On the Lateral Vibration of Bars of Uniform and Varying Sectional Area

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XLVII. *On the Lateral Vibration of Bars of Uniform and Varying Sectional Area.* By JOHN MORROW, M.Sc. (Vict.); M. Eng. (Liverpool); Lecturer in Engineering, University College, Bristol *.

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I. Introduction.

In the theory of vibrating rods expressions for the frequency and the form of the displaced bar are obtained from the equation

$$\frac{d^4y}{dx^4} = \mu^4 y,$$

in which y is the displacement at a distance x from the origin.

The solution is

$$y = A \sin \mu x + B \cos \mu x + C \sinh \mu x + D \cosh \mu x,$$

A, B, C, D being arbitrary constants to be determined by the end conditions.

In the case of a uniform bar supported at each end the numerical solution presents no difficulty. In other elementary cases the results are deduced from the values of the roots of equations which would be troublesome if not previously tabulated.

Lord Rayleigh† has shown that the natural period of a bar may be obtained approximately by simpler means. For example, he supposes that the curve assumed by a clamped-free bar whilst vibrating is the same as that which it would

* Read March 10, 1905.

† See Rayleigh's 'Sound,' vol. i. pp. 233-235.

take up if displaced from its position of equilibrium by a lateral force acting on the bar at some point in its length. He shows that the period is given with considerable accuracy if the point of application of the force is one quarter of the length from the free end.

More recently, Garrett* has investigated this subject and, from instantaneous photographs of the vibrating bar, has concluded that a better approximation to the centre-line of the bar is obtained if the lateral force is assumed to act at a point one-fifth of the length from the free end.

Rayleigh's method is, of course, applicable to rods under other conditions of support. It rests on the fact† that a close approach to the true period can be obtained by assuming a type of vibration which is admissible as an initial configuration, as a considerable departure from the true type leads to only a small error in the estimate of the frequency.

Garrett's method and results have been further investigated and compared with Lord Rayleigh's by Dr. Chree‡, and it is shown that, whilst Rayleigh's depends on a recognized dynamical principle, Garrett's apparently has no such basis.

These simple solutions may be of considerable importance in acoustics and mechanics. They have been employed in problems of the "whirling" and vibration of rotating shafts by Dunkerley§ and later by Chree||.

It is the object of the present paper to show that by assuming an equation which completely satisfies the end conditions a better approximation is obtained, and that, by a process of continuous approximation, the vibration curve and the period of the fundamental are given to any required degree of accuracy.

The treatment is capable of very general application, and will be chiefly useful in problems of which the solution has not previously been obtained. The first three cases dealt with will serve as illustrations under various types of terminal conditions. It is a noteworthy fact that, whilst Rayleigh's

* Phil. Mag. Nov. 1904.

† Rayleigh's 'Sound' vol. i. §§ 88, 89, 182.

‡ Phil. Mag. Jan. 1905, p. 134.

§ Phil. Trans. A, 1894, p. 279.

|| Phil. Mag. May 1904, p. 504.

approximations are necessarily overestimates, the method described below gives initially too small a value for the frequency.

When the density and flexural rigidity of a bar are variable from point to point in its length, its treatment by the ordinary method of analysis presents difficulties which have not yet been completely overcome. Many of the solutions are obtained easily by the method here given, and several special cases are considered.

2. *Approximate Method of Solution.*

If a bar is vibrating so that every point in it has the same period, the ratio of the acceleration to the displacement is constant for all points.

