VI. *The Influence of Flaws and Air-cavities on the Strength of Materials.* By J. LARMOR, M.A., Fellow of St. John's College, Cambridge*.

IN practical estimates of the strength of materials it is usual to take the greatest compressive or tensile stress which the material is found in experiment to sustain, and divide it by a factor of safety to insure against sudden applications and reversals of the load, and against flaws or sources of weakness that cannot be foreseen. Among the latter, cavities or air-bubbles in the material hold a place; and these may also be taken in a general way for purposes of calculation as the type of flaws consisting of a defect or weakening which is confined to a limited volume of the substance.

Thus, in the case of a shaft transmitting a torque or couple, the shearing-stress is annulled over the volume of the cavity, and this may lead to greater than average shearing-stress in some part of its immediate neighbourhood. In the case of a column supporting a load, the supporting thrust is absent over the part of the cross section occupied by the cavity, and this defect of support must be compensated by a greater thrust elsewhere.

When the cavity or flaw is at a great distance from the surface of the casting compared with its linear dimensions, the changes produced by it in the intensity of the stress are the same at corresponding points, whatever be the dimensions of the cavity. For when the latter is altered in linear dimensions but not in form, and the displacement of the material at corresponding points is altered in the same ratio, the components of the strain will maintain their intensities unaltered at corresponding points, and so will the components of the stress. Thus the traction over the surface of the cavity will be unaltered, and therefore remain zero; while the displacements over the surface will be changed in the above ratio. The practical statement of this principle of similarity is that the effect which is produced by a cavity on the strength of a piece under uniform stress is dependent on the form but not on the size of the cavity, provided the distance of the nearest part of the surface from it is at least two or three times its greatest diameter.

The amount of this increase of internal stress determines the theoretical factor of safety which the possibility of a flaw of the type in question would necessitate; and it is possible to arrive at an estimate for the case of spherical or cylindrical

* Communicated by the Author.

cavities, which may be of use as a general indication of the order of magnitude involved in other similar cases also. Even for the actual cases worked out the result is not, however, to be interpreted exactly. For, in the first place, to make calculation possible the proportionality of stress to strain (Hooke's law) is assumed, and this ceases to hold, the material sometimes even begins to flow, before the critical condition is attained; and, secondly, the conditions that produce a breakdown of the material are but vaguely understood.

A spherical portion of the mass becomes, when strained, an ellipsoid of which the principal axes determine the three principal elongations which constitute the strain. Now it is sometimes assumed that a simple change of volume by compression or expansion cannot produce or affect rupture, and therefore this ellipsoid need only be compared with the sphere of equal volume from which it is derived by three simple shears in mutually rectangular planes. The value of the greatest of these shears may then perhaps be taken to be the circumstance determining the limiting strength of the material. It may, however, be remarked that, as the forces of cohesion between the elements of the material are not infinite, it must be possible to break it down or pull it asunder by a tension uniform in all directions (say a negative hydrostatic pressure); and it is quite conceivable that a pressure equal in all directions may by the opposite displacement loosen the bonds of cohesion and so produce a plastic condition which will give other forces play to act. The experiments of W. Spring, in which an intimate mixture of two solid substances which do not combine chemically under ordinary circumstances is caused to combine by the application of great pressure, may have a bearing on this question. The fact that cast iron supports compression much better than tension is also in point. If it is, however, the case with any material that the range of tension uniform in all directions which it can stand is very much greater than the range of stresses involving shear, the rupture would depend for that material on the shears only, and the greatest of them might be taken to be its determining cause. Thus rupture would be determined by the difference between the greatest and least axes of the ellipsoid into which a sphere of unit radius is strained. When this supposition is not valid the greatest elongation would be a more likely criterion; but in any case the assumed law will be a sufficient indication for our purpose, because any more precise specification would be vitiated in its application by the causes above mentioned, which render elastic calculations illustrative rather than exact

when pushed towards the limit of strength of the materials. In the most important examples we shall be concerned only with shears.

It will appear on consideration that a small spherical cavity in a column or other mass under tension or compression cannot seriously affect its strength. For its strength could be reduced only by an increase of shear in the neighbourhood of the cavity. Now this shear must act all round it in the planes containing that diameter which lies along the direction of the stress, and at the free surface of the cavity it must be zero, in the absence of surface-tractions there; hence the shear is diminished in the neighbourhood of the cavity. The compression may be slightly increased in a ratio depending on that of the area of the section of the cavity to the area of the section of the shaft. The same argument applies of course to any symmetrical form of cavity, and generally to any cavity of regular shape.

