This article was downloaded by: [University of Tasmania]
On: 29 April 2015, At: 09:57
Publisher: Taylor & Francis
Informa Ltd Registered in England and Wales Registered Number:
1072954 Registered office: Mortimer House, 37-41 Mortimer Street,
London W1T 3JH, UK

Philosophical Magazine Series 6

Publication details, including instructions
for authors and subscription information:
<http://www.tandfonline.com/loi/tphm17>

I. On the flow of compressible fluid past an obstacle

Lord Rayleigh O.M. F.R.S.
Published online: 08 Apr 2009.

To cite this article: Lord Rayleigh O.M. F.R.S. (1916) I. On the flow of
compressible fluid past an obstacle , Philosophical Magazine Series 6, 32:187,
1-6, DOI: [10.1080/14786441608635539](https://doi.org/10.1080/14786441608635539)

To link to this article: <http://dx.doi.org/10.1080/14786441608635539>

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of
all the information (the "Content") contained in the publications
on our platform. However, Taylor & Francis, our agents, and our
licensors make no representations or warranties whatsoever as to the
accuracy, completeness, or suitability for any purpose of the Content.
Any opinions and views expressed in this publication are the opinions
and views of the authors, and are not the views of or endorsed by
Taylor & Francis. The accuracy of the Content should not be relied
upon and should be independently verified with primary sources of
information. Taylor and Francis shall not be liable for any losses,
actions, claims, proceedings, demands, costs, expenses, damages,
and other liabilities whatsoever or howsoever caused arising directly

or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at <http://www.tandfonline.com/page/terms-and-conditions>

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JULY 1916.

I. *On the Flow of Compressible Fluid past an Obstacle.*

By Lord RAYLEIGH, O.M., F.R.S.*

IT is well known that according to classical Hydrodynamics a steady stream of frictionless incompressible fluid exercises no resultant force upon an obstacle, such as a rigid sphere, immersed in it. The development of a "resistance" is usually attributed to viscosity, or when there is a sharp edge to the negative pressure which may accompany it (Helmholtz). In either case it would seem that resistance involves something of the nature of a *wake*, extending behind the obstacle to an infinite distance. When the system of disturbed velocities, although it may mathematically extend to infinity, remains as it were attached to the obstacle, there can be no resistance.

The absence of resistance is asserted for an *incompressible* fluid; but it can hardly be supposed that a small degree of compressibility, as in water, would affect the conclusion. On the other hand, high relative velocities, exceeding that of sound in the fluid, must entirely alter the conditions. It seems worth while to examine this question more closely, especially as the first effects of compressibility are amenable to mathematical treatment.

The equation of continuity for a compressible fluid in

* Communicated by the Author.

Phil. Mag. S. 6. Vol. 32. No. 187. July 1916.

B

steady motion is in the usual notation

$$u \frac{d\rho}{dx} + v \frac{d\rho}{dy} + w \frac{d\rho}{dz} + \rho \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \quad (1)$$

or, if there be a velocity-potential ϕ ,

$$\frac{d\phi}{dx} \frac{d \log \rho}{dx} + \frac{d\phi}{dy} \frac{d \log \rho}{dy} + \frac{d\phi}{dz} \frac{d \log \rho}{dz} + \nabla^2 \phi = 0. \quad (2)$$

In most cases we may regard the pressure p as a given function of the density ρ , dependent upon the nature of the fluid. The simplest is that of Boyle's law where $p = a^2 \rho$, a being the velocity of sound. The general equation

$$\int \frac{dp}{\rho} = C - \frac{1}{2} q^2, \quad (3)$$

where q is the resultant velocity, so that

$$q^2 = (d\phi/dx)^2 + (d\phi/dy)^2 + (d\phi/dz)^2 \quad (4)$$

reduces in this case to

$$a^2 \log \rho = C - \frac{1}{2} q^2,$$

or

$$a^2 \log (\rho/\rho_0) = -\frac{1}{2} q^2, \quad (5)$$

if ρ_0 correspond to $q=0$. From (2) and (5) we get

$$\nabla^2 \phi = \frac{1}{2a^2} \left\{ \frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} + \frac{d\phi}{dz} \frac{dq^2}{dz} \right\}. \quad (6)$$

When q^2 is small in comparison with a^2 , this equation may be employed to estimate the effects of compressibility. Taking a known solution for an incompressible fluid, we calculate the value of the right-hand member and by integration obtain a second approximation to the solution in the actual case. The operation may be repeated, and if the integrations can be effected, we obtain a solution in series proceeding by descending powers of a^2 . It may be presumed that this series will be convergent so long as q^2 is less than a^2 .

There is no difficulty in the first steps for obstacles in the form of spheres or cylinders, and I will detail especially the treatment in the latter case. If U , parallel to $\theta=0$, denote the uniform velocity of the stream at a distance, the velocity-potential for the motion of incompressible fluid is known to be

$$\phi = U(r + c^2/r) \cos \theta, \quad (7)$$

the origin of polar coordinates (r, θ) being at the centre

of the cylinder. At the surface of the cylinder $r=c$, $d\phi/dr=0$, for all values of θ .