If y_z be the displacement at a distance z from the point chosen as origin, so that $\frac{d^2 y_z}{dt^2}$ is the acceleration, and if y_1 be the displacement of any chosen point (say that at which the displacement is a maximum), then the above statement may be expressed by the equation

$$\frac{d^2 y_z}{dt^2} / y_z = \frac{d^2 y_1}{dt^2} / y_1$$

or

$$\ddot{y}_z = \frac{\ddot{y}_1}{y_1} y_z. \quad \dots \quad (1)$$

This is equivalent to the usual relation in frequency problems denoted by

$$y = \cos kt \cdot f(x).$$

Now, if we consider the reversed effective forces due to the acceleration of an increment of the length of the bar, we have, by the ordinary Euler-Bernoulli theory, that the couple at any section in the direction tending to increase $\frac{dy}{dx}$ is equal to the product of the curvature and the flexural rigidity at that section. That is, in view of the approximate straightness of the rod,

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}, \quad \dots \quad (2)$$

where M is the couple due to the reversed effective forces,
 E = Young's Modulus,
 and I = Moment of Inertia of the cross-section about the neutral axis.

Since M is a function of the mass per unit length, the end forces, and the accelerations, we can express it in terms of $\frac{\ddot{y}_1}{y_1} y_z$ and the density and cross-sectional area of the bar. And assuming for y_z a hypothetical type of vibration satisfying the end conditions, we can insert the value of M thus obtained in the equation (2). Solving for y we have a second approximation to the centre-line of the bar, and can calculate the period by

$$N = \frac{1}{2\pi} \sqrt{-\frac{\ddot{y}_1}{y_1}} \dots \dots \dots (3)$$

The next approximation consists in using this solution for y to find a more accurate expression for M , inserting this in equation (2) and again solving.

The process can be repeated to any extent, but in common with other methods it neglects the rotatory inertia of the rod.

In the following work the bars will be considered to be of uniform sectional area, density, and flexural rigidity except when otherwise stated.

3. *Clamped-Free Bar.*

Taking first the case of a rod "clamped" or "built-in" at one end and free at the other, let the origin, O , be at the free end.

If ρ = density of material of bar,
 ω = cross-sectional area,

then the Moment of the Inertia forces about X (distant x from O)

$$= \int_0^x \rho \omega \ddot{y}_z (x-z) dz,$$

and therefore the differential equation of the centre line of the beam is, by (1) and (2),

$$-\frac{d^2 y}{dx^2} = \frac{\rho \omega \ddot{y}_1}{EI y_1} \int_0^x y_z (x-z) dz, \dots \dots \dots (4)$$

$\rho\omega$ being taken from under the integral sign on account of the uniformity of the density and sectional area throughout the length.

The object is now to find the solution of this equation in ascending powers of x , and hence to obtain the period of vibration.

Assume a value of y_z which satisfies the end conditions. Thus let

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4.$$

The end conditions at the free end, $x=0$, are

$$\frac{d^2y}{dx^2} = 0, \quad \therefore C = 0,$$

$$\frac{d^3y}{dx^3} = 0, \quad \therefore D = 0,$$

$$y = y_1, \quad \therefore A = y_1,$$

and at the fixed end, $x=l$, $y=0$, $\frac{dy}{dx}=0$,

$$B = -\frac{4}{3}\frac{y_1}{l}, \quad E = \frac{y_1}{3l^4},$$

and the equation is

$$y = y_1 \left(1 - \frac{4x}{3l} + \frac{1}{3}\frac{x^4}{l^4} \right). \quad \dots \quad (5)$$

Substituting for y_z in equation (4)

$$\begin{aligned} -\frac{d^2y}{dx^2} &= \frac{\rho\omega}{EI} y_1 \int_0^x \left(1 - \frac{4z}{3l} + \frac{1}{3}\frac{z^4}{l^4} \right) (x-z) dz \\ &= \frac{\rho\omega}{EI} y_1 \left(.5x^2 - .2\frac{x^3}{l} + .01\frac{x^6}{l^4} \right), \end{aligned}$$

and performing the integrations, evaluating the constants by the end conditions,

$$\begin{aligned} -y &= \frac{\rho\omega}{EI} y_1 \left(.08194 l^4 - .112,698,41 l^3 x + .0416 x^4 \right. \\ &\quad \left. - .01\frac{x^5}{l} + .000,198,41 \frac{x^8}{l^4} \right), \quad \dots \quad (6) \end{aligned}$$

which is approximately the curve of vibration of the bar and gives

$$-y_1 = .08194 \frac{\rho\omega l^4}{EI} y_1,$$

or from (3)

$$N = \frac{3.493}{2\pi} \frac{kU}{l^2},$$

where k = radius of gyration of cross-section,

$$U = \sqrt{\frac{E}{\rho}} = \text{velocity of transmission of longitudinal vibrations in the rod.}$$

The expression given by Rayleigh may be put in the form

$$N = \frac{3.5160}{2\pi} \frac{kU}{l^2},$$

showing that the ratio between the two values of N is 0.9935.

