THE GROUP OF THE LINEAR CONTINUUM

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1. The linear continuum has already received a complete characterization in terms of order* and of limit.† Now, the author has shown that over a wide range of cases the notion of limit may be defined in terms of that of bicontinuous biunivocal transformation.‡ It is the purpose of this paper to develop a categorical theory of the structure of the line in terms of bicontinuous, biunivocal transformations, or, in other words, to give a complete postulational characterization of the analysis situs group of the line.

The set of postulates will be so framed that only one will have any direct effect on the dimensionality. All the other postulates together determine an analysis situs property of space which is shared by a large number of systems of a finite or infinite dimension number. A number of necessary conditions and a sufficient condition for a system to possess this property will be formulated.

INDEFINABLES.

2. Our indefinables are two in number—a set K of elements and a set Σ of one-one transformations of the whole of K into itself.

DEFINITIONS.

3. A sub-set E of K is said to have a *limit-element* A if A is invariant under every transformation belonging to Σ that leaves invariant every member of E except possibly A.

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* Cf. E. V. Huntington, "A Set of Postulates for Real Algebra," *Trans. Am. Math. Soc.* (1905); O. Veblen, "Definition in Terms of Order alone in the Linear Continuum," *ibid.*

† R. L. Moore, "The Linear Continuum in Terms of Point and Limit," *Annals of Mathematics* (1914–15).

‡ N. Wiener, "Limit in Terms of Continuous Transformation," *Bull. Soc. Math. de France* (1921–22).

A set E is *closed* if it contains all its limit-elements.

A set E is *connected* if, whenever it is divided into the two non-null sets, F and G, either F has a limit-element in G or G has a limit-element in F.

$\bar{E}$ is the set of all elements in K but not in E.

An *interior* element of E is one that is not a limit-element of $\bar{E}$.

An element A is *exterior* to E if it is interior to $\bar{E}$.

An element A is a *boundary-element** of E if it is at once a limit-element of E and of $\bar{E}$.

A *segment* is a closed, connected set with at least two boundary elements.

A *component*† of a set E is a greatest connected sub-set of E.

The transformation $\breve{R}$ is the inverse of R. $R|S$ is the transformation which consists in performing first S and then R.

POSTULATES.

4. I. K contains at least three distinct elements.

II. If R is a biunivocal transformation of the whole of K into itself that turns all closed sets into all closed sets and only into closed sets, then R belongs to Σ.

III. If R and S belong to Σ, so does $R|S$.

IV. If R belongs to Σ, so does $\breve{R}$.

V. If there is a transformation from Σ changing A and leaving every member of E invariant, while there is a transformation from Σ changing A and leaving every member of F invariant, then there is a transformation from Σ changing A and leaving every member of $E+F$ invariant.

VI. If A, B, C, and D belong to K, and $A \neq C$, $B \neq D$, then there is a transformation from Σ changing A to B and C to D.

VII. If E is any sub-set of K and A is an element of K not a limit-element of E, then there is a segment of which A is an interior element and which contains no element of E.

VIII. There is an at most denumerable sub-set K' of K such that no member of Σ except possibly the identity transformation leaves every member of K' invariant.

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* Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, p. 214. The notions of boundary element and *Randpunkt* are not identical.

† *Ibid.*, p. 245.

IX. If E and F are two connected sets, and two boundary elements of E are boundary elements of F, then every other element of E is an element of F.

DEFINITIONS OF SYSTEMS.

5. A system satisfying postulates I–IX inclusive will be called a system (Li). A system satisfying postulates I–VII inclusive will be called a system (Sp).

A system (T_1) will be defined as in the author's previous paper in the *Bull. Soc. Math. de France*, as a system satisfying Postulates II–IV. A system (R) will be defined as by Fréchet,* as a system satisfying the conditions of F. Riesz.

1. Every limit-element of a set E is a limit-element of every set containing E.

2. Every limit-element of the sum of two sets E and F is a limit-element of at least one of the two sets.

3. A set containing a single element has no limit-element.

4. If A is a limit-element of a set E and B is distinct from A, there is always at least one set which has A for a limit-element without having B for a limit-element.

It has been proved by the author† that in the case of a (T_1), the necessary and sufficient condition that the system should also be an (R) is that it should satisfy the following three conditions:—

2′. This is verbally identical with V.

3′. Given any two elements, A and B, there is a transformation from Σ changing A but leaving B invariant.

4′. If there is a set E not containing the element A, but such that every transformation from Σ that leaves all the elements of E invariant leaves A also invariant, then, given any element B distinct from A, there is a set F not containing A such that there is no transformation from Σ changing A but leaving each member of F invariant, while there is a transformation from Σ changing B but leaving F invariant.