On the right hand of (6)

$$\frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} = \frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta}; \quad \dots \quad (8)$$

and from (7)

$$\frac{q^2}{U^2} = \frac{1}{U^2} \left\{ \left(\frac{d\phi}{dr} \right)^2 + \frac{1}{r^2} \left(\frac{d\phi}{d\theta} \right)^2 \right\} = 1 + \frac{c^4}{r^4} - \frac{2c^2}{r^2} \cos 2\theta. \quad \dots \quad (9)$$

Also

$$\frac{1}{U} \frac{d\phi}{dr} = \left(1 - \frac{c^2}{r^2} \right) \cos \theta, \quad \frac{1}{U} \frac{d\phi}{r d\theta} = - \left(1 + \frac{c^2}{r^2} \right) \sin \theta;$$

$$\frac{1}{U^2} \frac{dq^2}{dr} = - \frac{4c^4}{r^5} + \frac{4c^2}{r^3} \cos 2\theta, \quad \frac{1}{U^2} \frac{dq^2}{r d\theta} = \frac{4c^2}{r^3} \sin 2\theta.$$

Accordingly

$$\nabla^2 \phi = \frac{2U^3 c^2}{a^2 r^3} \left\{ - \frac{c^2}{r^2} \left(2 - \frac{c^2}{r^2} \right) \cos \theta + \cos 3\theta \right\}. \quad (10)$$

The terms on the right of (10) are all of the form $r^p \cos n\theta$, so that for the present purpose we have to solve

$$\nabla^2 \phi = \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \frac{1}{r^2} \frac{d^2 \phi}{d\theta^2} = r^p \cos n\theta. \quad \dots \quad (11)$$

If we assume that ϕ varies as $r^m \cos n\theta$, we see that $m=p+2$, and that the complete solution is

$$\phi = \cos n\theta \left\{ Ar^m + Br^{-n} + \frac{r^{p+2}}{(p+2)^2 - n^2} \right\}, \quad \dots \quad (12)$$

A and B being arbitrary constants. In (10) we have to deal with $n=1$ associated with $p=-5$ and -7 , and with $n=3$ associated with $p=-3$. The complete solution as regards terms in $\cos \theta$ and $\cos 3\theta$ is accordingly

$$\begin{aligned} \phi = & (Ar + Br^{-1}) \cos \theta + (Cr^3 + Dr^{-3}) \cos 3\theta \\ & + \frac{2U^3 c^2}{a^2} \left[\cos \theta \left(- \frac{c^2}{4r^3} + \frac{c^4}{24r^5} \right) - \frac{\cos 3\theta}{8r} \right]. \quad \dots \quad (13) \end{aligned}$$

The conditions to be satisfied at infinity require that, as in (1), $A=U$, and that $C=0$. We have also to make $d\phi/dr$ vanish when $r=c$. This leads to

$$B = c^2 U + \frac{13U^3 c^2}{12a^2}, \quad D = \frac{U^3 c^4}{12a^2}. \quad \dots \quad (14)$$

Thus

$$\phi = U \left\{ r + \frac{c^2}{r} + \frac{13U^2c^2}{12a^2r} \right\} \cos \theta + \frac{U^3c^4}{12a^2r^3} \cos 3\theta \\ + \frac{U^3c^2}{a^2} \left\{ \cos \theta \left(-\frac{c^2}{2r^3} + \frac{c^4}{12r^5} \right) - \frac{\cos 3\theta}{4r} \right\} \quad (15)$$

satisfies all the conditions and is the value of ϕ complete to the second approximation.

That the motion determined by (15) gives rise to no resultant force in the direction of the stream is easily verified. The pressure at any point is a function of q^2 , and on the surface of the cylinder $q^2 = c^{-2}(d\phi/d\theta)^2$. Now $(d\phi/d\theta)^2$ involves θ in the forms $\sin^2 \theta$, $\sin^2 3\theta$, $\sin \theta \sin 3\theta$, and none of these are changed by the substitution of $\pi - \theta$ for θ ; the pressures on the cylinder accordingly constitute a balancing system.

There is no particular difficulty in pursuing the approximation so as to include terms involving the square and higher powers of U^2/a^2 . The right-hand member of (6) will continue to include only terms in the cosines of odd multiples of θ with coefficients which are simple powers of r , so that the integration can be effected as in (11), (12). And the general conclusion that there is no resultant force upon the cylinder remains undisturbed.