The next approximation gives a still closer value and a ratio of 0.9998. It is obtained by inserting y_2 from equation (6) in (4), and using the value just obtained for y_1 , whence

$$\begin{aligned} -\frac{d^2y}{dx^2} &= \frac{12.204\rho\omega}{EI l^4} \ddot{y}_1 \int_0^x \left(.08194 l^4 - .112,698,41 l^3 z + .0416 z^4 \right. \\ &\quad \left. - .01 \frac{z^5}{l} + .000,198,41 \frac{z^8}{l^4} \right) (x-z) dz \\ &= \frac{12.204\rho\omega}{EI l^4} \ddot{y}_1 \left(.040972 l^4 x^2 - .018,783,07 l^3 x^3 + .00138 x^6 \right. \\ &\quad \left. - .000,264,55 \frac{x^7}{l} + .000,002,21 \frac{x^{10}}{l^4} \right), \\ \therefore -y &= \frac{.12204\rho\omega \ddot{y}_1}{EI l^4} \left(.663084 l^8 - .912718 l^7 x + .341435 l^4 x^4 \right. \\ &\quad \left. - .093915 l^3 x^5 + .002480 x^8 \right. \\ &\quad \left. - .000367 \frac{x^9}{l} + .000002 \frac{x^{12}}{l^4} \right). \end{aligned}$$

This is a more accurate expression for the vibration-curve than is (6), but it can be seen that the difference is not great.

$$-y_1 = .0809227 \frac{\rho\omega l^4}{EI} \ddot{y}_1,$$

and

$$N = \frac{3.5153}{2\pi} \frac{kU}{l^2},$$

which is correct to 0.02 per cent. of the value given by exact methods.

4. Bar "Supported" at Both Ends.

This is the case in which the directions at the ends are free, but the ends themselves are constrained to remain at rest by "supporting" forces of the required magnitude.

For a uniform bar these forces must each equal one half of the total force due to the inertia of the bar, that is, taking 0 at one end,

$$\frac{1}{2}\rho\omega\frac{\ddot{y}_1}{y_1}\int_0^l y\,dx.$$

Equation (2) then becomes

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}\rho\omega\frac{\ddot{y}_1}{y_1}(l-x)\int_0^l y\,dx - \rho\omega\frac{\ddot{y}_1}{y_1}\int_x^l y_z(z-x)dz. \quad (7)$$

Taking $y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$,

$$x=0, \quad y = \frac{d^2y}{dx^2} = 0, \quad \therefore A = C = 0$$

$$x = \frac{l}{2} \quad \frac{dy}{dx} = 0, \quad \therefore 0 = B + \frac{3}{4}Dl^2 + \frac{1}{2}El^3 + \frac{5}{16}Fl^4$$

$$x = \frac{l}{2} \quad y = y_1, \quad \therefore y_1 = B\frac{l}{2} + D\frac{l^3}{8} + E\frac{l^4}{16} + F\frac{l^5}{32}$$

$$x=l \quad y=0, \quad \therefore 0 = Bl + Dl^3 + El^4 + Fl^5$$

$$x=l \quad \frac{d^2y}{dx^2} = 0, \quad \therefore 0 = 6Dl + 12El^2 + 20Fl^3;$$

and the last four of these give

$$B = \frac{16}{5}\frac{y_1}{l}, \quad D = -\frac{32}{5}\frac{y_1}{l^3}, \quad E = \frac{16}{5}\frac{y_1}{l^4}, \quad F = 0.$$

