind for degrees *less than* δ.

perpetuant syzygies of the $(n-1)$-th kind, we get the L.G.F. for the associated perpetuant syzygies of the n-th kind.*

Taking the sum, for all possible forms of doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ of these L.G.F.'s, we get the total L.G.F. for perpetuant syzygies of the n-th kind of that degree.†

We shall use C_κ to denote a perpetuant of degree κ, $S_\delta^{(n)}$ to denote an irreducible syzygant of kind n and degree δ, and $[C_\kappa S_\delta^{(n)}]$ to denote a syzygant of the $(n+1)$-th kind, resolving the compound perpetuant syzygies of the n-th kind $C_\kappa S_\delta^{(n)}$.

II. *Arrangement of Compound Perpetuant Syzygies of the n-th Kind.*

The sequence is according to the following rules :—

(i) A non-primary‡ syzygy always precedes a primary one.

(ii) Of two syzygies, both non-primary or both primary, with C_1^m and C_1^n as factors respectively, where $m < n$, that with C_1^m as a factor stands first.

(iii) Of two syzygies, whose precedence is not determined by (i) or (ii), that which has the fewest perpetuant factors stands first.

(iv) If $$A = C_{\kappa_1} C_{\kappa_2} \dots C_{\kappa_p} S^{(n)} \quad (\kappa_1 \leqslant \kappa_2 \leqslant \kappa_3 \dots \leqslant \kappa_p),$$
and $$B = C_{\kappa_1'} C_{\kappa_2'} \dots C_{\kappa_p'} S'^{(n)} \quad (\kappa_1' \leqslant \kappa_2' \leqslant \kappa_3' \dots \leqslant \kappa_p'),$$

have not their precedence determined by (i), (ii) or (iii), then A precedes B if the first of the differences $(\kappa_1 - \kappa_1')$, $(\kappa_2 - \kappa_2')$, ..., $(\kappa_p - \kappa_p')$ that does not vanish is *positive*.

(v) If $$\left.\begin{aligned} A &= C_{\kappa_1} C_{\kappa_2} \dots C_{\kappa_p} S^{(n)} \\ B &= C'_{\kappa_1} C'_{\kappa_2} \dots C'_{\kappa_p} S'^{(n)} \end{aligned}\right\} \quad (\kappa_1 \leqslant \kappa_2 \leqslant \kappa_3 \dots \leqslant \kappa_p),$$
and

have not their precedence determined by (i), (ii), (iii), or (iv), and if, of the

* For an example, see Section III or Section XI.

† For the result is, as before,

generating function for all compound perpetuant syzygies of the $(n-1)$-th kind

$-$,, ,, canonical ,, ,, ,,

‡ A primary perpetuant syzygy of the n-th kind is one derived by symbolical multiplication from the perpetuant syzygies of the n-th kind of lowest possible degree. See Section VI.

symbols $a_1, a_2, \ldots, a_\delta$ involved,

C_{κ_1} (or $C_{\kappa_1} \ldots C_{\kappa_s}$, if $\kappa_1 = \kappa_2 = \kappa_3 = \ldots = \kappa_s < \kappa_{s+1}$) contain $a_{r_1}, a_{r_2}, \ldots, a_{r_t}$,

and

C'_{κ_1} (or $C'_{\kappa_1} \ldots C'_{\kappa_s}$, ,, ,, ,,) ,, $a_{r'_1}, a_{r'_2}, \ldots, a_{r'_t}$,

where $r_1 < r_2 \ldots < r_t$ and $r'_1 < r'_2 \ldots < r'_t$,

then A precedes B if the first of the differences

$$(r_1 - r'_1),\ (r_2 - r'_2),\ \ldots,\ (r_t - r'_t),$$

that does not vanish is *negative*.

If C_{κ_1} and C'_{κ_1} contain exactly the same letters, consider C_{κ_2} and C'_{κ_2}, …, and so on.

It will not be necessary to go into further details of this kind, when all the above principles fail, to establish the results of the present paper. It is sufficient to know that of two syzygies we *can** always determine which stands first.

For perpetuant products, replace $S^{(n)}$ in the above by unity. Principle (i) does not apply.

III. *Perpetuant Syzygies of the Second Kind of Degree* 4.

We consider a perpetuant syzygy of the second kind as connecting a canonical and an uncanonical compound perpetuant syzygy of the first kind that resolve the same perpetuant product. This product must contain at least three factors, or it cannot be resolved in more than one way by a compound perpetuant syzygy of the first kind. Its degree, therefore, cannot be less than three.

Degree 3.—The only product of at least three factors is C_1^3, which is irresolvable, therefore *there are no perpetuant syzygies of the second kind of degree* 3.

Degree 4.—The only products to be considered are

$$\text{(i)}\ C_1^2 C_2, \quad \text{and} \quad \text{(ii)}\ C_1^4.$$

* By following out, for example, part of the classification of perpetuant products given in a paper on "Perpetuant Syzygies (of the First Kind)," by A. Young and P. W. Wood, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 230-233. We use the classification given in Section III, (ii), (iii) and (iv) [but *not* (i)], and apply it to the perpetuant factors of compound perpetuant syzygies of the n-th kind.

(i) $C_1^2 C_2$.—This is only resolvable* if one of the two C_1's is A_1, and if C_2 be of unit weight, *i.e.*, the generating function for resolvables is $3x$.

The number of compound perpetuant syzygies of the first kind of form $C_1[C_1 C_2]$ that resolve a form $C_1^2 C_2$ is,* however, given by $4x$; therefore the generating function for linearly independent perpetuant syzygies of the second kind is the difference

$$4x - 3x = x,$$

i.e., there is only one perpetuant syzygy of the second kind of degree 4. It is

$$\begin{aligned}(a_1 a_2 a_3 a_4) \equiv\ & A_1(a_2 a_3 a_4) - A_2(a_1 a_3 a_4) + A_3(a_1 a_2 a_4) - A_4(a_1 a_2 a_3) \\ \equiv\ & A_1\{A_2(a_3 a_4) - A_3(a_2 a_4) + A_4(a_2 a_3)\} \\ & -A_2\{A_1(a_3 a_4) - A_3(a_1 a_4) + A_4(a_1 a_3)\} \\ & +A_3\{A_1(a_2 a_4) - A_2(a_1 a_4) + A_4(a_1 a_2)\} \\ & -A_4\{A_1(a_2 a_3) - A_2(a_1 a_3) + A_3(a_1 a_2)\} \\ \equiv\ & 0.\end{aligned}$$

A_r is here written instead of C_1 when the symbolical letter involved is a_r.

IV. *Perpetuant Syzygies of the Third Kind of Degree* 5.

Since the lowest degree of a perpetuant syzygy of the first kind is 3, therefore the lowest degree of a doubly compound perpetuant syzygy of the first kind is 5; therefore *the lowest possible degree of a perpetuant syzygy of the third kind is* 5.

The only doubly compound perpetuant syzygy of the first kind of this degree is of form $C_1^2 S_3^{(1)}$.

This is only resolvable† in more than one way if the C_1's are A_1 and A_2, when we get two resolutions.

Therefore *there is only one perpetuant syzygy of the third kind of degree* 5, *i.e.*,

$$\begin{aligned}(a_1 a_2 a_3 a_4 a_5) \equiv A_1(a_2 a_3 a_4 a_5) - A_2(a_1 a_3 a_4 a_5) + A_3(a_1 a_2 a_4 a_5) \\ - A_4(a_1 a_2 a_3 a_5) + A_5(a_1 a_2 a_3 a_4).\end{aligned}$$

* These results are quoted from Young and Wood's paper. They can easily be obtained independently.

† Since $(a_1 a_2 a_3 a_4)$ resolves $A_1(a_2 a_3 a_4)$.

V. *Perpetuant Syzygies of the n-th Kind of Degree* $(n+2)$.

Consider the zero determinant of $(n+2)$ rows and columns:—

$$\begin{vmatrix} A_1 & A_1 & \ldots\ldots & A_1 & a_1 \\ A_2 & A_2 & \ldots\ldots & A_2 & a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n+2} & A_{n+2} & \ldots\ldots & A_{n+2} & a_{n+2} \end{vmatrix}.$$

Denote this by $(a_1 a_2 \ldots a_{n+2})$.