The corresponding problem for the *sphere* is a little more complicated, but it may be treated upon the same lines with use of Legendre's functions $P_n(\cos \theta)$ in place of cosines of multiples of θ . In terms of the usual polar coordinates (r, θ, ω) , the last of which does not appear, the first approximation, as for an incompressible fluid, is

$$\phi = U \cos \theta \left(r + \frac{c^3}{2r^2} \right) = U \left(r + \frac{c^3}{2r^2} \right) P_1, \quad (16)$$

c denoting the radius of the sphere. As in (8),

$$\Sigma \frac{d\phi}{dx} \frac{dq^2}{dx} = \frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta} = U^3 \left\{ \left(-\frac{36c^6}{5r^7} + \frac{9c^9}{2r^{10}} \right) P_1 \right. \\ \left. + \left(\frac{6c^3}{r^4} - \frac{24c^6}{5r^7} + \frac{3c^9}{2r^{10}} \right) P_3 \right\}, \quad (17)$$

on substitution from (16) of the values of ϕ and q^2 . This gives us the right-hand member of (6).

In the present problem

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \right), \quad (18)$$

while P_n satisfies

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP_n}{d\theta} \right) + n(n+1)P_n = 0; \quad (19)$$

so that

$$\nabla^2 \phi = r^2 P_n \quad (20)$$

reduces to

$$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \frac{n(n+1)}{r^2} \phi = r^2. \quad (21)$$

The solution, corresponding to the various terms of (17), is thus

$$\phi = \frac{r^{p+2} P_n}{(p+2)(p+3) - n(n+1)}. \quad (22)$$

With use of (22), (6) gives

$$\begin{aligned} \phi = \frac{U^3}{a^2} \left\{ -\frac{c^6 P_1}{5r^5} + \frac{c^3 P_1}{24r^3} - \frac{3c^3 P_3}{10r^2} - \frac{3c^6 P_3}{10r^5} + \frac{3c^9 P_3}{176r^8} \right\} \\ + ArP_1 + Br^{-2}P_1 + Cr^3P_3 + Dr^{-4}P_3, \quad (23) \end{aligned}$$

A, B, C, D being arbitrary constants. The conditions at infinity require $A=U$, $C=0$. The conditions at the surface of the sphere give

$$B = \frac{c^3}{2} \left(U + \frac{2U^3}{3a^2} \right), \quad D = \frac{27c^5 U^3}{55a^2}; \quad (24)$$

and thus ϕ is completely determined to the second approximation.

The P 's which occur in (23) are of *odd* order, and are polynomials in $\mu (= \cos \theta)$ of *odd* degree. Thus $d\phi/dr$ is odd (in μ) and $d\phi/d\theta = \sin \theta \times$ even function of μ . Further,

$$\begin{aligned} q^2 &= \text{even function} + \sin^2 \theta \times \text{even function} = \text{even function}, \\ dq^2/dr &= \text{even function}, \\ dq^2/d\theta &= \sin \theta \times \text{odd function}. \end{aligned}$$

Accordingly

$$\frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta} = \text{odd function of } \mu,$$

and can be resolved into a series of P 's of odd order. Thus not only is there no resultant force discovered in the second

approximation, but this character is preserved however far we may continue the approximations. And since the coefficients of the various P 's are simple polynomials in μ , the integrations present no difficulty in principle.

Thus far we have limited ourselves to Boyle's law, but it may be of interest to make extension to the general adiabatic law, of which Boyle's is a particular case. We have now to suppose

$$p/p_0 = (\rho/\rho_0)^\gamma, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

making

$$\frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = a^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \quad . \quad . \quad . \quad (26)$$

if a denote the velocity of sound corresponding to ρ_0 . Then by (3)

$$\frac{a^2}{\gamma-1} \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = C - \frac{1}{2}q^2. \quad . \quad . \quad . \quad (27)$$

If we suppose that ρ_0 corresponds to $q=0$, $C=a^2/(\gamma-1)$, and

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = 1 - \frac{(\gamma-1)q^2}{2a^2}, \quad . \quad . \quad . \quad (28)$$

whence

$$\frac{d \log \rho}{dx} = - \frac{dq^2/dx}{2a^2 - (\gamma-1)q^2}. \quad . \quad . \quad . \quad (29)$$

The use of this in (2) now gives

$$\nabla^2 \phi = \frac{1}{2a^2 - (\gamma-1)q^2} \left\{ \frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} + \frac{d\phi}{dz} \frac{dq^2}{dz} \right\}, \quad (30)$$

from which we can fall back upon (6) by supposing $\gamma=1$. So far as the first and second approximations, the substitution of (30) for (6) makes no difference at all.

As regards the general question it would appear that so long as the series are convergent there can be no resistance and no wake as the result of compressibility. But when the velocity U of the stream exceeds that of sound, the system of velocities in front of the obstacle expressed by our equations cannot be maintained, as they would be at once swept away down stream. It may be presumed that the passage from the one state of affairs to the other synchronizes with a failure of convergency. For a discussion of what happens when the velocity of sound is exceeded, reference may be made to a former paper*.

* Proc. Roy. Soc. A. vol. lxxxiv. p. 247 (1910); 'Scientific Papers,' vol. v. p. 608.