Hence the type to be assumed as a first approximation is

$$y = \frac{16}{5}\frac{y_1}{l^4}(l^3x - 2lx^3 + x^4),$$

and (7) becomes

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\rho\omega\ddot{y}_1}{EI} \left\{ \frac{l-x}{2} \int_0^l \frac{16}{5l^4}(l^3x - 2lx^3 + x^4)dx \right. \\ &\quad \left. - \frac{16}{5l^4} \int_x^l (l^3z - 2lz^3 + z^4)(z-x)dz \right\} \\ &= \frac{16}{5} \frac{\rho\omega\ddot{y}_1}{EI l^4} \left(\frac{l^5x}{10} - \frac{l^3x^3}{6} + \frac{lx^5}{10} - \frac{x^6}{30} \right). \end{aligned}$$

Integrating and keeping the end conditions in view,

$$y = \frac{16}{5} \frac{\rho \omega \ddot{y}_1}{EI l^4} (\cdot 016 l^5 x^3 - \cdot 0083 l^3 x^5 + \cdot 002,380,95 l x^7 \\ - \cdot 000,595,24 x^9 - \cdot 010,119,05 l^7 x), \quad \dots \quad (8)$$

which is the approximate vibration-curve, and

$$-\frac{\ddot{y}_1}{y_1} = 97 \cdot 1098 \frac{EI}{\rho \omega l^4}.$$

For the second approximation, using (8) in (7) and simplifying as before,

$$-\frac{d^2 y}{dx^2} = \cdot 310,751 \frac{\rho \omega \ddot{y}_1}{EI l^6} (-1 \cdot 02513 l^3 x + 1 \cdot 68650 l^5 x^3 \\ - \cdot 83 l^7 x^5 + \cdot 19841 l^9 x^7 - \cdot 03307 l^{11} x^9 + \cdot 00662 l^{13} x^{11}) \\ \therefore -y = \cdot 310751 \frac{\rho \omega \ddot{y}_1}{EI l^8} (\cdot 10386 l^{11} x - \cdot 17086 l^9 x^3 \\ + \cdot 08433 l^7 x^5 - \cdot 01984 l^5 x^7 \\ + \cdot 00276 l^3 x^9 - \cdot 00030 l x^{11} + \cdot 00005 x^{13}).$$

This is a close approximation to the equation of the centre-line, and gives

$$-y_1 = \cdot 0102725 \frac{\rho \omega l^4}{EI} \ddot{y}_1 \\ \therefore N = \frac{9 \cdot 866}{2\pi} \frac{kU}{l^2}.$$

The correct value of the numerical constant in the numerator is π^2 , so that the ratio of the two values is 0.9996.

The last expression for the displacement may be conveniently compared with the exact type. Thus it may be written

$$y = Ax \left\{ 1 - \frac{\cdot 17086 x^2}{\cdot 10386 l^2} + \frac{\cdot 08433 x^4}{\cdot 10386 l^4} \dots - \frac{\cdot 00030 x^{10}}{\cdot 10386 l^{10}} + \frac{\cdot 00005 x^{11}}{\cdot 10386 l^{11}} \right\},$$

whilst the method indicated at the commencement of this paper gives, by the insertion of the end conditions,

$$y \propto \sin \frac{\pi x}{l},$$

or

$$y = A' x \left\{ 1 - \frac{\pi^2 (x/l)^2}{[3]} + \frac{\pi^4 (x/l)^4}{[5]} \dots - \frac{\pi^{10} (x/l)^{10}}{[11]} \right\}.$$

Comparing the coefficients of x^2 , x^4 , &c. inside the bracket shows how the accuracy alters in passing from the lower to the higher powers. For example,

$$\begin{array}{ll} \frac{\cdot 17086}{\cdot 10386} = 1\cdot 645. & \frac{\pi^2}{6} = 1\cdot 645. \\ \frac{\cdot 08433'}{\cdot 10386} = \cdot 8120. & \frac{\pi^4}{5} = \cdot 8117. \\ \frac{\cdot 00030}{\cdot 10386} = \cdot 000289. & \frac{\pi^{10}}{11} = \cdot 000235. \end{array}$$