The case is different, however, when the cavity exists in a shaft which transmits a couple. If we suppose the cavity to consist of a narrow tunnel bored down the length of the shaft, we may make use of the result of St. Venant's torsion problem. The distribution of the shear across the section of the shaft is simply and succinctly expressed by hydrodynamical analogy*. If a cylindrical shell of the same form of cross section as the shaft is filled with frictionless fluid and is set in rotation, the velocity of the fluid relative to the shell will at each point represent the shear, in direction and magnitude; and the momentum of the fluid relative to the shell, which must necessarily have no linear component, will be proportional to the torsional rigidity of the shaft. For the present purpose it is convenient to state the proposition in a form less practically realizable: suppose the shell fixed and the fluid circulating inside it with uniform vorticity, the velocity at each point will represent the shear, and its resultant momentum (angular) will be proportional to the rigidity of the shaft.

Now the result of boring a small tunnel will be to modify the velocity system in the neighbourhood in the same way as a solid cylinder changes the velocities in a stream flowing past it. The velocities in front and rear are reduced to zero, while those at the sides are doubled. A tunnel of this kind therefore halves the strength of the portion of the shaft in which it is situated; and the same statement practically applies to any cavity of elongated form and circular section which lies parallel to the axis of the shaft. The possibility of

* Thomson and Tait's 'Natural Philosophy,' § 705.

a flaw of this kind near the part of the cross section where the shear is greatest will therefore necessitate the use of a factor of safety equal to two. As the cavity is taken shorter in proportion to its diameter, its effect might at first sight be taken to diminish till we come to the spherical form, which is again amenable to calculation, though with considerable intricacy: we might perhaps expect for it a factor considerably less than two. The result of the mathematical investigation for a spherical cavity which follows, for which I am indebted to Mr. A. E. H. Love, gives, however, a factor which is never very far from the value two, unless the material is but slightly compressible, like a jelly. If we now suppose the spherical cavity to elongate in a direction perpendicular to the shear the factor may be expected still to diminish; and when it is so long as to be sensibly cylindrical the shear is itself diminished in its neighbourhood, for reasons specified above. But if it elongates in the direction perpendicular to the axis of the shaft, and in the plane of the shear, the factor two is recovered.

If the cylindrical cavity is of flat cross section the hydrodynamical analogy shows that its action is intensified. If it were absolutely flat with a sharp edge the strain would be infinite there and rupture would take place, unless in the test there is a chance of smoothing the edge of the flaw by a local flow or adjustment of the material.

A semicircular groove, running along the surface of a shaft, would (in the absence of local flow) nearly halve its torsional strength.

Adaptation of St. Venant's Solution for a Shaft.

The displacement in St. Venant's solution is

$$u = \omega yz, \quad v = -\omega xz, \quad w = f(x, y);$$

where u, v represent a simple torsion round the axis of z , and w represents the warping of the cross section which is necessary to annul the shear in a plane normal to the free boundary.

The value of this shear is $\frac{dw}{dn} - \omega p$, where p is the perpendicular from the axis on the tangent plane to the boundary, and dn is an element of the normal. Thus the boundary condition is

$$\frac{dw}{dn} = \omega p;$$

and these displacements maintain internal equilibrium provided

$$\nabla^2 w = 0.$$

These equations show that w is the velocity-potential of the absolute motion in space of liquid contained in a box rotating with angular velocity ω .

The tractions exerted across the section of the shaft are (with unit rigidity) .

$$X = \omega y + \frac{dw}{dy}, \quad Y = -\omega x + \frac{dw}{dx}.$$

These must vanish when integrated over the area of the section ; therefore the box containing liquid must be supposed to rotate round an axis through the centre of gravity of the section.

The couple transmitted across the section is

$$G = -\int (Xy - Yx) dS \\ = \int \omega (x^2 + y^2) dS - \int \left(y \frac{dw}{dx} - x \frac{dw}{dy} \right) dS ;$$

it is therefore less than the couple due to simple torsion by the absolute angular momentum of the liquid.

Also X, Y are the component velocities of liquid circulating in a fixed box with vorticity ω ; its resultant velocity represents the shear at each point, and its angular momentum represents the couple transmitted. Its linear momentum is null.

