337. The Bisectors of the Angles of a Triangle ABC Meet the Opposite Sides in X, Y, Z. The Circle XYZ Passes through the Point of Contact of the In- and Nine-Point Circles

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Plainly the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ lies on the Curve. The tangent thereat is without difficulty found to be

$$\Sigma x \left(\frac{1}{b} - \frac{1}{c} \right) = 0, \dots\dots\dots (2)$$

which passes through O . Now O lies on the Cubic by (1). Hence $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ is the point of contact of the fourth tangent from O to the Cubic.

W. P. MILNE.

336. [L¹. 4. 7.] *Tangents and foci of conics. Alternative proof of standard theorems.*

Let P, Q be two points on an ellipse of which the foci are S and H .

Since $SP + PH = SQ + QH$, a circle can be drawn to touch the quadrilateral formed by the focal distances. Let T be the centre of this circle.

Let SQ, HP cut at X , and let Z, U be the excentres opposite X of SPX, HQT .

The direction of the chord PQ is between those of ZT, TU . Therefore, in the limit when P, Q, T, X coincide, the tangent at P is ZU , the exterior bisector of SPH .

Again, in the figure PT, TQ are now seen to be the tangents at P and Q , and it follows that the tangents from T subtend equal angles at either focus.

Finally, because T is the excentre of SPX ,

$$\angle PTS = \frac{1}{2} \angle PXS = \frac{1}{2} \angle QXH = \angle QTH,$$

and the tangents from T are equally inclined to the focal distances of T .

F. J. W. WHIPPLE.

337. [K. 2. e.] *The bisectors of the angles of a triangle ABC meet the opposite sides in X, Y, Z .*

The circle XYZ passes through the point of contact of the in- and nine-point circles.

If $L=0, M=0, N=0$ are the equations to the sides of any triangle inscribed in the triangle of reference, then the equation of any conic circumscribing the triangle LMN may be written $paL + q\beta M + r\gamma N = 0$. For this equation is plainly satisfied by $\{M=0, N=0\}$, etc., since $a=0$ when $M=0$ and $N=0$.

The equation to the circle XYZ may therefore be written

$$p\alpha(\alpha - \beta - \gamma) + q\beta(\beta - \gamma - \alpha) + r\gamma(\gamma - \alpha - \beta) = 0.$$

Applying the conditions that this should represent a circle :

$$\begin{aligned} q \sin^2 C + r \sin^2 B + (q+r) \sin B \sin C \\ = r \sin^2 A + p \sin^2 C + (r+p) \sin C \sin A \\ = p \sin^2 B + q \sin^2 A + (p+q) \sin A \sin B. \end{aligned}$$

From which we deduce

$$p : q : r = a + 2(a+b+c) \cos A : b + 2(a+b+c) \cos B : c + 2(a+b+c) \cos C.$$

The equation to the circle XYZ is therefore

$$\begin{aligned} (a + 4s \cos A)\alpha(\alpha - \beta - \gamma) + (b + 4s \cos B)\beta(\beta - \gamma - \alpha) \\ + (c + 4s \cos C)\gamma(\gamma - \alpha - \beta) = 0. \end{aligned}$$

The above theorem now follows from the identity,

$$\begin{aligned} \sin^2 \frac{1}{2}(B-C) - \sin^2 \frac{1}{2}(C-A) - \sin^2 \frac{1}{2}(A-B) \\ = 2 \cos \frac{1}{2}(B-C) \sin \frac{1}{2}(C-A) \sin \frac{1}{2}(A-B). \end{aligned}$$

For the coordinates of the point of contact of the inscribed and nine-point circles are given by

$$\alpha' : \beta' : \gamma' = \sin^2 \frac{1}{2}(B-C) : \sin^2 \frac{1}{2}(C-A) : \sin^2 \frac{1}{2}(A-B).$$

$$\begin{aligned} \text{Hence } \alpha'(\alpha' - \beta' - \gamma') &= \sin(B-C) \sin \frac{1}{2}(B-C) \sin \frac{1}{2}(C-A) \sin \frac{1}{2}(A-B) \\ &= \kappa \sin(B-C), \text{ say.} \end{aligned}$$

$$\text{Similarly } \beta'(\beta' - \gamma' - \alpha') = \kappa \sin(C-A), \quad \gamma'(\gamma' - \alpha' - \beta') = \kappa \sin(A-B).$$

$$\begin{aligned} \text{Therefore } \Sigma(a + 4s \cos A) \alpha'(\alpha' - \beta' - \gamma') \\ = \kappa \Sigma a \sin(B-C) + 4\kappa s \Sigma \cos A \sin(B-C), \end{aligned}$$

and both terms in the latter expression vanish.

JOHN H. LAWLOR.

338. [K. 6. a.] *On the Centre of a Circle in Trilinears.*

The equation to the circle in trilinears may be written

$$\begin{aligned} \lambda_1^2(x_2^2 + x_3^2 + 2x_2x_3 \cos A_1) + \lambda_2^2(x_3^2 + x_1^2 + 2x_3x_1 \cos A_2) \\ + \lambda_3^2(x_1^2 + x_2^2 + 2x_1x_2 \cos A_3) + \lambda^2(a_1x_1 + a_2x_2 + a_3x_3)^2 = 0. \end{aligned}$$

- For
- (1) it is homogeneous of the second degree ;
 - (2) it contains three independent constant ratios ;
 - (3) it is satisfied by the circular points at infinity.

This form shows that the circle is the locus of a point whose pedal triangle has three sides p_1, p_2, p_3 obeying the relation

$$\lambda_1^2 p_1^2 + \lambda_2^2 p_2^2 + \lambda_3^2 p_3^2 + 4\lambda^2 \Delta^2 = 0,$$

i.e. a point whose distances from the vertices obey the relation

$$\delta_1^2 \sin^2 A_1 \lambda_1^2 + \delta_2^2 \sin^2 A_2 \lambda_2^2 + \delta_3^2 \sin^2 A_3 \lambda_3^2 + 4\Delta^2 \lambda^2 = 0.$$

Hence the centre of the circle is the mean centre of the vertices for multiples

$$\lambda_1^2 \sin^2 A_1 : \lambda_2^2 \sin^2 A_2 : \lambda_3^2 \sin^2 A_3.$$