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* "Sur la notion de voisinage dans les ensembles abstraits," *Bulletin des Sciences Mathématiques*, May 1918.

† *Loc. cit.*

A system (H) is one in which neighbourhoods are so defined as to satisfy Hausdorff's "Umgebungsaxiome":*

(A) Given any point x, there is at least one neighbourhood U_x, of which x is a member.

(B) If U_x and V_x are two neighbourhoods of x, then there is a neighbourhood W_x contained in both.

(C) If y belongs to U_x, there is a neighbourhood of y, U_y, contained in U_x.

(D) If x and y are two points, then neighbourhoods U_x and Y_x can be so chosen as not to overlap.

In a system (H) a set E is said to have a limit-point A if every neighbourhood U_A of A contains an infinity of points of E.†

A *vector-system*, or system (Ve), is defined as in my previous paper‡ as a system K of elements (represented by capitals), associated with entities called vectors (represented by Greek letters), real numbers (represented by lower case letters), and the operations $\odot$, $\oplus$, and $\| \; \|$ by the following laws:—

(1) If ξ and η are vectors, $\xi \oplus \eta$ is a vector.

(2) If ξ is a vector and $n \geqslant 0$, $n \odot \xi$ is a vector.

(3) If ξ is a vector, $\| \xi \|$ is a non-negative real number.

(4) $n \odot (\xi \oplus \eta) = (n \odot \xi) \oplus (n \odot \eta)$.

(5) $m \odot (n \odot \xi) = mn \odot \xi$.

(6) $(m \odot \xi) \oplus (n \odot \xi) = (m+n) \odot \xi$.

(7) $\| m \odot \xi \| = m \| \xi \|$.

(8) $\| \xi \oplus \eta \| \leqslant \| \xi \| + \| \eta \|$.

(9) If A and B belong to K, there is associated with them a unique vector AB.

(10) $\| AB \| = \| BA \|$.

(11) Given an element A of K and a vector ξ, there is an element B of K such that $AB = \xi$.

* *Loc. cit.*, p. 213.

† *Ibid.*, p. 219, definition of *β-Punkt*.

‡ *Loc. cit.*

(12) $AC = AB \oplus BC$.

(13) $\| AB \| = 0$ when and only when $A = B$.

(14) If $AB = CD$, $DC = BA$.

A system (Vr), or a *restricted vector system* is a vector system of at least two elements in which the sum of two vectors is independent of their order, and in which, if A, B, and C are any three distinct elements such that $\| AB \| = \| AC \|$, then there is a finite set $B_1, B_2, \ldots, B_n$ of elements such that

(1) $B_1 = B$, $B_n = C$.

(2) For all k, $\| AB_k \| = \| AB \|$.

(3) For all k, $\| B_k B_{k+1} \| < \| AB \|$.

We shall say that a set E has A for a limit-element if, for all the B's that belong to E, the lower bound of $\| AB \|$ is zero.

Relations of Systems.

6. We shall say that a system of one of our classes belongs to another of our classes if a translation into the language of the second class is always possible in such a manner as to keep limit properties invariant. We have already seen that every (Sp) or (Li) is a (T_1), and every (Li) is clearly an (Sp); we shall prove the further relations:

(1) Every (Sp) is an (R).

(2) Every (Sp) is an (H).

(3) Every (Vr) is an (Sp).

Proof of (1).

All that we need to prove is contained in propositions 3′ and 4′ of § 5. If there are at least three elements, 3′ is a consequence of VI. Now, there are at least three elements, by I.

As to 4′ it is enough to show that, given any two elements A and B, there is a set E having A but not B as a limit-element. It follows from VII, I, and 3′, that there is at least one set F_1 which has limit-elements without having the whole of K for the class of its limit-elements.

Let A_1 be a limit-element of this set, and B_1 an element not a limit-element of the set. By VI, there is a transformation from Σ changing A_1 to A and B_1 to B. Let this transformation change F_1 to F. Then, as a result of III and IV, F will have A for a limit-element, but not B.

Proof of (2).

Let a neighbourhood U_A consist of all the interior elements of some set E of which A is an interior element. By I, 3′, and VII at least one element has a neighbourhood, and by the use of VI, III, and IV, as above, every element will have at least one neighbourhood. Indeed, it may be shown by I, 3′, and VII that there is at least one set with both interior and exterior elements, so that this same argument may be used to show that any two elements will have two mutually exclusive neighbourhoods, thus proving that Hausdorff's conditions (A) and (D) are satisfied. (C) is an obvious result of the definition of neighbourhood, for a neighbourhood is a neighbourhood of any of its elements. As to (B), the interior elements of a set E that are also interior to a set F are interior to the common part of E and F; this follows from condition 2 that our set be a set (R).