The analysis of this well-known result has been here indicated in full, partly in order to point out that in the first form of the analogy in which the box is made to rotate, the velocity of the liquid relative to the box represents the shear whatever be the axis of rotation, but the angular momentum of the liquid represents the correction to the rigidity only when taken about that axis for which its value is least, viz. the axis through the centre of gravity of the section. If the motion is referred to any other axis, as in the case of a rectangle bounded by two concentric arcs and two radii (Thomson and Tait, § 707), then from the angular momentum round that axis must be subtracted the moment of the linear momentum of the whole mass of fluid supposed collected at its centre of gravity.

Suppose, now, a cylindrical tunnel of small circular section bored down the shaft at a place where the velocity of the rotational fluid motion is V ; the stream function near it will be changed from the form $\psi_1 = Vy$ to the form $\psi_2 = Vy - V \frac{a^2}{r^2} y$,

because the boundary of the tunnel, $r=a$, must become a stream line, and therefore give a constant value to ψ_2 . The velocity along the tunnel is $-\frac{d\psi_2}{dn}$, and is therefore $2V$ at the sides, as stated above.

The angular momentum of the fluid is altered, owing to the tunnel, by

$$\iint -\frac{d(\psi_2 - \psi_1)}{dy} dx dy,$$

that is $\int (\psi_2 - \psi_1) \frac{dx}{ds} ds$ round the boundary. For a circular boundary this is equal to $Va \int \cos^2 \theta ds$, or $\pi a^2 V$. The rigidity of the surrounding parts is therefore diminished by the presence of the cavity, just as if the shearing over the material which originally occupied its place were reversed in direction; the loss of rigidity is due in equal proportion to the removal of the matter and the release on the constraint of the surrounding parts.

The case when the section of the cavity is an elliptic cylinder is of interest, as it illustrates the effect of making it more and more flat until it is finally a mere crack for which the strain is theoretically infinite at the edge. The corresponding hydrodynamical problem has been solved by Prof. Lamb*: his value of the stream function ψ , which may easily be verified, is

$$\psi = -V \left(\frac{a+b}{a-b} \right)^{\frac{1}{2}} c^{-\eta} \sin \xi - Vx;$$

where a, b are the semi-axes of the ellipse, V is the velocity of the stream past it parallel to the axis b , and ξ, η are the conjugate functions given by

$$x + iy = c \sin (\xi + i\eta).$$

The velocity at the end of the longer axis is the value of $d\psi/dx$ when $y=0$, that is, when $\xi = \frac{1}{2}\pi$, and is found to be

$$V \left(1 + \frac{a}{b} \right).$$

Thus in the elastic problem the increase of shear produced by the cavity is the original shear multiplied by a/b .

* Quart. Journ. of Math. 1875; "Fluid Motion," p. 90.

Analysis for Spherical Cavity. (By Mr. Love.)

To investigate the strain in an infinite solid containing a spherical cavity, the displacements at an infinite distance being

$$u = \alpha y, \quad v = 0, \quad w = 0.$$

From the spherical harmonic solutions of Thomson and Tait it can be shown that the forms of the displacements at a point (x, y, z) at a distance r from the centre of the cavity are

$$\left. \begin{aligned} u &= A \frac{y}{r^3} + (B + Cr^2) \frac{d}{dx} \left(\frac{xy}{r^5} \right) + \alpha y, \\ v &= A \frac{x}{r^3} + (B + Cr^2) \frac{d}{dy} \left(\frac{xy}{r^5} \right), \\ w &= (B + Cr^2) \frac{d}{dz} \left(\frac{xy}{r^5} \right), \end{aligned} \right\} \quad \dots \quad (1)$$

where A, B, C are constants to be determined.

The cubical dilatation δ is

$$\delta = -6(A + C) \frac{xy}{r^5} \dots \dots \dots (2)$$

The equations of equilibrium are three such as

$$(\lambda + \mu) \frac{d\delta}{dx} + \mu \nabla^2 u = 0 \dots \dots \dots (3)$$

which gives

$$-(\lambda + \mu)6(A + C) - 10\mu C = 0$$

or

$$3(\lambda + \mu)A + (3\lambda + 8\mu)C = 0 \dots \dots \dots (4)$$

The remaining equations to determine the constants are to be found from the condition that the surface $r = a$ of the cavity is free from stress. It is shown in Thomson and Tait's Nat. Phil., Part ii., art. 737, that if F, G, H be the component surface-tractions parallel to the axes across a spherical surface whose centre is the origin and radius is r , then

$$Fr = \lambda x \delta + \mu \left(\frac{d\xi}{dx} + r \frac{du}{dr} - u \right),$$

where $\xi = ux + vy + wz$, and similar equations hold for G and H.

