LXXXI. Pendulum motion and spherical trigonometry

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It will be seen that fused silica has qualities which commend it for use as a material for standards of length. A silica standard metre is on the point of completion at the National Physical Laboratory, and there is good reason to believe that its adoption will be attended with success.

The National Physical Laboratory,
Teddington.

LXXXI. *Pendulum Motion and Spherical Trigonometry.*
By G. GREENHILL *.

MR. ROSE-INNES has developed the relation between the revolution of a pendulum in a plane and the projection of the motion on a spherical surface, and he shows that the argument of the elliptic function required can be represented by an area on the sphere which grows uniformly with the time (Phil. Mag. June 1910).

In a change to the polar reciprocal, the time will then be represented by a spherical arc, as discussed here in § 9.

1. Consider a circle AQD on the vertical diameter AD, and a particle Q circulating round it under gravity with velocity due to the depth KQ below a horizontal line HK; the motion of Q will represent a pendulum making complete revolutions, like a bicycle-wheel on its ball-bearings, put out of balance by an iron bar in the spokes (fig. 1).

The lettering and notation is that employed in my 'Elliptic Functions,' fig. 13, where, with ADQ = ϕ ,

$$(1) \quad KQ = AE - AN = AE - AD \sin^2 \phi \\ = AE(1 - \kappa^2 \sin^2 \phi) = AE \cdot \Delta^2 \phi, \quad \kappa^2 = \frac{AD}{AE},$$

$$(2) \quad (\text{vel. of } Q)^2 = \left(AD \frac{d\phi}{dt} \right)^2 = 2g \cdot KP \\ = 2g \cdot AE \cdot \Delta^2 \phi, \quad \left(\frac{d\phi}{dt} \right)^2 = \frac{g}{AC} \cdot \frac{\Delta^2 \phi}{\kappa^2},$$

$$(3) \quad \int_0 \frac{d\phi}{\Delta \phi} = \frac{nt}{\kappa} = u, \quad \phi = \text{am } u, \quad n = \sqrt{\frac{g}{AC}};$$

so that n/π is the number of beats per second in small oscillation; and the elliptic argument u grows uniformly with the time t , starting from the lowest point A.

Draw the circle, centre E and radius EB, orthogonal to

* Communicated by the Author.

Fig. 1.

the circle AQD, cutting AD in L, the limiting point or Landen point ; then

- (4) $ED \cdot EA = EL^2$, and $HQ \cdot HQ' = HL^2$,
 (5) $QL^2 = (HQ - HL)^2 = HQ^2 - 2HQ \cdot HL + HQ \cdot HQ'$
 $= (HQ + HQ' - 2HL)HQ = (2HM - 2HL)HQ$
 $= 2LM \cdot HQ = 2LC \cdot KQ$
 $= 2LC \cdot AE \cdot \Delta^2\phi = AL^2 \cdot \Delta^2\phi$.

Fig. 2.

2. Turn the vertical circle about the diameter AD through a right angle, so as to bring DB to DO in fig. 2, and project

in Mr. Rose-Innes's manner on a sphere, by lines drawn from the centre O ; corresponding points on the sphere in fig. 4 and the plane AQD of fig. 1 may be represented without confusion by the same letter, in most cases. Then

$$(1) \quad \kappa^2 = \frac{AD}{AE} = \frac{AD^2}{AO^2} = \sin^2 AOD;$$

so that AOD is the modular angle, denoted by c ; and $EO = EL$, and OL bisects the angle AOD .

Invert with respect to O , making $OQ \cdot OQ' = OD^2$; the

Fig. 3.

vertical plane AQD inverts into a sphere on the diameter OD , and the circle AQD into another circle $A'Q'D$ in a plane perpendicular to OA , so that these circles are circular sections of the cone, vertex O and base AQD .

Now AQ is perpendicular to the plane ODQ , so that the planes OAQ , ODQ are at right angles, and the angle AQD on the sphere in fig. 4 is a right angle.

If DX is the perpendicular on the tangent at Q , $QDX = QDA = \phi$ in figs. 1 and 4; so also in fig. 4, if AY is the perpendicular from A on the tangent at Q , $QAY = QAD = \phi'$, but ϕ' is the angle $Q'A'D$ in fig. 2, or ADQ' in fig. 1.