Use a similar notation for the first minors of the first column. Then expanding

$$(a_1 a_2 \ldots a_{n+2}) = A_1(a_2 a_3 \ldots a_{n+2}) - A_2(a_1 a_3 \ldots a_{n+2}) + A_3(a_1 a_2 a_4 \ldots a_{n+2}) \ldots .$$

If we expand the first minors in terms of the constituents of *their* first columns and first minors, we shall have the original determinant expanded in terms of its first two columns. As these are the same, the result is identically zero.

Losing sight of the determinants, and quoting only this last identity, it follows that, if $(a_{r_1} a_{r_2} \ldots a_{r_n})$ be a perpetuant syzygy of the $(n-2)$-th kind, and $(a_{r_1} a_{r_2} \ldots a_{r_n} a_{r_{n+1}})$ be a perpetuant syzygy of the $(n-1)$-th kind, and $A_1 A_2 \ldots A_{n+2}$ denote, as usual, quantics of infinite order, then $(a_1 a_2 \ldots a_{n+2})$ is a perpetuant syzygy of the n-th kind.

But we have seen that $(a_1 a_2 a_3 a_4)$ is a perpetuant syzygy of the second kind, and $(a_1 a_2 a_3 a_4 a_5)$ is a perpetuant syzygy of the third kind. Therefore $(a_1 a_2 \ldots a_{n+2})$ *is a perpetuant syzygy of the n-th kind of degree* $(n+2)$.

Generating Function for Perpetuant Syzygies of the n-th Kind of Degree $(n+2)$.

To find the generating function for perpetuant syzygies of the n-th kind of degree δ, we take all doubly compound perpetuant syzygies of the $(n-2)$-th kind of that degree. The excess of the generating function for compound perpetuant syzygies of the $(n-1)$-th kind (some of them uncanonical) that resolve these, over the generating function for resolvable perpetuant syzygies of the $(n-2)$-th kind of degree δ, will be the generating function for the perpetuant syzygies of the n-th kind of degree δ. If $(a_1 \ldots a_{n+1})$ is the only perpetuant syzygy of the $(n-1)$-th kind of degree $(n+1)$, and if there are no perpetuant syzygies of the $(n-1)$-th kind of lower degrees; if, also $(a_1 \ldots a_n)$ is the only perpetuant syzygy of the $(n-2)$-th kind of degree n, and if there are no perpetuant syzygies of the

$(n-2)$-th kind of lower degrees; then there are no doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree $< (n+2)$; therefore, *there are no perpetuant syzygies of the n-th kind of degree* $<(n+2)$. This is true for kinds 2 and 3, and therefore it is true in general.

For degree $(n+2)$, we have only to consider $C_1^2 S_n^{(n-2)}$. This is resolvable in two ways only if the C_1's be A_1 and A_2; therefore, *there is only one perpetuant syzygy of the n-th kind of degree* $(n+2)$, *i.e.* $(a_1 \dots a_{n+2})$. This is true for kinds 2 and 3, and therefore in general.

VI. *Symbolical Multiplication and Primary Perpetuant Syzygies.*

$$A_1(a_2 a_3) - A_2(a_1 a_3) + A_3(a_1 a_2) \equiv 0,$$

or, as it is often written,

$$(a_2 a_3) - (a_1 a_3) + (a_1 a_2) \equiv 0,$$

is a perpetuant syzygy of the first kind.

If we multiply throughout by the symbols $(a_1 a_4)^\omega$, we get

$$(a_1 a_4)^\omega (a_2 a_3) - (a_1 a_4)^\omega (a_1 a_3) + (a_1 a_4)^\omega (a_1 a_2) \equiv 0,$$

which is another perpetuant syzygy of the first kind. Denote it by $J^{(1)}[(a_1 a_4)^\omega (a_1 a_2 a_3)]$. In fact, if the result of multiplying each term of a sum of perpetuant products (expressed symbolically) by some further symbols is to give another sum of *products** of perpetuants, and the original sum is a syzygant, so is the new one. This follows from the first principles of the symbolical notation.

In the sum of compound perpetuant syzygies of the first kind

$$(a_1 a_5)^\omega (a_2 a_3 a_4) - A_2 J^{(1)}[(a_1 a_5)^\omega (a_1 a_3 a_4)] + A_3 J^{(1)}[(a_1 a_5)^\omega (a_1 a_2 a_4)]$$
$$- A_4 J^{(1)}[(a_1 a_5)^\omega (a_1 a_2 a_3)],$$

if we replace each perpetuant syzygy of the first kind by its equivalent sum of perpetuant products, the result will be found to be identically zero. The expression is therefore a perpetuant syzygy of the second kind.

Now, if this replacement is carried out in full, and then compared with the proof† that $(a_1 a_2 a_3 a_4)$ itself is a perpetuant syzygy of the second kind, it will be seen that one piece of work may be obtained from the

* They must be products; otherwise we get, not a syzygy, but merely a relation between reducible forms.

† See end of Section III.

other by multiplication of the symbols throughout by $(a_1 a_5)^\omega$ and omitting the capital A's.

On this account we denote the perpetuant syzygy of the second kind by $J^{(2)}[(a_1 a_5)^\omega (a_1 a_2 a_3 a_4)]$.

Similarly the sum of compound perpetuant syzygies of the second kind

$$(a_1 a_6)^\omega (a_2 a_3 a_4 a_5) - A_2 J^{(2)}[(a_1 a_6)^\omega (a_1 a_3 a_4 a_5)] + A_3 J^{(2)}[(a_1 a_6)^\omega (a_1 a_2 a_4 a_5)] - \dots,$$

when expressed as a sum of compound perpetuant syzygies of the first kind, will be found to be zero.

The work closely corresponds to that with

$$(a_2 a_3 a_4 a_5) - (a_1 a_3 a_4 a_5) + (a_1 a_2 a_4 a_5) - \dots .$$

Thus from the existence of a perpetuant syzygy of the third kind $(a_1 a_2 a_3 a_4 a_5)$ we infer that of another, which we call

$$J^{(3)}[(a_1 a_6)^\omega (a_1 a_2 a_3 a_4 a_5)].$$

This process, which we may call *symbolical multiplication*, evidently applies to the n-th kind also, starting from $(a_1 a_2 \dots a_{n+2})$. It is clear that other syzygies can be formed by choosing more elaborate symbolical factors as multipliers, such as, for example,

$$(a_1 a_{n+3})^\lambda (a_1 a_{n+4})^\mu (a_2 a_{n+5})^\nu.$$

Primary Perpetuant Syzygies.

DEF.—We shall call a perpetuant syzygy of the n-th kind *primary* if it is (1) the simple one of degree $(n+2)$, *i.e.* $(a_1 a_2 \dots a_{n+2})$, or (2) derived from this by *symbolical multiplication.*

NOTATION.—We shall use

$$S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p),$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots \geqslant \lambda_p \ (\geqslant 2)$ and $p \leqslant (n+2)$,

to denote such a primary perpetuant syzygy of the n-th kind as that formed by the symbolical multiplication of $(a_1 a_2 \dots a_{n+2})$ by the product of p perpetuants

$$(a_1 a_{n+3})^{\rho_{1,1}} (a_1 a_{n+4})^{\rho_{1,2}} \dots (a_1 a_{n+\lambda_1+1})^{\rho_{1,\lambda_1-1}} \text{ of degree } \lambda_1,$$

$$(a_2 a_{n+\lambda_1+2})^{\rho_{2,1}} (a_2 a_{n+\lambda_1+3})^{\rho_{2,2}} \dots (a_2 a_{n+\lambda_1+\lambda_2})^{\rho_{2,\lambda_2-1}} \text{ of degree } \lambda_2,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(a_p a_{n+\sum_1^{p-1}\lambda_r - p+4})^{\rho_{p,1}} (a_p a_{n+\sum_1^{p-1}\lambda_r - p+5})^{\rho_{p,2}} \dots (a_p a_{n+\sum_1^{p}\lambda_r - p+2})^{\rho_{p,\lambda_p-1}} \text{ of degree } \lambda_p.$$

Each factor in this product contains *one* symbolical letter (and one only) of those contained in $(a_1 a_2 \ldots a_{n+2})$.