It remains to show that limit in a system (H) corresponds to limit in a system (Sp). It is a result of our definition of neighbourhood that if E is a set having A as a limit-element, every neighbourhood of A contains some element of E other than A. It results from Riesz's condition 2 that every neighbourhood of A contains a set of elements of E having A as a limit-element. From 2 and 3 together it follows that every such set is infinite. Hence every (Sp)-limit is an (H)-limit. The converse relation follows from VII.

Proof of (3).

Let Σ consist of all biunivocal, bicontinuous transformations in our system (Vr). That this will give the same notion of limit as that defined in a system (Sp) I have proved in my previous paper. Postulates I, II, III, IV, and V demand no discussion. VII will be obvious if we consider that a "sphere" with its boundary-elements will answer to our definition of a segment, for it is closed, has at least two boundary-elements, and is connected, for any point is connected with the centre by a radius. Moreover, the centre is an interior point. VII will then follow from our definition of limit.

There remains only condition VI. It is clear that any element A of K can be changed to any other member B of K by a transformation from Σ, for it will follow from II and the various properties of vectors that the transformation which turns C into the element D such that $CD = AB$ belongs to Σ. In a similar way, it may be shown that the transformation which consists in holding an element A fast and "multiplying" all the vectors AB by the same numerical factor also belongs to Σ. We shall establish our theorem, then, if we show that if AB and AC are two vectors such that $\| AB \| = \| AC \|$, there is a transformation belonging to Σ holding A fixed and changing B into C, for every transformation of a point-pair into another may be reduced, as in ordinary geometry, into a " translation," an " expansion," and a " rotation." Our special hypothesis for a (Vr) enables us, moreover, without essentially limiting our problem, to suppose $\| BC \| < \| AB \|$.

Let us consider the vector transformation which turns ξ into

$$\xi \oplus \left\{ \frac{\| \xi \|}{\| AB \|} \odot BC \right\}.$$

This transformation is clearly univocal; it is, moreover, biunivocal. To prove this, let us make use of the fact that it results from our assumptions that if $\xi \oplus \eta = \vartheta$, η is uniquely determined by ϑ and ξ, and may be written $\vartheta \ominus \xi$. Now suppose that

$$\xi \oplus \left\{ \frac{\| \xi \|}{\| AB \|} \odot BC \right\} = \eta \oplus \left\{ \frac{\| \eta \|}{\| AB \|} \odot BC \right\}.$$

It results that
$$\xi \ominus \eta = \frac{\| \xi \| - \| \eta \|}{\| AB \|} \odot BC,$$

or
$$\| \xi \ominus \eta \| = \{ \| \xi \| - \| \eta \| \} \frac{\| BC \|}{\| AC \|}.$$

Now, by our hypothesis, $\| BC \| / \| AC \| < 1$. Hence either

$$\| \xi \ominus \eta \| = 0, \quad \text{or} \quad \| \xi \ominus \eta \| < \| \xi \| - \| \eta \|.$$

If we write this latter proposition in the form

$$\| \xi \ominus \eta \| + \| \eta \| < \| (\xi \ominus \eta) \oplus \eta \|,$$

it will be seen to contradict (8) in the definition of a (Ve). Hence

$$\| \xi \ominus \eta \| = 0,$$

or what results from (13), $\xi = \eta$.

Let us consider the point-transformation which retains A invariant and changes every other element P into the element P' such that

$$AP' = AP \oplus \left\{ \frac{\| AP \|}{\| AB \|} \odot BC \right\}.$$

It results from what has been said and the properties of vectors that this is biunivocal; let us consider how it affects the magnitude of vectors. If P is transformed into P' and Q into Q' by our transformation, we wish to determine a relation between PQ and $P'Q'$.

Now, as an immediate consequence of the commutative law and the definition of our transformation,

$$P'Q' = PQ \oplus \left\{ \frac{\| AQ \| - \| AP \|}{\| AB \|} \odot BC \right\}.*$$

As a consequence,

$$\begin{aligned} \| P'Q' \| &\leqq \| PQ \| + \frac{\| BC \|}{\| AB \|} \big| \| AQ \| - \| AP \| \big| \\ &\leqq 2 \| PQ \|. \end{aligned}$$

On the other hand, it may readily be proved that

$$\begin{aligned} \| P'Q' \| &\geqq \left| \| PQ \| - \left| \frac{\| BC \|}{\| AB \|} \right| \| AQ \| - \| AP \| \right| \\ &\geqq \| PQ \| \left\{ 1 - \frac{\| BC \|}{\| AB \|} \right\}. \end{aligned}$$