5. *Free-Free Bar.*

In the case in which both ends are free and no external forces act on the bar, the conditions at each end are

$$y = y_1, \quad \frac{d^2 y}{dx^2} = 0, \quad \frac{d^3 y}{dx^3} = 0.$$

Assuming

$$y = A + Bx + Cx^3 + Dx^3 + Ex^4 + Fx^5 + Gx^5,$$

the conditions at $x=0$ give

$$A = y_1, \quad C = D = 0,$$

and those at $x=l$ give

$$B = -\frac{1}{2} Gl^5, \quad E = \frac{5}{2} Gl^2, \quad F = -3 Gl.$$

To determine G we have that, since the total force on the bar vanishes,

$$\rho \omega \frac{y_1}{y_1} \int_0^l y \, dx = 0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is

$$\begin{aligned} \int_0^l (y_1 - \frac{1}{2} Gl^5 x + \frac{5}{2} Gl^2 x^4 - 3 Gl x^5 + G x^5) dx &= 0 \\ \therefore G &= \frac{28 y_1}{3 l^8}, \end{aligned}$$

and hence the equation to be assumed is

$$y = y_1 \left(1 - \frac{14}{3} \frac{x}{l} + \frac{70}{3} \frac{x^4}{l^4} - 28 \frac{x^5}{l^5} + \frac{28}{3} \frac{x^5}{l^8} \right).$$

Equation (2) becomes for this case

$$-\frac{d^2 y}{dx^2} = \frac{\rho \omega}{EI} \frac{\ddot{y}_1}{y_1} \int_0^x y_z (x-z) dz,$$

and substituting for y_z

$$\begin{aligned} -\frac{d^2 y}{dx^2} &= \frac{\rho \omega}{EI} \ddot{y}_1 \int_0^x \left(1 - \frac{14}{3} \frac{z}{l} + \frac{70}{3} \frac{z^4}{l^4} - 28 \frac{z^5}{l^5} + \frac{28}{3} \frac{z^6}{l^6} \right) (x-z) dz \\ &= \frac{\rho \omega}{EI} \ddot{y}_1 \left(.5 x^2 - .7 \frac{x^3}{l} + .7 \frac{x^6}{l^3} - .6 \frac{x^7}{l^5} + .16 \frac{x^8}{l^6} \right). \end{aligned}$$

Integrating twice and evaluating the second constant by virtue of (9)

$$\begin{aligned} -y &= \frac{\rho \omega}{EI} \ddot{y}_1 \left(.001,993,14 l^4 - .00925 l^3 x + .0416 x^4 - .038 \frac{x^5}{l} \right. \\ &\quad \left. + .0138 \frac{x^6}{l^3} - .00925 \frac{x^6}{l^5} + .00185 \frac{x^{10}}{l^6} \right). \end{aligned}$$

and

$$N = \frac{22.399}{2\pi} \frac{kU}{l^2}.$$

The more correct value is 22.373, so that in this case the first approximation gives a ratio of the two values as near as 0.9988.

6. *Clamped-Free Bar of Varying Breadth.* ($b = Ax$.)

Let b be the breadth of the bar, and d its depth in the plane in which the vibration occurs.

Let $d = \text{constant}$, and $b = Ax$. Equation (4) then becomes

$$\begin{aligned} -\frac{d^2 y}{dx^2} &= \frac{1}{EI} \int_0^x \rho \omega_z \ddot{y}_z (x-z) dz \\ &= \frac{2\rho \ddot{y}_1}{Ed^2} \left(x^2 - \frac{2}{3} \frac{x^3}{l} + \frac{1}{21} \frac{x^6}{l^4} \right), \end{aligned}$$

by virtue of (1) and (5).