We use the notation $S^{(n)}(\lambda_1 \lambda_2 \ldots \lambda_p)$ to denote also any of the primary perpetuant syzygies of the n-th kind derived from the above by interchanging the suffixes of the symbolical letters involved.

Properties of Primary Perpetuant Syzygies.

(i) $S^{(n)}(\lambda_1 \ldots \lambda_p)$ is clearly irreducible, for no two of the compound perpetuant syzygies of the $(n-1)$-th kind that compose it have a common perpetuant factor.

By multiplication by a perpetuant or product of perpetuants, we can form compound primary perpetuant syzygies of the n-th kind.

(ii) $S^{(n)}(\lambda_1 \ldots \lambda_p)$ resolves $C_{\lambda_1} S^{(n-1)}(\lambda_2 \ldots \lambda_p)$, a compound primary perpetuant syzygy of the $(n-1)$-th kind, which has only a single perpetuant factor, of degree not less than 2 [unless $S^{(n)}$ be simply $(a_1 \ldots a_{n+2})$]. $S^{(n-1)}(\lambda_2 \ldots \lambda_p)$ resolves, in its turn, $C_{\lambda_2} S^{(n-2)}(\lambda_3 \ldots \lambda_p)$.

It follows that every irreducible primary perpetuant syzygy of the n-th kind comes from a doubly compound primary perpetuant syzygy of the $(n-2)$-th kind *with only two perpetuant factors.* It is proved further on* that the perpetuant syzygy of the n-th kind that arises from a primary doubly compound perpetuant syzygy of the $(n-2)$-th kind is itself necessarily primary. Therefore there is a complete correspondence between irreducible primary perpetuant syzygies of the n-th kind, and doubly compound primary perpetuant syzygies of the $(n-2)$-th kind *with only two perpetuant factors.* This is true whatever the degree.

VII. LEMMA 1.—*A perpetuant syzygy of the first kind* (*of degree greater than* 3) *that resolves a product with a factor C_1 is compound.*

By our arrangement scheme, a syzygy is never to be regarded as resolving a product with a factor C_1 if there is any term without such a factor. It follows that a syzygy resolving a C_1 product must consist entirely of C_1 products. Hence we wish to prove that a syzygy consisting entirely of C_1 products must be compound.

* In Section X.

This is shown by the following more general theorem:—

Let F be a rational integral function of the $\binom{\delta}{2}$ variables $x_{r,s}$,

$$r = 1, 2, \ldots, \delta, \quad s = 1, 2, \ldots, \delta;$$

but

$$r \neq s \quad (x_{r,s} = -x_{s,r}).$$

Each term is to be of the same total degree in the x's taken all together.

Further, let F have the property of vanishing when all such relations as

$$x_{p,q} + x_{q,r} + x_{r,p} = 0$$

hold good. We shall call such a function *a null function*. Then we shall prove that, if F be such that each *term* involves at most $(\delta-1)$ suffixes, F can be expressed as the sum of a number of *null functions*, each of which likewise involves at most $(\delta-1)$ suffixes. *E.g.*,

$$F \equiv x_{2,4}(x_{1,2} + x_{2,3} + x_{3,4} - x_{1,4})$$

can be written

$$x_{2,4}(x_{2,3} + x_{3,4} - x_{2,4}) + x_{2,4}(x_{1,2} - x_{1,4} + x_{2,4}),$$

i.e., as the sum of two null functions, one involving only the suffixes 2, 3, and 4; the other involving only 1, 2, and 4.

PROOF.—Take any term of F, say

$$x_{p,q}^{\lambda}\, x_{r,s}^{\mu} \ldots,$$

where the suffix omitted is t.

If $t \neq \delta$, subtract

$$f \equiv x_{p,q}^{\lambda}\, x_{r,s}^{\mu} - \ldots - (x_{p,\delta} - x_{q,\delta})^{\lambda}\, (x_{r,\delta} - x_{s,\delta})^{\mu} \ldots,$$

f is a null function not containing the suffix t.

If $t = \delta$, subtract

$$f' \equiv x_{p,q}^{\lambda}\, x_{r,s}^{\mu} - \ldots - (x_{p,\delta-1} - x_{q,\delta-1})^{\lambda}\, (x_{r,\delta-1} - x_{s,\delta-1})^{\mu} \ldots.$$

Treating every term of F in this manner, we finally get a new null function

$$F - \Sigma f - \Sigma f'$$

$=$ a rational integral function of $x_{1,\delta}\, x_{2,\delta} \ldots x_{\delta-1,\delta}$ from each term of which at least one of the suffixes $1 \ldots \delta-1$ is absent

$+$a rational integral function of $x_{1,\delta-1}\, x_{2,\delta-1} \ldots x_{\delta-2,\delta-1}$.

Write
$$A \equiv x_{1,\delta-1}, \qquad a \equiv x_{1,\delta},$$
$$B \equiv x_{2,\delta-1}, \qquad b \equiv x_{2,\delta},$$
$$Z \equiv x_{\delta-2,\delta-1}, \qquad z \equiv x_{\delta-2,\delta},$$
and
$$\epsilon \equiv -x_{\delta-1,\delta}.$$

Then
$$\phi \equiv F-\Sigma f-\Sigma f' = \psi(a, b, \ldots, z, \epsilon)+\chi(A, B, \ldots, Z),$$
where at least one of $a, b, \ldots, z$ or ϵ is absent from each term of ψ. That is
$$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\psi = 0.$$

Now ϕ is a null function, *i.e.*, $\psi(a, b, \ldots, z, \epsilon)+\chi(A, B, \ldots, Z)$ vanishes in consequence of such relations as
$$A = a+\epsilon, \quad B = b+\epsilon, \quad \ldots.$$

Hence
$$\psi(a, b, \ldots, z, \epsilon) = -\chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon);$$
therefore
$$\phi = \chi(A, B, \ldots, Z)-\chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon).$$
But
$$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\psi = 0;$$
therefore
$$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon) = 0.$$
Put
$$a+\epsilon = a', \quad b+\epsilon = b', \quad \ldots,$$
$$\frac{\partial}{\partial \epsilon} = \Sigma\frac{\partial a'}{\partial \epsilon}\frac{\partial}{\partial a'} = \Sigma\frac{\partial}{\partial a'},$$
while
$$\frac{\partial}{\partial a} = \frac{\partial a'}{\partial a}\frac{\partial}{\partial a'} = \frac{\partial}{\partial a'}, \quad \ldots.$$

Hence
$$\frac{\partial}{\partial a'}\frac{\partial}{\partial b'}\cdots\frac{\partial}{\partial z'}\left(\frac{\partial}{\partial a'}+\frac{\partial}{\partial b'}+\ldots+\frac{\partial}{\partial z'}\right)\chi(a', b', \ldots, z') = 0,$$
so
$$\frac{\partial}{\partial A}\frac{\partial}{\partial B}\cdots\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial A}+\frac{\partial}{\partial B}+\ldots+\frac{\partial}{\partial Z}\right)\chi(A, B, \ldots, Z) = 0.$$

The most general solution of this equation is
$$\chi = \Omega_1+\Omega_2+\ldots+\Omega_{\delta-2}+\Delta;$$
where Ω_1 does not involve A, Ω_2 does not involve B, and so on, while Δ is

a function of the *differences* of $A, B, \ldots, Z$. Hence

$$\begin{aligned}\phi = & \ \chi(A, B, \ldots, Z) - \chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon) \\ = & \ \Omega_1(B, C, \ldots, Z) - \Omega_1(b+\epsilon, c+\epsilon, \ldots, z+\epsilon) \\ & +\Omega_2(A, C, \ldots, Z) - \Omega_2(a+\epsilon, c+\epsilon, \ldots, z+\epsilon) \\ & +\ldots \\ & +\Delta(A, B, \ldots, Z) - \Delta(a, b, \ldots, z),\end{aligned}$$

for $$\Delta(a+\epsilon, b+\epsilon, \ldots z+\epsilon) = \Delta(a, b, \ldots, z).$$