It follows from these inequalities that, to put it roughly, $P'Q'$ is small when and only when PQ is small, and that a set of elements approaching indefinitely close to a given element is transformed into a set approaching indefinitely close to the transform of the given element, and *vice versa*. In other words, our transformation leaves limit-properties invariant in both directions, and so belongs to Σ. Moreover, our transformation leaves A invariant and changes B into the element D such that

$$AD = AB \oplus \left\{ \frac{\| AB \|}{\| AB \|} \odot BC \right\} = AB \oplus BC = AC,$$

or, in other words, into C. We thus have completed our proof of the equivalence of point-pairs by the consideration of "rotations."

EXAMPLES OF SETS (Vr).

7. (1) The system consists of all n-partite numbers $(x_1, x_2, \ldots, x_n)$. If $A = (x_1, x_2, \ldots, x_n)$ and $B = (y_1, y_2, \ldots, y_n)$, AB shall be the n-partite number $(x_1-y_1, y_2-y_2, \ldots, x_n-y_n)$, and every n-partite number shall be a vector. If

$$\xi = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_n),$$

$$\| \xi \| = \sqrt{(u_1^2+u_2^2+\ldots+u_n^2)}, \quad k \odot \xi = (ku_1, ku_2, \ldots, ku_n),$$

and

$$\xi \oplus \eta = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n).$$

* $(-n) \odot UV$ is to be understood as $n \odot VU$.

The independence of addition on order is immediately obvious. The other specifically (Vr) property results from the fact that any arc of a circle can be traversed with a finite number of chords each less in length than ϵ, for any given ϵ.

(2) The system of elements and that of vectors alike consist in all ∞-partite numbers $(x_1, x_2, \ldots, x_k, \ldots)$ such that there is a finite X such that for all k, $|x_k| \leqslant X$. If

$$A = (x_1, x_2, \ldots, x_k, \ldots) \quad \text{and} \quad B = (y_1, y_2, \ldots, y_k, \ldots),$$

$$AB = (x_1 - y_1, x_2 - y_2, \ldots, x_k - y_k, \ldots).$$

If $$\xi = (u_1, u_2, \ldots, u_k, \ldots) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_k, \ldots),$$

$$\| \xi \| = \text{least upper bound } |u_k|,$$

$$m \odot \xi = (mu_1, mu_2, \ldots, mu_k, \ldots),$$

and $$\xi \oplus \eta = (u_1 + v_1, u_2 + v_2, \ldots, u_k + v_k, \ldots).$$

The commutative law is obvious; the other condition for a (Vr) can be demonstrated if we show that given ξ and η such that $\|\xi\| = \|\eta\| \neq 0$, there is a chain of vectors, $\xi_1 = \xi, \xi_2, \ldots, \xi_n = \eta$, such that for all j,

$$\| \xi_j \| = \| \xi \| \quad \text{and} \quad \| \xi_{j+1} \ominus \xi_j \| < \| \xi \|.$$

Such a chain may be constructed as follows; let ζ be the vector $(z_1, z_2, \ldots, z_k, \ldots)$, such that for all k, z_k is the larger of the two quantities u_k and v_k if they differ, and their common value, if they agree. Then

$$\| \zeta \| = \| \xi \|.$$

Let $$\frac{\| \zeta \ominus \xi \|}{\| \xi \|} = p, \quad \text{and} \quad \frac{\| \zeta \ominus \eta \|}{\| \xi \|} = q.$$

Let r be any integer larger than both p and q. Then the sequence of vectors

$$\xi,\ \xi \oplus \left\{ \frac{1}{r} \odot (\zeta \ominus \xi) \right\},\ \ldots,\ \xi \oplus \left\{ \frac{h}{r} \odot (\zeta \oplus \xi) \right\},\ \ldots,$$

$$\zeta,\ \zeta \oplus \left\{ \frac{1}{r} \odot (\eta \ominus \zeta) \right\},\ \ldots,\ \zeta \oplus \left\{ \frac{h}{r} \odot (\eta \ominus \zeta) \right\},\ \ldots,\ \eta,$$

may readily be shown to satisfy the conditions for a chain $\{\xi_j\}$.

