A NEW THEOREM IN ANALYTIC CONICS.

By **Alan S. Hawkesworth** (Pittsburgh, Pa.)

Adunanza del 14 novembre 1915.

If PU be the normal of any point P on a central conic. And through P chords PQ , PR be drawn at right angles; cutting the conic again in Q , R respectively. And chord QR cuts the normal PU in U .

Then will the magnitude PU ever bear the same fixed ratio to the semi diameter conjugate to the diameter through P , that *twice the product* of the semi axis of the conic bears to the *sum* [if an Ellipse], or the *difference* [if a Hyperbola], of the *squares* of those semi-axi.

Let x_1 , y_1 be the coördinates of the point P on the conic $b^2 x^2 \pm a^2 y^2 - a^2 b^2 = 0$. [with the conic's axi for the rectangular axi. When the signs are ambiguous; the upper is that for the Ellipse; the lower for the Hyperbola].

Then the equation of the normal at P is

$$y - y_1 = \frac{\pm a^2 y_1}{b^2 x_1} (x - x_1)$$

or

$$y = \frac{\pm a^2 y_1}{b^2 x_1} x \mp \frac{a^2 y_1}{b^2 x_1} x_1 + \frac{b^2 x_1 y_1}{b^2 x_1}.$$

Now, by a known theorem, if perpendicular chords PQ , PR be drawn through any point P on any conic; cutting it again in Q , R . Then will chord QR cut the normal of P in a point U ; fixed for that normal. For convenience, then, let chords PQ , PR be drawn parallel to the axi. When chord QR , cutting normal PU in U , will be a diameter; symmetric to that through P . Whereupon the equation of diameter QR will be $y_1 x + x_1 y = 0$ or $y = \frac{-y_1}{x_1} x$.

Hence

$$\frac{-y_1}{x_1} x = \frac{\pm a^2 y_1}{b^2 x_1} x \mp \frac{a^2 y_1}{b^2 x_1} x_1 + \frac{b^2 x_1 y_1}{b^2 x_1} = \frac{y_1 (\pm a^2 x \mp a^2 x_1 + b^2 x_1)}{b^2 x_1}.$$

Dividing by $\frac{y_1}{x_1}$ we obtain

$$-x = \pm \frac{a^2}{b^2} (x - x_1) + x_1$$

or

$$(\pm a^2 + b^2)x = (\pm a^2 - b^2)x_1.$$

So that the coördinates x_2, y_2 of point U , wherein diameter QUR cuts the normal PU , are

$$x_2 = \left(\frac{\pm a^2 - b^2}{\pm a^2 + b^2} \right) x_1$$

and

$$y_2 = - \left(\frac{\pm a^2 - b^2}{\pm a^2 + b^2} \right) y_1.$$

Hence the magnitude PU must be

$$\begin{aligned} & \sqrt{\left(x_1 - \frac{\pm a^2 - b^2}{\pm a^2 + b^2} x_1 \right)^2 + \left(y_1 + \frac{\pm a^2 - b^2}{\pm a^2 + b^2} y_1 \right)^2} \\ &= \sqrt{\frac{(\pm a^2 x_1 + b^2 x_1 \mp a^2 x_1 + b^2 x_1)^2 + (\pm a^2 y_1 + b^2 y_1 \pm a^2 y_1 - b^2 y_1)^2}{(\pm a^2 + b^2)^2}} \\ &= \frac{2}{|\pm a^2 + b^2|} \sqrt{b^4 x_1^2 + a^4 y_1^2}. \end{aligned}$$

But the coördinates x_3, y_3 of the semidiameter conjugate to that through $P = x_1, y_1$ being $x_3 = \mp \frac{a}{b} y_1; y_3 = \frac{b}{a} x_1$. It follows that its magnitude is $\sqrt{\frac{a^4 y_1^2 + b^4 x_1^2}{a^2 b^2}}$ so that the fixed ratio of the magnitude PU of the normal at P is ever to that conjugate semidiameter as $\frac{2ab}{\pm a^2 + b^2}$ or $\frac{2ab}{a^2 \pm b^2}$; the ratio being an absolute value.

NOTE I. — In the Hyperbola, the triangle of points PQR can obviously all lie on the conjugate curve.

NOTE II. — When the asymptotal angle is less than right; then PU lies within the curve. When greater, then externally. While in the Equilateral Hyperbola $a^2 - b^2$ disappears; PU is thus infinite; and parallel to the conjugate semi diameter.

NOTE III. — In the Parabola chord PQ being drawn perpendicular, and diameter QU parallel to its axis. It is plain that magnitude PU is bisected by the axis; and $QU = p$, the Parabola's parameter. So that θ being the angle in which normal PU cuts the axis, $PU = p \sec \theta$.

Thus the unknown infinite diameters at infinity, conjugate to those through say P and P_1 respectively, must be to each other in the ratio of $\frac{\sec \theta}{\sec \theta_1}$ or inversely, as $\frac{\cos \theta_1}{\cos \theta}$.

Pittsburgh, 14th novembre 1915.

ALAN S. HAWKESWORTH.