Reverting to our x notation, we see that ϕ is thus proved to be expressible as the sum of $(\delta-1)$ null functions, the first of which does not contain the suffix 1, the second of which does not contain the suffix 2, and so on; the $(\delta-2)$-th of which does not contain the suffix $(\delta-2)$, while the last

$$\Delta(x_{1,\delta-1} \ldots x_{\delta-2,\delta-1}) - \Delta(x_{1,\delta} \ldots x_{\delta-2,\delta}),$$

say $$\Sigma(x_{p,\delta-1}-x_{q,\delta-1})^{\lambda}(x_{r,\delta-1}-x_{s,\delta-1})^{\mu} \ldots$$

$$-\Sigma(x_{p,\delta}-x_{q,\delta})^{\lambda}(x_{r,\delta}-x_{s,\delta})^{\mu} \ldots,$$

is equal to

$$\Sigma\left[(x_{p,\delta-1}-x_{q,\delta-1})^{\lambda}(x_{r,\delta-1}-x_{s,\delta-1})^{\mu} \ldots - x_{p,q}^{\lambda} x_{r,s}^{\mu} \ldots\right]$$

$$+\Sigma\left[x_{p,q}^{\lambda} x_{r,s}^{\mu} \ldots - (x_{p,\delta}-x_{q,\delta})^{\lambda}(x_{r,\delta}-x_{s,\delta})^{\mu} \ldots\right],$$

say $$N_{\delta} + N_{\delta-1},$$

where N_t is a null function not containing the suffix t. Hence

$$\phi \equiv F - \Sigma f - \Sigma f' = \sum_{1}^{\delta} N_t,$$

i.e., $$F = \Sigma f + \Sigma f' + \Sigma N_t$$

= the sum of a number of null functions, each quite free from at least one suffix, which is our theorem.

Example.—Take $F \equiv \ x_{1,3} x_{2,3} x_{1,2}$ (suffix 4 omitted from this term)

$-x_{1,4} x_{2,4} x_{1,2}$ („ 3 „ „)

$+x_{1,4} x_{3,4} x_{1,3}$ („ 2 „ „)

$-x_{2,4} x_{3,4} x_{2,3}$ („ 1 „ „),

F is easily seen to be a null function, or the following work shows it.

(N.B.—No confusion will arise from writing x_{13} instead of $x_{1,3}$, so we will drop these commas.)

Subtract the null functions

$$f_1 \equiv -x_{24}x_{34}x_{23}+x_{24}x_{34}(x_{24}-x_{34}),$$
$$f_2 \equiv x_{14}x_{34}x_{13}-x_{14}x_{34}(x_{14}-x_{34}),$$
$$f_3 \equiv -x_{14}x_{24}x_{12}+x_{14}x_{24}(x_{14}-x_{24}),$$
$$f' \equiv x_{13}x_{23}x_{12}-x_{13}x_{23}(x_{13}-x_{23}).$$

We get

$$\begin{aligned}\phi &\equiv F-f_1-f_2-f_3-f' \\ &= -x_{24}x_{34}(x_{24}-x_{34}) = \quad b\epsilon(b+\epsilon) \\ &\quad +x_{14}x_{34}(x_{14}-x_{34}) \quad -a\epsilon(a+\epsilon) \\ &\quad -x_{14}x_{24}(x_{14}-x_{24}) \quad -ab(a-b) \\ &\quad +x_{13}x_{23}(x_{13}-x_{23}) \quad +AB(A-B),\end{aligned}$$

where

$$A \equiv x_{13}, \qquad a \equiv x_{14},$$
$$B \equiv x_{23}, \qquad b \equiv x_{24},$$
$$\epsilon \equiv -x_{34},$$

so

$$\begin{aligned}\phi &= \psi(a, b, \epsilon)+\chi(A, B) \\ &= -\chi(a+\epsilon, b+\epsilon)+\chi(A, B),\end{aligned}$$

as is easily verified. Also $\chi(A, B)$, *i.e.*, $AB(A-B)$, is equal to

$$\Omega_1+\Omega_2+\Delta,$$

where $\Delta \equiv -\frac{1}{3}(A-B)^3, \quad \Omega_1 \equiv -\frac{1}{3}B^3, \quad \Omega_2 \equiv \frac{1}{3}A^3$

so

$$\begin{aligned}\phi &= -\tfrac{1}{3}B^3+\tfrac{1}{3}A^3-\tfrac{1}{3}(A-B)^3+\tfrac{1}{3}(b+\epsilon)^3-\tfrac{1}{3}(a+\epsilon)^3+\tfrac{1}{3}(a-b)^3 \\ &= \tfrac{1}{3}[(x_{24}-x_{34})^3-x_{23}^3]+\tfrac{1}{3}[x_{13}^3-(x_{14}-x_{34})^3] \\ &\quad -\tfrac{1}{3}[(x_{13}-x_{23})^3-x_{12}^3]+\tfrac{1}{3}[(x_{14}-x_{24})^3-x_{12}^3]\end{aligned}$$

(the two terms in x_{12}^3 cancelling)

of form $N_1+N_2+N_4+N_3$.

So, finally, $F=f_1+f_2+f_3+f'+N_1+N_2+N_3+N_4$,

i.e., a sum of null functions each with one suffix omitted.

VIII. LEMMA 2.—*To prove that a perpetuant syzygy of the n-th kind that resolves a primary compound perpetuant syzygy of the $(n-1)$-th kind must itself be primary.*

Since, by our scheme of arrangement, a non-primary syzygy is always to precede a primary one, the theorem amounts to :—*A perpetuant syzygy of the n-th kind, made up entirely of primary compound perpetuant syzygies of the* $(n-1)$*-th kind, is itself primary.*

By definition, a perpetuant syzygy of the n-th kind is such a linear function of compound perpetuant syzygies of the $(n-1)$-th kind, that when these are replaced by the linear functions of compound perpetuant syzygies of the $(n-2)$-th kind that they respectively represent, the resulting expression, regarded as a linear function of compound perpetuant syzygies of the $(n-2)$-th kind, is identically zero.

In the present case, when the perpetuant syzygies of the $(n-1)$-th kind are all primary, they are all symbolical (or actual) products of perpetuant syzygies of the $(n-1)$-th kind of the form $(a_{r_1} a_{r_2} \dots a_{r_{n+1}})$.

Let the first term be* $\alpha_1 C_1^m C_{\lambda_1} C_{\lambda_2} \dots C_{\lambda_p} S^{(n-1)}(\mu_1 \mu_2 \dots \mu_q)$ (where no λ or μ is < 2).

This resolves a perpetuant syzygy of the $(n-2)$-th kind that is the symbolical product of $(a_{r_2} \dots a_{r_{n+1}})$ by $C_1^m C_{\lambda_1} C_{\lambda_2} \dots C_{\lambda_p} C_{\mu_1} C_{\mu_2} \dots C_{\mu_q}$.

This must be cancelled by another perpetuant syzygy of the $(n-2)$-th kind, equal except in sign, and this can only arise from a perpetuant syzygy of the $(n-1)$-th kind, which is the symbolical product of $C_1^m C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$ and $(a_\kappa a_{r_2} \dots a_{r_{n+1}})$, a_κ being some other of the letters involved.

Consider any other perpetuant syzygy of the $(n-2)$-th kind included in either of these two perpetuant syzygies of the $(n-1)$-th kind. We see that there will be a third perpetuant syzygy of the $(n-1)$-th kind, that is, the symbolical product of $C_1^m C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$, and, say $(a_\kappa a_\rho a_{r_3} \dots a_{r_{n+1}})$. In the preceding, the factors $C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$ are exactly the same in each case, but the C_1^m's are not.

Proceeding in this way, we see that, unless the syzygy is the sum of two or more others, every perpetuant syzygy of the $(n-1)$-th kind in it must have a symbolical factor $C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_p}$.