$$\begin{aligned} \therefore -y &= \frac{2\rho \ddot{y}_1}{Ed^2} \left(.122,619,04 l^4 - .173,469,38 l^3 x + .083 x^4 \right. \\ &\quad \left. - .03 \frac{x^5}{l} + .000,850,34 \frac{x^8}{l^4} \right) \\ -\frac{\ddot{y}_1}{y_1} &= 4.07767 \frac{Ed^2}{\rho l^4}. \end{aligned}$$

In the next approximation the differential equation reduces to

$$-\frac{d^2y}{dx^2} = \frac{97.86408\rho\ddot{y}_1}{Ed^2l^4} \left(.020,436,51 l^4 x^2 - .014,455,78 l^3 x^3 \right. \\ \left. + .001,984,13 x^6 - .000,59523 \frac{x^7}{l} + .000,007,73 \frac{x^{10}}{l^4} \right);$$

whence

$$-y = \frac{.9786408\rho\ddot{y}_1}{Ed^2l^4} \left(.240053 l^4 - .340800 l^3 x + .170304 l^4 x^4 \right. \\ \left. - .072279 l^3 x^3 + .003543 x^6 - .000827 \frac{x^7}{l} + .000006 \frac{x^{12}}{l^4} \right) \\ - \frac{\ddot{y}_1}{y_1} = 4.256 \frac{Ed^2}{\rho l^4},$$

and

$$N = \frac{2.063}{2\pi} \frac{dU}{l^2}.$$

All clamped-free bars in which the depth is constant and the breadth proportional to the distance from the free end will vibrate with a period which is independent of the breadth at the fixed end (that is of the value of A), as given by the above formula.

7. *Clamped-Free Bar of Varying Breadth.* ($b = Ax^2$.)

$$\text{Here} \quad -\frac{d^2y}{dx^2} = \frac{12\rho}{Ed^2x^2} \frac{\ddot{y}_1}{y_1} \int_0^x z^2(x-z)y_z dz \\ = \frac{\rho\ddot{y}_1}{Ed^2} \left(x^2 - .8 \frac{x^3}{l} + .07142857 \frac{x^5}{l^3} \right),$$

$$\therefore -y = \frac{\rho\ddot{y}_1}{Ed^2} \left(.098,928,57 l^4 - .143,537,41 l^3 x + .083 x^4 \right. \\ \left. - .04 \frac{x^5}{l} + .001,275,51 \frac{x^8}{l^4} \right),$$

and

$$-\frac{\ddot{y}_1}{y_1} = 10.1083 \frac{Ed^2}{\rho l^4}.$$

For the second approximation

$$-\frac{d^2y}{dx^2} = \frac{121.3\rho\ddot{y}_1}{Ed^2l^4x^2} \int_0^x z^2(x-z) \left(.09892857 l^4 - .14353741 l^3 z \right. \\ \left. + .083 z^4 - .04 \frac{z^5}{l} + .001,275,51 \frac{z^8}{l^4} \right) dz \\ = \frac{.1213\rho\ddot{y}_1}{Ed^2l^4} \left(8.24405 l^4 x^2 - 7.17687 l^3 x^3 + 1.48809 x^6 \right. \\ \left. - .5 \frac{x^7}{l} + .00966 \frac{x^{10}}{l^4} \right),$$

$$\therefore -y = \frac{1213\rho\ddot{y}_1}{E d^2 l^4} \left(.75074 l^8 - 1.09782 l^7 x + .68700 l^6 x^2 \right. \\ \left. - .35884 l^5 x^3 + .02657 x^4 - .00772 \frac{x^5}{l} + .00007 \frac{x^{12}}{l^7} \right) \\ - \frac{\ddot{y}_1}{y_1} = 10.9812 \frac{E d^2}{\rho l^4}.$$