(3) The system of all points and the system of all vectors consist alike

in all ∞-partite numbers $(x_1, x_2, \ldots, x_k, \ldots)$ such that the series

$$x_1^2+x_2^2+\ldots+x_n^2+\ldots$$

converges. AB, $m \odot \xi$, and $\xi \oplus \eta$ are defined as in (2). If

$$\xi = (u_1, u_2, \ldots, u_k, \ldots),$$

$$\| \xi \| = \sqrt{(u_1^2+u_2^2+\ldots+u_k^2+\ldots)}.$$

To show that our system is a (Vr), let us introduce a few considerations from the trigonometry of infinitely many dimensions. If

$$\xi = (u_1, u_2, \ldots, u_k, \ldots) \quad \text{and} \quad \eta = (v_1, v_2, \ldots, v_k, \ldots),$$

let us define $< \xi\eta$ as

$$\cos^{-1} \frac{u_1 v_1 + u_2 v_2 + \ldots + u_n v_n + \ldots}{\| \xi \| \cdot \| \eta \|}.$$

The first question to arise is under what circumstances $< \xi\eta$ will exist. It may easily be shown that if Σu_n^2 and Σv_n^2 converge, $\Sigma(u_n+v_n)^2$ and $\Sigma(u_n-v_n)^2$ will converge.* It results that $\Sigma \frac{1}{4} \{(u_n+v_n)^2-(u_n-v_n)^2\}$ will converge, or that $\Sigma u_n v_n$ will converge. Furthermore, it is obvious that to multiply ξ or η by a positive constant will not affect the magnitude or existence of $< \xi\eta$. We may thus assume $\| \xi \| = \| \eta \|$, which gives us

$$< \xi\eta = \cos^{-1} \frac{u_1 v_1 + u_2 v_2 + \ldots + u_n v_n + \ldots}{u_1^2+u_2^2+\ldots+u_n^2}.$$

Now, consider the inequality $\Sigma(u_n-v_n)^2 \geqslant 0$. We may write this

$$\Sigma u_n^2 - 2\Sigma u_n v_n + \Sigma v_n^2 \geqslant 0.$$

Making use of the fact that $\Sigma u_n^2 = \Sigma v_n^2$, this becomes

$$2\Sigma u_n v_n \leqslant 2\Sigma u_n^2.$$

It may be proved in precisely the same manner that

$$-2\Sigma u_n v_n \leqslant 2\Sigma u_n^2.$$

Hence $< \xi\eta$ is the anticosine of a number not greater in absolute value than 1, and consequently exists.

As in ordinary geometry,

$$\| \xi \ominus \eta \|^2 = \| \xi \|^2 + \| \eta \|^2 - 2 \| \xi \| \cdot \| \eta \| \cos < \xi\eta.$$

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* Cf. Hausdorff, *loc. cit*, p. 287.

This may be proved by writing the formula out at length, when it will reduce to an identity. All the series involved will be absolutely convergent, so there is no difficulty about changing the order of terms.

Let us suppose, as above, that $\|\xi\| = \|\eta\|$, and let us consider $\cos < \xi\, \{\xi \oplus \eta\}$. This will be

$$\frac{u_1(u_1+v_1)+u_2(u_2+v_2)+\ldots+u_n(u_n+v_n)+\ldots}{\sqrt{(u_1^2+u_2^2+\ldots+u_n^2+\ldots)}\,\sqrt{\{(u_1+v_1)^2+(u_2+v_2)^2+\ldots+(u_n+v_n)^2+\ldots\}}},$$

By our previous remarks this is an essentially positive quantity. We shall moreover get the identity

$$\begin{aligned}
&\cos 2 < \xi\,\{\xi\oplus\eta\} \\
&\quad = 2\cos^2 < \xi\,\{\xi\oplus\eta\} - 1 \\
&\quad = \frac{2\,\{\Sigma(u_n^2+u_nv_n)\}^2-[\Sigma u_n^2][\Sigma(u_n+v_n)^2]}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&\quad = \frac{2\Sigma u_n^2\Sigma u_m^2+4\Sigma u_n^2\Sigma u_mv_m+2\Sigma u_nv_n\Sigma u_mv_m-\Sigma u_n^2\Sigma u_m^2 -2\Sigma u_n^2\Sigma u_mv_m-\Sigma u_n^2\Sigma v_m^2}{[\Sigma u_n^2][\Sigma(u_n+v_m)^2]} \\
&\quad = \frac{2\Sigma u_n^2\Sigma u_mv_m+2\Sigma u_nv_n\Sigma u_mv_m}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&\quad = \frac{[\Sigma u_mv_m]\,\{\Sigma u_n^2+2\Sigma u_nv_n+\Sigma v_n^2\}}{[\Sigma u_n^2][\Sigma(u_m+v_m)^2]} \\
&\quad = \frac{\Sigma u_mv_m}{\Sigma u_m^2} = \cos\xi\eta.
\end{aligned}$$

It results from this that $< \xi\,(\xi\oplus\eta)$ is the half of $< \xi\eta$ in the first or fourth quadrant.