If we divide this out (a symbolical operation, of course) we are left with a new perpetuant syzygy of the n-th kind of the form

$$\Sigma \beta_r C_1^m (a_{r_1} \dots a_{r_{n+1}}) = 0.$$

The next step is to show that this is primary. By our arrangement scheme $C_1^m(a_{r_1} a_{r_2} \dots a_{r_{n+1}})$ precedes $C_1^m(a_{r'_1} a_{r'_2} \dots a_{r'_{n+1}})$, where

$$r_1 < r_2 \dots < r_{n+1} \quad \text{and} \quad r'_1 < r'_2 \dots < r'_{n+1},$$

* In the following, the α's, β's, γ's and δ's are numerical constants.

if the first of the differences $r_1 - r_1'$, $r_2 - r_2'$, ..., $r_{n+1} - r_{n+1}'$, that does not vanish is *positive.*

Arrange the terms of $\Sigma\beta_r C_1^m (a_{r_1} a_{r_2} \ldots a_{r_{n+1}})$ in this way. Take the symbolical letters involved to be $a_1 a_2 \ldots a_{m+n+1}$. If the first term be $\beta_s C_1^m (a_{s_1} a_{s_2} \ldots a_{s_{n+1}})$, subtract $\beta_s C_1^{m-1} (a_{s_1 - 1} a_{s_1} a_{s_2} \ldots a_{s_{n+1}})$, a perpetuant syzygy of the n-th kind involving $a_1 \ldots a_{m+n+1}$.

The difference resolves a perpetuant syzygy of the $(n-1)$-th kind that follows $C_1^m (a_{s_1} a_{s_2} \ldots a_{s_{n+1}})$ in the sequence.

Remove this by a fresh subtraction of a suitable perpetuant syzygy of the n-th kind of the same general form as before.

Now there are only a finite number $\binom{m+n}{m-1}$ of perpetuant syzygies of the $(n-1)$-th kind, of the form of $C_1^m (a_{s_1} a_{s_2} \ldots a_{s_{n+1}})$ in which $s_1 > 1$; therefore, by the repetition of the above process, we shall, sooner or later, arrive at a stage where every term left is of the form

$$\gamma_r C_1^m (a_1 a_{r_2} a_{r_3} \ldots a_{r_{n+1}}).$$

Let $\gamma_t C_1^m (a_1 a_{t_2} \ldots a_{t_{n+1}})$ be the first (in the fixed sequence) of these terms. Then, in the expanded form of the syzygy as an identically vanishing linear function of perpetuant syzygies of the $(n-2)$-th kind, there is nothing to cancel the term $\gamma_t C_1^{m+1} (a_{t_2} \ldots a_{t_{n+1}})$. Therefore $\gamma_t = 0$, *i.e.*, all the γ's are zero. Therefore $\Sigma\beta_r C_1^m (a_{r_1} a_{r_2} \ldots a_{r_{n+1}})$ is expressible as a sum of perpetuant syzygies of the n-th kind, like

$$\Sigma\delta_s C_1^{m-1} (a_{s_1 - 1} a_{s_1} a_{s_2} \ldots a_{s_{n+1}}),$$

i.e., is *primary*. Therefore the syzygy from which we started, being derived from the last by symbolical multiplication, is also primary, *i.e.*, *a perpetuant syzygy of the n-th kind that resolves a primary perpetuant syzygy of the* $(n-1)$*-th kind is itself primary.*

IX. LEMMA 3.—*No perpetuant syzygy of the n-th kind of degree* $> (n+2)$ *can be irreducible if it resolve a perpetuant syzygy of the* $(n-1)$*-th kind with* C_1 *as a factor.*

Let $S^{(n)}$ be an irreducible perpetuant syzygy of the n-th kind resolving $C_1^m C_{\kappa_1} \ldots C_{\kappa_p} S^{(n-1)}$ (where $m \neq 0$ and no κ is unity).

To prove that $m = 1$ and $\kappa_1 = \kappa_2 = \ldots = \kappa_p = 0$, and $S^{(n-1)}$ is of the form $(a_1 a_2 \ldots a_{n+1})$, so $S^{(n)}$ is simply of the form $(a_1 a_2 \ldots a_{n+2})$, of degree $(n+2)$.

Assume the theorem true for all kinds lower than n. We shall see it is then true for n.

Let $S^{(n-1)}$ resolve $C_{\lambda_1} \dots C_{\lambda_q} S^{(n-2)}$.

Unless $S^{(n-1)}$ be $(a_1 \dots a_{n+1})$, in which case it resolves $C_1(a_1 \dots a_n)$, no λ can be unity, by hypothesis.

Now the sum of compound perpetuant syzygies of the $(n-1)$-th kind which $S^{(n)}$ represents must contain one, at least, of the forms

$$C_{\mu_1} \dots C_{\mu_i} [C_{\mu_i+1} \dots C_{\mu_r} C_1 S^{(n-2)}],$$

where $C_{\mu_1} \dots C_{\mu_r}$ are simply the other $(m-1)$ C_1's, and $C_{\kappa_1} \dots C_{\kappa_p}$, $C_{\lambda_1} \dots C_{\lambda_q}$ in some order; for if this were not the case, every term of $S^{(n)}$ would be divisible by C_1^m, and so $S^{(n)}$ would be *compound*, not irreducible.

Now by our scheme of arrangement, of two compound perpetuant syzygies of the $(n-1)$-th kind, containing a power of C_1 as a factor, that whose power of C_1 is the lower stands first (provided both the $S^{(n-1)}$'s are primary or both non-primary).

The number of C_1's in $C_{\mu_1} \dots C_{\mu_i}$ is at most $(m-1)$, since no κ or λ is unity. Also, from Lemma 2, both of the $S^{(n-1)}$'s in the next line are primary, or both non-primary. Therefore

$$C_{\mu_1} \dots C_{\mu_i} [C_{\mu_i+1} \dots C_{\mu_r} C_1 S^{(n-2)}] \quad \text{precedes} \quad C_1^m C_{\kappa_1} \dots C_{\kappa_p} S^{(n-1)}.$$

Therefore $S^{(n)}$ does *not* resolve $C_1^m C_{\kappa_1} \dots C_{\kappa_p} S^{(n-1)}$.

This contradicts a fact with which we started. Therefore the assumption that no λ is unity is false; therefore, by our hypothesis for kind $(n-1)$, $S^{(n-1)}$ is of form $(a_1 \dots a_{n+1})$ and resolves $C_1 S^{(n-2)}$, where $S^{(n-2)}$ is of form $(a_1, \dots, a_n)$; therefore, from Lemma 2, $S^{(n)}$ is primary.

Therefore as $m \neq 0$, $S^{(n)}$ is the simple perpetuant syzygy of the n-th kind of degree $(n+2)$ and of form $(a_1 a_2 \dots a_{n+2})$ [for we can easily see that no other *primary irreducible* perpetuant syzygy of the n-th kind resolves a compound with a factor C_1], and

$$\kappa_1 = \kappa_2 = \dots = \kappa_p = 0.$$

The theorem is true, by Lemma 1, for kind unity, therefore we have proved it in general, by induction.

X. THEOREM.—*All perpetuant syzygies of the n-th kind of degree* $< (2n+2)$ *are primary.*

The proof is by induction.

Assume the theorem to be true for all kinds less than n. We shall deduce its truth for kind n.

For an irreducible perpetuant syzygy of the $(n-2)$-th kind to be *non-primary*, its degree, by hypothesis, must be at least $(2n-2)$. Therefore the total degree of the perpetuant factors of a *doubly compound* perpetuant syzygy of the $(n-2)$-th kind of total degree $< (2n+2)$, of which the irreducible part is *non-primary*, is $(2n+1)-(2n-2) = 3$, at most.

Therefore the perpetuant factors are $C_1 C_2$, C_1^3, or C_1^2.

Now, by Lemma 3, no irreducible perpetuant syzygy of the $(n-1)$-th kind of degree $> (n+1)$ resolves a compound perpetuant syzygy of the $(n-2)$-th kind with a factor C_1.