This gives
$$N = \frac{3.313}{2\pi} \frac{dU}{l^2}.$$

For a third approximation the differential equation reduces to

$$-\frac{d^2 y}{dx^2} = \frac{15.984\rho\ddot{y}_1}{E d^2 l^8} \left(.062562 l^8 x^2 - .054891 l^7 x^3 + .012268 l^6 x^4 \right. \\ \left. - .004984 l^5 x^7 + .000201 x^{10} - .000049 \frac{x^{11}}{l} \right), \\ \therefore -y = \frac{15984\rho\ddot{y}_1}{E d^2 l^8} \left(.5655 l^2 - .8275 l^1 x + .52135 l^4 x^4 \right. \\ \left. - .27446 l^7 x^5 + .02191 l^4 x^8 - .00692 l^3 x^9 \right. \\ \left. + .00015 x^{12} - .00003 \frac{x^{13}}{l} \right).$$

Whence
$$-\frac{\ddot{y}_1}{y_1} = 11.063 \frac{E d^2}{\rho l^4},$$

and
$$N = \frac{3.326}{2\pi} \frac{dU}{l^2}.$$

This is again independent of the breadth of the bar.

8. *Clamped-Free Bar of Varying Depth.* ($d=Ax$.)

The exact form of the equation first assumed for the type of vibration is immaterial to the final result, but, by choosing a suitable equation, the labour involved in obtaining this result to any required degree of accuracy is reduced.

In the present case (as, indeed, in the last two) $\frac{d^2 y}{dx^2}$ is not necessarily zero when $x=0$, for there the area of the section of the bar also vanishes.

Hence we may start with an equation

$$y = y_1 \left(1 - \frac{2x}{l} + \frac{x^2}{l^2} \right), \quad \dots \dots (10)$$

9. *Kirchhoff's Investigation.*

The solution for some cases of non-uniform cross-section was considered by Kirchhoff in 1879 *. He remarked that the general equation

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 y}{dx^2} \right) + \rho \omega \frac{d^2 y}{dt^2} = 0 \quad \dagger,$$

can be solved in general terms when the depth and breadth of the cross-section are of the form

$$d = x^m f_1(\chi), \quad b = x^n f_2(\chi),$$

χ being any variable quantity.

The only cases he investigates fully are

$$\begin{array}{lll} (a) & m=1, & n=0, & \chi = \text{constant.} \\ (b) & m=1, & n=1, & \chi = \text{constant.} \end{array}$$

The former corresponds to that in the last section and the numerical result obtained by Kirchhoff may be written

$$N = \frac{1.534 \text{ AU}}{2\pi} \frac{1}{l}.$$

Agreeing well with equation (11) above.

University College, Bristol,
January 1905.

DISCUSSION.

Dr. CHREE pointed out that the Author's formulæ for the frequency contained factors of the type $\sqrt{\ddot{y}/y}$. y varied, however, as $\cos kt$, and so $\ddot{y}/y$ was negative. The explanation was that Mr. Morrow's "inertia forces" ought really to be taken with the reversed sign and ought to be the "reversed effective forces" ordinarily employed in dynamical problems. He remarked that the results to which the method led were very close approximations indeed, but that the successive approximations entailed laborious

* *Berliner Monatsberichte*, 1879, p. 815.

† Rayleigh's 'Theory of Sound,' vol. i. art. 187.

algebra. The value of the method would be much enhanced if it could be shown that the phenomenon exhibited by all the cases treated by Mr. Morrow of the calculated frequency being too low was universally true. If that were the case, one would know that the correct value was intermediate between that given by Mr. Morrow's first approximation and by Lord Rayleigh's method. Dr. Chree also pointed out that Kirchhoff had obtained exact solutions for a variety of forms of varying section, and that one of his cases, discussed in Todhunter and Pearson's '*History of Elasticity*,' was the same as the last of the Author's cases. In this instance the frequency obtained by Mr. Morrow's third approximation came very near Kirchhoff's exact result.

Mr. MORROW, in reply, said that by his method the first value for the frequency was sometimes an appreciable amount out, but that the true value was gradually approached by successive approximations. By making two approximations, therefore, it was possible in any case to determine whether the initial value of the frequency was too high or too low. The method employed by Mr. Garrett gave one approximation only, and there was no continuous approach to the true frequency.