Now, let ξ and η be any two vectors of equal magnitude, provided only that neither is made up entirely of 0's. Form the vector ξ_3, which shall be a positive multiple of $\xi\oplus\eta$ with the same magnitude as ξ. In a similar manner, interpolate ξ_2 between ξ and ξ_3, and ξ_4 between ξ_3 and η, and let us know ξ and η as ξ_1 and ξ_5, respectively. We have

$$\cos < \xi\xi_3 = \cos < \xi_3\eta = \sqrt{\{\tfrac{1}{2}(1+\cos < \xi\eta)\}} \geqslant 0.$$

Hence

$$\begin{aligned}
\cos < \xi_1\xi_2 = \cos < \xi_2\xi_3 = \cos < \xi_3\xi_4 &= \cos < \xi_4\xi_5 \\
&= \sqrt{\{\tfrac{1}{2}(1+\cos < \xi\xi_3)\}} \geqslant \tfrac{1}{2}\sqrt{2}.
\end{aligned}$$

It follows from the law of cosines that

$$\begin{aligned}\| \xi_h-\xi_{h+1} \| &= \sqrt{(\| \xi_h \|^2+\| \xi_{h+1} \|^2-2 \| \xi_h \| . \| \xi_{h+1} \| \cos \sphericalangle \xi_h\xi_{h+1})} \\ &\leqslant \| \xi \| \sqrt{(2-\sqrt{2})} \\ &< \| \xi \|\end{aligned}$$

(4) The system of all elements and the system of all vectors both consist of all continuous functions of a real variable defined over a given closed interval. The vector fg is the function $f(x)-g(x)$. If $\xi=f(x)$ and $\eta=g(x)$, $\| \xi \| = \max |f(x)|$, $k \odot \xi = kf(x)$, and $\xi \oplus \eta = f(x)+g(x)$. The proof that this system is a (Vr) proceeds as in (2).

It may be noted that systems (1), (3), and (4) satisfy VIII.*

CONSISTENCY OF POSTULATES I–IX.

8. The following system satisfies Postulates I–IX: K consists of all the points on a line, and Σ consists of all bicontinuous, biunivocal transformations of the whole line into itself.

DEDUCTIONS FROM POSTULATES I–IX.

9. THEOREM I.—*If A and B are any two distinct members of K, there is a unique closed set (A, B), completely characterized by the facts that it is connected and that A and B are boundary elements of it.*

Proof.

It follows from Postulates I, VI, and VII that there is at least one set with at least two boundary elements. By VI, these can be transformed by a transformation from Σ into A and B, and by III and IV, this transformation will leave every connected set connected. By IX, this set is uniquely determined except as to whether it contains A and B. Adjoin to it its limit-elements, and it will clearly remain connected, while it will contain A and B.

THEOREM II.—*A and B are the only boundary-elements of (A, B).*

Proof.

Let D be any element not in (A, B). Consider the component† E of

* Hausdorff, *loc. cit.*, pp. 288, 289.

† Since we have proved that our system satisfies Hausdorff's axioms, we may take advantage of his proof of the existence of components.

(A, B) to which D belongs. This must have a limit-element P in (A, B), for otherwise the segment (D, A), which exists, by Theorem I, would not be connected. P is then a boundary-element of (A, B) which is the limit of the connected set E in (A, B).

Now, let C be any boundary-element of (A, B) other than A and B. If C were the limit of a connected set F in (A, B), then F would either have A for a limit-element, or B for a limit-element, or neither A nor B. In the first two cases it results from IX that F must coincide with (A, B), which is impossible. In the third case, it follows from V that A and B are boundary-elements of the connected set $(A, B)+F$, which hence must coincide with (A, B), by IX. This is again impossible. It follows that there is no such set as F.

Let Q be any boundary-element of (A, B) other than C and P. By IX, we may write (A, B) as (Q, C) or as (Q, P). Now, by VI, there is a transformation from Σ leaving Q invariant and changing P into C. By III and IV, this changes (Q, P) into (Q, C), and changes every connected set in (Q, P) having P as a limit-element into a connected set in (Q, C) having C as a limit-element. As the existence of sets of the latter sort has been disproved, while the existence of sets of the former sort has been proved, it follows that either our assumption of the existence of C or our assumption of the existence of P is inadmissible. If either assumption is incorrect, (A, B) has only two boundary-elements, which must be A and B.

THEOREM III.—*If (A, B) and (A, C) have an element in common other than A, either (A, B) contains (A, C) or vice versa.*

Proof.