Therefore $C_1 C_2 S^{(n-2)}$ is never resolvable in more than one way at most (*i.e.*, by $C_1 [C_2 S^{(n-2)}]$) by a *compound* perpetuant syzygy of the $(n-1)$-th kind, while $C_1^3 S^{(n-2)}$ and $C_1^2 S^{(n-2)}$ are never resolvable at all by a compound perpetuant syzygy of the $(n-1)$-th kind, if $S^{(n-2)}$ is non-primary [and hence of degree $\geqslant (2n-2)$, *i.e.*, $\geqslant n$, as $(n-2) \geqslant 0$]. Therefore no perpetuant syzygies of the n-th kind arise from these doubly compound perpetuant syzygies of the $(n-2)$-th kind.

In forming the generating function for perpetuant syzygies of the n-th kind of degree $<(2n+2)$, *we need only consider those doubly compound perpetuant syzygies of the* $(n-2)$*-th kind which are primary.*

Therefore a perpetuant syzygy of the n-th kind of degree $< (2n+2)$ connects two compound perpetuant syzygies of the $(n-1)$-th kind that themselves resolve primary perpetuant syzygies of the $(n-2)$-th kind, *i.e.*, from Lemma 2, a perpetuant syzygy of the n-th kind of degree $< (2n+2)$ connects two *primary* compound perpetuant syzygies of the $(n-1)$-th kind.

We now have only to prove that such a perpetuant syzygy of the n-th kind is necessarily primary.

The perpetuant syzygy of the n-th kind in question is formed as follows :—

Subtract the two compound primary perpetuant syzygies of the $(n-1)$-th kind [of course, as these resolve the same perpetuant syzygies of the $(n-2)$-th kind, not more than one of them can be canonical], so that their difference is free from the doubly compound perpetuant syzygies of the $(n-2)$-th kind from which we started.

This difference is a perpetuant syzygy of the $(n-1)$-th kind that resolves another doubly compound perpetuant syzygy of the $(n-2)$-th kind.

But this last must be resolvable by a member of the canonical set [for, if not, the difference of a canonical and an uncanonical syzygy is one linearly independent of the canonical set, which is absurd, as by definition all syzygies of a given kind and degree are expressible linearly in terms of the canonical set].

We subtract this canonical perpetuant syzygy of the $(n-1)$-th kind, so that the resulting difference (if not zero) resolves yet another doubly compound primary perpetuant syzygy of the $(n-2)$-th kind.

Continue this process as far as possible.

Now since the number of perpetuant products of given degree and weight is necessarily finite, the same is true of the perpetuant syzygy of the first kind connecting them, and therefore of the second kind connecting these, and so on; therefore for perpetuant syzygies of the $(n-2)$-th kind.

Thus the above process of subtraction must stop in time. But we have shown that we cannot come to an irresolvable, and whenever a resolvable appears we go on.

Therefore finally the difference is zero.

Each perpetuant syzygy of the $(n-1)$-th kind involved must be *primary*, as it resolves a *primary* perpetuant syzygy of the $(n-2)$-th kind, and as this last is *doubly compound*, the perpetuant syzygy of the $(n-1)$-th kind must be *compound*, for an irreducible primary syzygy resolves only *singly compounds*.

Thus we have formed the perpetuant syzygy of the n-th kind in question and shown that it connects only *primary* compound perpetuant syzygies of the $(n-1)$-th kind.

Therefore, by Lemma 2, it is itself primary.

This completes the induction.

But the theorem is true for kind 1, for the only perpetuant syzygy of the first kind of degree < 4 is the Jacobian identity.

Therefore it is true universally that *all perpetuant syzygies of the n-th kind of degree $<(2n+2)$ are primary.*

XI. *Note.*—In what follows we shall use the symbol $\binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p}$ to denote the number of ways of picking, from δ things, a group of p sub-groups, containing $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ things respectively. If of the p λ's, r are equal to one integer, s to another, t to yet another, &c., and the rest are all unequal, then clearly

$$\binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p} = \frac{1}{r!\, s!\, t! \dots} \frac{\delta!}{\lambda_1!\, \lambda_2! \dots \lambda_p! \left(\delta - \sum_1^p \lambda_\kappa\right)!}.$$

We shall also, to save space, denote the expression

$$\frac{x^N}{(1-x)^{\sum_1^p \lambda_p - p}},$$

where $$N = \sum_1^p 2^{\lambda_r - 1} - p + 1,$$

by the symbol $F(\lambda_1 \lambda_2 \dots \lambda_p)$; and make frequent use of the identity

$$\frac{x^{2^{\lambda_1 - 1} - 1}}{(1-x)^{\lambda_1 - 1}} F(\lambda_2 \dots \lambda_p) = F(\lambda_1 \lambda_2 \dots \lambda_p).$$

"*Well Ordered*" *Syzygies. Their Generating Functions.*

To distinguish one particular primary perpetuant syzygy of the n-th kind from others of the same general form $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$, we shall sometimes use a different notation. Thus

$$J^{(n)}\left[(a_1 a_{n+3})^{\rho_1} (a_2 a_{n+4})^{\rho_2} (a_1 a_2 a_3 \dots a_{n+2})\right]$$

will be used to denote the result of symbolical multiplication of the perpetuant syzygies of the n-th kind $(a_1 a_2 a_3 \dots a_{n+2})$ by $(a_1 a_{n+3})^{\rho_1} (a_2 a_{n+4})^{\rho_2}$.

This is of form $S^{(n)}(2, 2)$.

If in $J^{(n)}\left[(a_r a_{r_1})^{\rho_{r_1}} (a_r a_{r_2})^{\rho_{r_2}} (a_r a_{r_3})^{\rho_{r_3}} \dots (a_s a_{s_1})^{\rho_s} \dots (a_t a_{t_1})^{\rho_{t_1}} \dots (a_r a_s a_t \dots)\right]$,

where the letters are $a_1 a_2 \dots a_\delta$ in some order

$$r < r_1 < r_2 < \dots,$$
$$s < s_1 < s_2 < \dots,$$
$$t < t_1 < t_2 < \dots,$$

we call the syzygy *well ordered*.

Consider $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ a *well ordered* primary perpetuant syzygy of the n-th kind. We shall prove by induction that, given

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_p \geqslant 2,$$

the *generating function* for all such well ordered ones is

$$\binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p} F(\lambda_1 \lambda_2 \dots \lambda_p),$$

provided $\delta < (2n+2)$.

Assume this for kinds $< n$.

(I) If $\lambda_1 >$ any other λ, $S^{(n)}$ resolves $C^{\lambda}{}_1 S^{(n-1)}(\lambda_2 \dots \lambda_p)$, where $S^{(n-1)}$ is also well ordered.

Conversely, $C_{\lambda_1} S^{(n-1)}(\lambda_2 \dots \lambda_p)$ is always resolvable by a well ordered perpetuant syzygy of the n-th kind if $S^{(n-1)}$ be well ordered, and $\lambda_1 >$ any other λ; therefore the generating function for $S^{(n)}$ is the same as that for the compound perpetuant syzygy of the $(n-1)$-th kind, *i.e.*,

$$\binom{\delta}{\lambda_1}\frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}}\binom{\delta-\lambda_1}{\lambda_2\ldots\lambda_p}F(\lambda_2\ldots\lambda_p)=\binom{\delta}{\lambda_1\lambda_2\ldots\lambda_p}F(\lambda_1\lambda_2\ldots\lambda_p).$$

(II) If $\lambda_1=\lambda_2=\lambda_3=\ldots=\lambda_\kappa>\lambda_{\kappa+1}$, we have κ λ's equal to λ (say). $S^{(n)}(\lambda_1\lambda_2\ldots\lambda_p)$ will have κ terms of form

$$C_\lambda S^{(n-1)}[\lambda\lambda\ldots\text{ to }(\kappa-1)\text{ terms}\ldots\lambda_p].$$

Let us arrange the κ C_λ's according to our fixed sequence, and call the first C_{λ_1}, the second C_{λ_2}, and so on.

With this convention $S^{(n)}(\lambda_1\lambda_2\ldots\lambda_p)$ still resolves $C_{\lambda_1}S^{(n-1)}(\lambda_2\lambda_3\ldots\lambda_p)$, a well ordered compound perpetuant syzygy of the $(n-1)$-th kind, and conversely, if this last is well ordered, it is always resolvable by $S^{(n)}(\lambda_1\lambda_2\ldots\lambda_p)$, a well ordered perpetuant syzygy of the n-th kind.