Now

$$x\delta = -6(A+C) \frac{x^2y}{r^5},$$

$$\xi = (2A-3C) \frac{xy}{r^3} - 3B \frac{xy}{r^5} + \alpha xy,$$

$$\frac{d\xi}{dx} = (2A-3C) \frac{y}{r^3} - 3B \frac{y}{r^5} - (6A-9C) \frac{x^2y}{r^5} + 15B \frac{x^2y}{r^7} + \alpha y,$$

$$r \frac{du}{dr} - u = -3(A+C) \frac{y}{r^3} - 5B \frac{y}{r^5} + 15C \frac{x^2y}{r^5} + 25B \frac{x^2y}{r^7}.$$

Hence the equations

$$-\left(A+6C+\frac{8B}{r^2}\right) \frac{y}{r^3} + \alpha y - \left[\frac{\lambda}{\mu} 6(A+C) + 6A - 24C - \frac{40B}{r^2}\right] \frac{x^2y}{r^5} = 0,$$

$$-\left(A+6C+\frac{8B}{r^2}\right) \frac{x}{r^3} + \alpha x - \left[\frac{\lambda}{\mu} 6(A+C) + 6A - 24C - \frac{40B}{r^2}\right] \frac{xy^2}{r^5} = 0;$$

and similarly from the z equation

$$-\left[\frac{\lambda}{\mu} 6(A+C) + 6A - 24C - \frac{40B}{r^2}\right] \frac{xyz}{r^5} = 0.$$

These three equations hold when $r=a$.

Hence

$$\frac{40B}{a^2} - 6(A+C) \frac{\lambda+\mu}{\mu} + 30C = 0,$$

and

$$\frac{8B}{a^2} + (A+6C) = \alpha a^3.$$

From which and (4) we find

$$\left. \begin{aligned} A &= \frac{3\lambda+8\mu}{9\lambda+14\mu} \alpha a^3, \\ B &= \frac{3(\lambda+\mu)}{9\lambda+14\mu} \alpha a^5, \\ C &= -\frac{3(\lambda+\mu)}{9\lambda+14\mu} \alpha a^3. \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

The shear $\frac{du}{dy} + \frac{dv}{dx}$ is given by

$$\frac{du}{dy} + \frac{dv}{dx} = \alpha + \frac{2}{r^3} \left(A + C + \frac{B}{r^2} \right) - \frac{x^2 + y^2}{r^5} \left(8C + 3A + \frac{10B}{r} \right) + \frac{x^2 y^2}{r^7} \left(50C + \frac{70B}{r^2} \right).$$

To see the magnitude of this take $x=0$, $y=0$, $r=a$, then

$$\begin{aligned} \frac{du}{dy} + \frac{dv}{dx} &= \alpha \left[1 + \frac{10\mu + 6(\lambda + \mu)}{9\lambda + 14\mu} \right] \\ &= \alpha \frac{15\lambda + 30\mu}{9\lambda + 14\mu}, \end{aligned}$$

which depends on the value of λ/μ , but for all known isotropic materials differs little from 2α .

When $\lambda/\mu = \infty$, or the material is incompressible, the shear is

$$\frac{5}{3}\alpha.$$

When $\mu/\lambda = \infty$, or the stretch-squeeze ratio vanishes, the shear is

$$\frac{15}{7}\alpha.$$

When $\lambda = \mu$ (Poisson's condition) the shear is

$$\frac{45}{23}\alpha.$$

St. John's Coll., Cambridge,
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VII. *A Galvanometric Method of Measuring H.* By R. A. LEHFELDT, B.A., B.Sc., Demonstrator at Firth College, Sheffield*.

IN the ordinary magnetometric, and in Kohlrausch's method of determining the earth's horizontal magnetic force, the magnetic moment or the strength of current is involved, and two observations are necessary to give two equations between which the unknown may be eliminated. If, however, it is assumed that a current can be measured absolutely, the problem of measuring H is simplified. This may fairly be assumed now, for not only are there very reliable direct-reading ammeters to be had, but the E.M.F of the Clark cell

* Communicated by the Author.