Let E consist of all elements in (A, B) but not in (A, C), and let F be the component of E containing B. As (A, C) is connected, F has some limit-element D in (A, C). If A is the only limit-element of F in (A, C), $A+F$ is a connected set containing the boundary-elements A and B, and hence coincides with (A, B), which hence, contrary to assumptions, contains no other term than A in common with (A, C). By Theorem II, the only other value which D can have is C. Now, consider the set $F+(A, C)$. It is connected, and, by V, has A as a boundary element. By V, either B is a boundary-element or B belongs to (A, C). If B belongs to (A, C), then every element of (A, B) does likewise, for otherwise, as (A, C) has only two boundary-elements, E can have only A and C as limit-elements in (A, C). If B differs from C, this is clearly impossible, while if B coincides with C, $(A, B) = (A, C)$.

The only other possibility is that E contains no elements. In this case, (A, C) is contained in (A, B).

THEOREM IV.—*If C is interior to (A, B), $(A, B) = (A, C)+(C, B)$, and (A, C) shares with (C, B) no other element than C.*

Proof.

By Theorem III, (A, B) contains (A, C) and (C, B). If B belonged to (A, C), by Theorem III, (A, C) would contain, and hence coincide with (A, B). This contradicts our assumption. Hence, by Postulate V, B is a boundary-element of $(A, C)+(C, B)$. The same argument applies to A. Moreover, being the sum of two overlapping, closed, connected sets, by V, $(A, C)+(C, B)$ is closed and connected. Hence, by Theorem II,

$$(A, C)+(C, B) = (A, B).$$

If (A, C) and (C, B) had in common any other element than C, then, by Theorem III, either (A, C) would contain (C, B), or *vice versa*. In this case, either (A, C) or (C, B) would contain (A, B). Hence, by Theorem II, C would coincide with either A or B, and would not be an interior element of (A, B).

Definition.—If C is interior to (A, B), we shall write ACB. It is obvious that if ABC, A, B, and C are all different, and it is also obvious that ABC and CBA are equivalent. Furthermore, by Theorem III, ABC and ACB are incompatible.

THEOREM V.—*If ABC and ACD, then BCD.*

Proof.—By Theorem IV, ABD. Hence, by Theorem IV, either ACB or BCD. ACB, however is incompatible with ABC, by Theorem III.

THEOREM VI.—*ABC, ABD, and CBD are incompatible.*

Proof.—By Theorem IV, either ACB, $B = C$, or BCD. As Theorem III excludes the first two suppositions, which are incompatible with ABC, there remains only the last possibility, which, by III, is incompatible with BCD.

THEOREM VI.—*Either ABC, BAC, or ACB, if A, B and C are distinct.*

Proof.—Suppose the first two alternatives are not fulfilled. Then, by Theorem III, (A, C) and (B, C) have only C in common, $(A, C)+(B, C)$ is connected, and by Postulate V, has A and B as boundary-elements. Hence $(A, C)+(C, B) = (A, B)$, or, in other words, ACB.

THEOREM VII.—*If ABC and BCD, then ACD.*

Proof.—By Theorem VI, we have DAC, ADC, or ACD. If DAC and ABC, then by Theorem IV, DBC, which, by Theorem III, contradicts BCD. If ADC, then, by Theorem IV, ABD or DBC. DBC, by Theorem III, contradicts BCD. If ABD and BCD, then, by Theorem IV, ACD.

Definition.—$AB \mid CD$ shall mean any one of the following sets of relations :

(1) ACD, ABD.

(2) ACD, $B = D$.

(3) ACD, ADB.

(4) $A = C$, ABD.

(5) $A = C$, $B = D$, $A \neq B$.

(6) $A = C$, ADB.

(7) CAD, CAB.

(8) $A = D$, CAB.

(9) CDA, CAB.

THEOREM VIII.—*If $AB \mid CD$ and $BP \mid CD$, then $AP \mid CD$.*

Proof.—This involves merely the tabulation of the 81 possible cases and the application of Theorems III–VII in the instances in which they are appropriate.

THEOREM IX.—*If $AB \mid CD$, $A \neq B$ and $C \neq D$.*

Proof.—This follows from the fact that if ABC, $A \neq B \neq C$, and the definition of $AB \mid CD$.

THEOREM X.—*If $A \neq B$, $C \neq D$, then either $AB \mid CD$ or $BA \mid CD$.*

Proof.—This follows from Theorems VI, IV, V, and VII, as may be shown by tabulating the relations between A, B, C, and D, which are possible on the basis of Theorem VI.

THEOREM XI.—*If $AB \mid CD$ and APB, then $AP \mid CD$ and $PB \mid CD$.*

Proof.—As above, by tabulating the possible cases, and making use of Theorems IV–VII.