However $C_{\lambda_r}S^{(n-1)}(\lambda_1\lambda_2\ldots\lambda_{r-1}\lambda_{r+1}\ldots\lambda_p)$ is clearly not so resolvable if $r>1$.

Therefore the generating function for $S^{(n)}(\lambda_1\lambda_2\ldots\lambda_p)$ is the same as that for $C_{\lambda_1}S^{(n-1)}(\lambda_2\ldots\lambda_p)$, *i.e.*,

$$\frac{1}{\kappa}\binom{\delta}{\lambda_1}\frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}}\binom{\delta-\lambda_1}{\lambda_2\ldots\lambda_p}F(\lambda_2\ldots\lambda_p)$$

$$=\frac{1}{\kappa}\binom{\delta}{\lambda_1}\binom{\delta-\lambda_1}{\lambda_2\ldots\lambda_p}F(\lambda_1\lambda_2\ldots\lambda_p)$$

$$=\frac{1}{\kappa}\,\frac{\delta!}{\lambda_1!(\delta-\lambda_1)!}\,\frac{1}{(\kappa-1)!\,s!\,t!\ldots}\,\frac{(\delta-\lambda_1)!}{\lambda_2!\,\lambda_3!\ldots\lambda_p!\left(\delta-\sum_1^p\lambda_r\right)!}F(\lambda_1\lambda_2\ldots\lambda_p)$$

$$=\binom{\delta}{\lambda_1\lambda_2\ldots\lambda_p}F(\lambda_1\lambda_2\ldots\lambda_p).$$

Therefore in both cases the generating function for well ordered perpetuant syzygies of the n-th kind of form $S^{(n)}(\lambda_1\lambda_2\ldots\lambda_p)$ is this expression.

So if this formula holds good for kind $(n-1)$, it holds for kind n.

Now the formula can, by hypothesis, only fail for kind $(n-1)$ when degree $\geqslant 2n$.

Therefore the formula can only fail for kind n when degree $\geqslant 2n+\lambda_1$, *i.e.*, when degree $\geqslant (2n+2)$ [for, if $\lambda_1=1$ degree of $S^{(n-1)}$ must, by

Lemma III, be $(n+1)$, and in this case formula does not fail, so for failure $\lambda_1 \geqslant 2$].

Finally the formula holds good for perpetuant syzygies of the first kind of degree 3 (*i.e.* < 4) ; and therefore the induction shows that the generating function for $S^{(n)}(\lambda_1\lambda_2\lambda_3 \dots \lambda_p)$, if well ordered and of degree $\delta < (2n+2)$, is always

$$\binom{\delta}{\lambda_1\lambda_2\lambda_3 \dots \lambda_p} F(\lambda_1\lambda_2\lambda_3 \dots \lambda_p).$$

If $\delta \geqslant (2n+2)$, in some cases the syzygies of low weight degenerate. This is due ultimately to the fact that the perpetuant syzygy of the first kind $J^{(1)}[(a_1a_4)^\rho(a_1a_2a_3)]$ and $J^{(1)}[(a_2a_3)^\rho(a_2a_1a_4)]$ are equal when $\rho = 1$. In consequence the generating function, when expanded in powers of x, has some of the coefficients at the beginning less than those given by the above formula.

We can obtain our linearly independent sets, as linear functions of the members of which any other perpetuant syzygy of the n-th kind can be expressed, by taking all primary perpetuant syzygies of the n-th kind [of degree $< (2n+2)$] to be well ordered.

The proof is this :—

It is clear that any well ordered perpetuant syzygy of the n-th kind resolves a compound well ordered perpetuant syzygy of the $(n-1)$-th kind, which in turn resolves a well ordered doubly compound perpetuant syzygy of the $(n-2)$-th kind. Thus every well ordered perpetuant syzygy of the n-th kind arises from a well ordered doubly compound perpetuant syzygy of the $(n-2)$-th kind. We must prove the converse of this to justify the statement at the head of the paragraph. To do this, we find the generating function for perpetuant syzygies of the n-th kind arising from well ordered doubly compound perpetuant syzygies of the $(n-2)$-th kind, and notice that they are the same as those for well ordered perpetuant syzygies of the n-th kind.

Unless $S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$ is of degree n, for perpetuant syzygies of the n-th kind to arise from $C_{\lambda_1}C_{\lambda_2}S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$, λ_1 and λ_2 must, save in one case, be $\geqslant 2$. Therefore if degree of $S^{(n)}$ be $< (2n+2)$, that of $S^{(n-2)}$ is $< (2n-2)$. The one exception is when $S^{(n-2)}$ is of degree n, which is also $< (2n-2)$. Therefore in every case we may use the generating function found above.

Consider then $C_{\lambda_1}C_{\lambda_2}S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$, where $\lambda_3 \geqslant \lambda_4 \geqslant \dots \geqslant \lambda_p \geqslant 2$ and $S^{(n-2)}$ is well ordered.

The following cases must be distinguished :—

(I) $\lambda_1 > \lambda_2 >$ any other λ.

(II) $\lambda_1 = \lambda_2 >$ any other λ.

(III) $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3 = \lambda_4 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$.

(IV) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$.

(V) One (or both) of λ_1 and λ_2 less than the greatest of the other λ's.

(I) $\lambda_1 > \lambda_2 >$ any other λ.

$C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \dots \lambda_p)$ is *always* resolvable in two ways, *i.e.*, by $C_{\lambda_1} S^{(n-1)}(\lambda_2 \lambda_3 \dots \lambda_p)$ and by $C_{\lambda_2} S^{(n-1)}(\lambda_1 \lambda_3 \dots \lambda_p)$. Therefore the generating function for perpetuant syzygies of the n-th kind is the same as that for $C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \dots \lambda_p)$, *i.e.*,

$$\binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2} \frac{x^{2^{\lambda_2-1}-1}}{(1-x)^{\lambda_2-1}} \binom{\delta-\lambda_1-\lambda_2}{\lambda_3 \dots \lambda_p} F(\lambda_3 \dots \lambda_p)$$
$$= \binom{\delta}{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p),$$

which is exactly that for well ordered perpetuant syzygies of the n-th kind of form $S^{(n)}(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p)$.

(II) $\lambda_1 = \lambda_2 >$ any other λ.

Again, $C_\lambda^2 S^{(n-2)}(\lambda_3 \dots \lambda_p)$ is always resolvable in two ways. Therefore the generating function for perpetuant syzygies of the n-th kind is

$$\frac{1}{2} \binom{\delta}{\lambda_1} \binom{\delta-\lambda_1}{\lambda_2} \binom{\delta-\lambda_1-\lambda_2}{\lambda_3 \dots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p) \quad \left[\frac{1}{2}, \text{ since } \lambda_1 = \lambda_2\right]$$
$$= \frac{1}{2!} \frac{\delta!}{\lambda_1! \, \lambda_2! \, \lambda_3! \dots \lambda_p! \left(\delta - \sum_1^p \lambda_\kappa\right)!} F(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p)$$
$$= \binom{\delta}{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_p), \quad \text{as } \lambda_1 = \lambda_2.$$

This is also the generating function for well ordered perpetuant syzygies of the n-th kind.

(III) $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3 = \lambda_4 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$.

$C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \dots \lambda_p)$ is always resolvable by $C_{\lambda_2} S^{(n-1)}(\lambda_1 \lambda_3 \dots \lambda_p)$, but by $C_{\lambda_1} S^{(n-1)}(\lambda_2 \lambda_3 \dots \lambda_p)$ only if C_{λ_2} happens to precede $C_{\lambda_3}, C_{\lambda_4}, \dots, C_{\lambda_p}$, in our fixed sequence.

Therefore the generating function for perpetuant syzygies of the n-th kind equals the generating function for these extra resolutions

$$= \binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2\lambda_3 \dots \lambda_p} F(\lambda_2\lambda_3 \dots \lambda_p)$$

$$= \binom{\delta}{\lambda_1\lambda_2\lambda_3 \dots \lambda_p} F(\lambda_1\lambda_2\lambda_3 \dots \lambda_p) \text{ (as } \lambda_1 > \text{any other } \lambda)$$

= generating function for the well ordered ones.