THEOREM XII.—*If $AP \mid CD$ and $PB \mid CD$, then APB.*

Proof.—As above, by tabulation.

THEOREM XIII.—*If M and N are two classes of elements exhausting K, and such that there are two fixed elements C and D such that if A belongs to M and B belongs to N, $AB|CD$, then there is an element P such that if Q belongs to M and R belongs to N and $Q \neq P \neq R$, QPR.*

Proof.

Suppose that X and Y belong to M, and that XZY Either $XY|CD$ or $YX|CD$, by Theorem X. Similarly, either $XZ|CD$ or $ZX|CD$, and either $YZ|CD$ or $ZY|CD$. Making use of Theorems XII and VI, it turns out that the only admissible combinations of hypotheses are $XZ|CD$, $ZY|CD$, $XY|CD$ and $YZ|CD$, $ZX|CD$, $YX|CD$. Since we have $XB|CD$ and $YB|CD$ for all B in N, we have, by Theorem VIII, $ZB|CD$ in both cases. It follows then from Theorems VIII and IX that Z does not belong to N, so that it must belong to M. In other words, if M contains X and Y, it contains every element in (X, Y), so that M is connected. Likewise, N is connected.

It follows from Postulate IX and Theorem I that there is just one element P which is a limit-element of M belonging to N or a limit-element of N belonging to M. Let Q belong to M and R to N. As (Q, R) is connected, it must contain P.

THEOREM XIV.—*There is a denumerable set K' of elements such that if A and B are any two elements, there is an element C from K' such that ACB.*

Proof.

Let K' be the set to which reference is made in Postulate VIII. Then every element is a limit-element of K'. It follows from the fact that a single element has no limit-element and Postulate V that a segment has interior elements. Hence every segment contains at least one element of K'.

THEOREM XV.—*There is no element A such that for all $B \neq A$, $AB|CD$, and there is no element A such that for all $B \neq A$, $BA|CD$.*

Proof.—This follows directly from Postulate VI.

THEOREM XVI.—*K can be put into (1, 1)-correspondence with the set of all real numbers, in such a way that two elements C and D can be selected such that $AB|CD$ when and only when the correspondent of A is larger than the correspondent of B.*

Proof.—By Theorems VIII, IX, and X, order as defined by $AB \mid CD$ is serial. By Theorems XI, XII, and XIII, it is what Russell calls "Dedekindian." By XI, XII, and XIV, it contains a denumerable "median class." Hence, by a well known theorem,* it is ordinally similar to the series of reals.

THEOREM XVII.—*In the correspondence of Theorem XVI, Σ goes over into the set of all bicontinuous biunivocal transformations of the series of reals.*

Proof.—In the transformation of Theorem XVI, a segment goes over into a segment (Theorems XI, XII). Now, by Postulate VII, and Theorem I, the limit of a set E consists of all those elements A such that every segment (C, D) of which A is an element other than C and D contains a member of E. Hence limit goes over into limit, and in virtue of Postulates II, III, and IV, a transformation from Σ, which is precisely a transformation keeping limit-properties invariant, goes over into a bicontinuous, biunivocal transformation of the number-line, and every bicontinuous, biunivocal transformation of the number-line may be thus obtained.

Theorem XVII is equivalent to the statement that our set of postulates is a categorical set of postulates for the analysis-situs group of the line.

CONSIDERATIONS OF INDEPENDENCE.

10. Up to the present, the author has been unable to solve the question of the independence of Postulates IV, V, VII, and VIII. Each of the other postulates is independent of all the rest. The examples given below satisfy all the postulates except the one whose number they are given.

I. K consists of one element; Σ contains only the identity transformation.

II. K consists of all points on a line; Σ consists of all biunivocal, bicontinuous transformations that preserve direction.

III. K consists of all points on a line; Σ consists of all biunivocal, bicontinuous transformations, together with the transformations that displace all points with rational coordinates a rational distance in one direction, and all points with irrational coordinates a rational distance in the other.

* Whitehead and Russell, *Principia Mathematica*, Vol. 3, ✻275.

VI. K consists of all points on two mutually exclusive lines; Σ consists of all biunivocal, bicontinuous transformations of K.

IX. K consists of all points on a circle; Σ consists of all biunivocal, bicontinuous transformations of K.

It may be said that the independence of VIII would be proved if we could produce a closed homogeneous* series with a number of terms greater than $2^{\aleph_0}$. Homogeneous series with more than $2^{\aleph_0}$ terms are known, but they are not closed.

* Hausdorff, *loc. cit.*, p. 173.

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