(IV) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$.

$S^{(n)}(\lambda_1\lambda_2 \dots \lambda_p)$ has κ terms of form $C_\lambda S^{(n-1)}[\lambda\lambda \dots \text{to } (\kappa-1) \text{ terms} \dots \lambda_p]$.

Arrange these in accordance with our sequence, and call the first $C_{\lambda_1} S^{(n-1)}$, the second $C_{\lambda_2} S^{(n-2)}$, and so on.

With this convention, i and j being both $< \kappa$,

$$C_{\lambda_i} C_{\lambda_j} S^{(n-2)} [\lambda \dots \text{to } (\kappa-2) \text{ terms} \dots \lambda_p]$$

is, in general, irresolvable.

If $i = 1$, it is resolvable by $C_{\lambda_j} S^{(n-1)}(\lambda_1\lambda_2 \dots \lambda_{j-1}\lambda_{j+1} \dots \lambda_p)$, but in no other way, unless $j = 2$, when it is resolvable also by $C_{\lambda_1} S^{(n-1)}(\lambda_2\lambda_3 \dots \lambda_p)$.

The generating function for perpetuant syzygies of the n-th kind is that for these extra resolutions

$$= \frac{1}{\kappa} \binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2\lambda_3 \dots \lambda_p} F(\lambda_2\lambda_3 \dots \lambda_p)$$

$$= \frac{1}{\kappa} \frac{\delta!}{\lambda_1!\,(\delta-\lambda_1)!} \frac{1}{(\kappa-1)!\,s!\,t! \dots} \frac{(\delta-\lambda_1)!}{\lambda_2!\,\lambda_3! \dots \lambda_p! \left(\delta - \sum_1^p \lambda_r\right)!} F(\lambda_1\lambda_2 \dots \lambda_p)$$

$$= \frac{1}{\kappa!\,s!\,t! \dots} \frac{\delta!}{\lambda_1!\,\lambda_2! \dots \lambda_p! \left(\delta - \sum_1^p \lambda_r\right)!} F(\lambda_1\lambda_2 \dots \lambda_p)$$

$$= \binom{\delta}{\lambda_1\lambda_2 \dots \lambda_p} F(\lambda_1\lambda_2 \dots \lambda_p),$$

the same as for well ordered ones.

(V) If one or both of λ_1 and λ_2 be less than the greatest of the remaining λ's, it is clear that $C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \dots \lambda_p)$ can never be resolvable in more than one way.

Therefore there are no perpetuant syzygies of the n-th kind arising here.

There are also no well ordered perpetuant syzygies of the n-th kind of the form $S^{(n)}(\lambda_1\lambda_2\lambda_3 \ldots \lambda_p)$, where λ_1 or λ_2 is less than a following λ, by the convention with which we started.

To sum up, the canonical set of perpetuant syzygies of the n-th kind of any degree $< (2n+2)$ *may be taken to be composed solely of well ordered primary ones (irreducible and compound).*

Of course not *all* the compounds of degree δ of well ordered primary perpetuant syzygies of the n-th kind of degrees $< \delta$ belong to the canonical set. Those that do may be distinguished thus :—

Let $C_{\lambda_{i+1}} C_{\lambda_{i+2}} \ldots C_{\lambda_p}$ be the perpetuants that are symbolical factors of a well ordered primary perpetuant syzygy of the n-th kind $S^{(n)}(\lambda_{i+1} \ldots \lambda_p)$.

Then $C_{\lambda_1} \ldots C_{\lambda_i} S^{(n)}(\lambda_{i+1} \ldots \lambda_p)$, of degree δ $[< (2n+2)]$, belongs to the canonical set for kind n and degree δ if, in the fixed sequence, no one of $C_{\lambda_1} \ldots C_{\lambda_i}$ precede the first of $C^{\lambda}{}_{i+1} \ldots C_{\lambda_p}$.

XII. *Formation of Non-Primary Perpetuant Syzygies of the n-th Kind of Degree* $(2n+2)$.

Let large letters $S_1 S_2 \ldots S_n$ denote perpetuant syzygies of the first kind, and small letters $s_1 s_2 \ldots s_n$ the corresponding sums of perpetuant products.

The S's are such that no one has more than one symbolical letter in common with all the others put together.

Denote by $s_1 S_2$ the result of symbolical multiplication of S_2 by s_1. This will be a sum of *compound* perpetuant syzygies of the first kind, as s_1 and S_2 have only one symbol in common.

Then it is clear that $(s_1 S_2 - s_2 S_1)$ is a perpetuant syzygy of the second kind.

This may be written $\begin{vmatrix} s_1 & S_1 \\ s_2 & S_2 \end{vmatrix}$.

Similarly $s_1(s_2 S_3 - s_3 S_2) + s_2(s_3 S_1 - s_1 S_3) + s_3(s_1 S_2 - s_2 S_1)$

is a perpetuant syzygy of the third kind, since it is a linear function of compound perpetuant syzygies of the second kind that vanishes when expressed as a linear function of compound perpetuant syzygies of the first kind. This may be written

$$\begin{vmatrix} s_1 & s_1 & S_1 \\ s_2 & s_2 & S_2 \\ s_3 & s_3 & S_3 \end{vmatrix}.$$

In general, if we denote the vanishing determinant

$$\begin{vmatrix} s_1 & s_1 & \dots & s_1 & S_1 \\ s_2 & & & & \\ \vdots & & & & \\ s_n & & & & S_n \end{vmatrix}$$

by

$$[(s_1 s_2 \dots s_{n-1} S_n)],$$

then expanding it in terms of the constituents of the first column and their first minors, we get

$$\Sigma (-1)^{p+1} s_p [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)],$$

which must vanish identically if we replace the first minors by *their* expansions in terms of the constituents of *their* first columns and the corresponding first minors, since the original determinant will be now expanded in terms of the constituents of its first two columns, which are identical.

Assume that $[(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{q-1} s_{q+1} \dots s_{n-1} S_n)]$ is a perpetuant syzygy of the $(n-2)$-th kind, and $[(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)]$ is a perpetuant syzygy of the $(n-1)$-th kind.

Then, since all the s's are sums of perpetuant products, and no s has more than one letter common with all the others put together, it follows that $s_p s_q [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{q-1} s_{q+1} \dots s_{n-1} S_n)]$ is a sum of doubly compound perpetuant syzygies of the $(n-1)$-th kind, and

$$s_p [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)]$$

is a sum of compound perpetuant syzygies of the $(n-1)$-th kind.

The vanishing of the sum of expansions of each term of the expansion of $[(s_1 \dots s_{n-1} S_n)]$ shows that this last is a perpetuant syzygy of the n-th kind.

Therefore the device gives a perpetuant syzygy of the n-th kind for kind n, if it does so for kinds $(n-1)$ and $(n-2)$.

But we have seen that it does so for kinds 2 and 3; therefore it does so in general.

We can take S_1 to be the Stroh syzygy $(a_1 a_2 a_4 a_3)_w$, and $S_r (r > 1)$ the Jacobian identity $(a_4 a_{2r+1} a_{2r+2})$.

In this case, since a perpetuant syzygy of the n-th kind that contains a non-primary compound perpetuant syzygy of the $(n-1)$-th kind is itself non-primary, $[(s_1 s_2 \dots S_n)]$ is non-primary if $[(s_1 s_2 \dots S_{n-1})]$ is.

But $[(s_1 S_2)]$ is clearly non-primary, as it contains a compound Stroh syzygy.

Therefore $[(s_1 s_2 \dots S_n)]$ is a non-primary perpetuant syzygy of the n-th kind, of degree $(2n+2)$.

Thus non-primary perpetuant syzygies of the n-th kind do exist for degree $(2n+2)$, *although not for lower degrees.*

In conclusion, I wish to express my best thanks to Dr. A. Young for his kindness in suggesting the subject of syzygies to me, and also to him and to Mr. J. H. Grace for much helpful interest and encouragement in the course of the work.