3. In fig. 4, by Spherical Trigonometry,

$$(1) \quad \sin AQ = \sin AD \sin ADQ = \kappa \sin \phi, \quad \cos AQ = \Delta \phi,$$

$$(2) \quad \sin DQ = \kappa \sin \phi', \quad \cos DQ = \Delta \phi',$$

$$(3) \quad \cos AQ \cos DQ = \cos AD = \cos c, \quad \Delta \phi \Delta \phi' = \kappa',$$

$$(4) \quad \phi' = \text{am}(K - u), \quad \text{with } \phi = \text{am } u.$$

$$(5) \quad du = \frac{d\phi}{\Delta\phi} = \frac{\Delta\phi' d\phi}{\kappa'} = \frac{\cos DQ}{\cos c} d\phi = \frac{\text{spherical area } QM'm'q}{\cos c},$$

$$(6) \quad -du = \frac{d\phi'}{\Delta\phi'} = \frac{\Delta\phi d\phi'}{\kappa'} = \frac{\cos \text{AQ}}{\cos c} d\phi' \\ = \frac{\text{spherical area QMmq}}{\cos c},$$

$$(7) \quad u = \frac{\text{area AE'M'Q}}{\cos c}, \quad K - u = \frac{\text{area DEMQ}}{\cos c},$$

$$(8) \quad K = \frac{\text{area AE'JDQA}}{\cos c}.$$

$$(9) \quad \cos AQ \cos DQ = \sin QM \sin QM' = \cos e,$$

(Salmon, 'Solid Geometry,' Chap. X.), and the tangent UQU' intercepted by the cyclic arcs is bisected at Q , and

cuts off a constant area $UJU' = \pi - 2c$; so that the angles JUU' , $JU'U$ are equal to the angle UJU' , and then

$$JQ = QU = QU', JM = MU, JM' = M'U'.$$

Since AU is a quadrant and AYU a right angle, YU is a quadrant, and so also is XU' ; and $XU' = YU$, $QX = QY$.

If DX cuts AM in W , the spherical triangles XQW , YQA are equal, and

$$(10) \quad DWQ = QAY = DAQ = \phi', \quad DW = DA,$$

$$(11) \quad DW = DX + XW = DX + AY = DA = c.$$

4. So much for the geometry of the sphero-conic AQD , as developed in Salmon's 'Solid Geometry' and by Mr. Rose-Innes; returning to the vertical circle AQD , draw another interior circle aqd , centre c , with the same limiting point L and radical axis HEK , cutting QLQ' in q, q' ; then (fig. 3)

$$(1) \quad Hq \cdot Hq' = HL^2 = HQ \cdot HQ',$$

$$\begin{aligned} (2) \quad Qq \cdot Qq' &= (HQ - Hq)(HQ - Hq') \\ &= HQ^2 - (Hq + Hq')HQ + HQ \cdot HQ' \\ &= (HQ + HQ' - Hq - Hq')HQ \\ &= 2Mm \cdot HQ = 2Cc \cdot KQ. \end{aligned}$$

If QT is the tangent to this inner circle, cutting the outer circle again in Q_1 ,

$$(3) \quad QT^2 = 2Cc \cdot KQ, \quad QL^2 = 2CL \cdot KQ,$$

$$(4) \quad \frac{QT^2}{QL^2} = \frac{Cc}{CL} = \frac{Q_1T^2}{Q_1L^2},$$

and LT bisects the angle QLQ_1 ; also $IT = IL$, if QQ_1 cuts HE in I , giving a simple construction of the inner circle for a given QQ_1 .

As the tangent QTQ_1 moves round, cutting the outer circle at equal angles,

$$(5) \quad \frac{\text{vel. of } Q_1}{\text{vel. of } Q} = \frac{Q_1T}{QT} = \sqrt{\frac{K_1Q_1}{KQ}},$$

and this is the ratio of the velocity under gravity of two particles, Q and Q_1 , describing the circle in the same manner, so that Q and Q_1 will remain simultaneous positions of the particles if QQ_1 is a tangent of the inner circle; and putting $ADQ_1 = \psi = \text{am } u_1$, then

$$(6) \quad u_1 - u = w, \text{ a constant.}$$

5. Draw gcf in fig. 2 through c perpendicular to the plane AQD , cutting OA in f , and OL in g ; the circle, centre f and

radius fO will pass through a and d , since

$$(1) \quad Ea \cdot Ed = EL^2 = EO^2,$$

and

$$(2) \quad aOL = EOL - EOa = ELO - EdO = dOL,$$

so that OL bisects the angle aOd , and passes through g on the circle round Oad .

Then $gL a$, gaO are similar triangles, and $gL \cdot gO = ga^2$, so that L and O are inverse points with respect to the sphere, centre g and radius ga ; thence $LT : TO$ is a constant ratio, equal to $La : aO$, and similarly $LQ : QO$ is the constant ratio $LA : AO$ round the circle AQD , and $OQ = OA \cdot \Delta\phi$.

The inverse of the circle aTd with respect to O is another circle $a'T'd'$ parallel to $A'Q'D$; for $L'T' : OL' = LT : OT$, a constant ratio, so that $L'T'$ is constant, L' being the point inverse to L .

Conversely the inverse of a system of parallels of latitude on a sphere with respect to a point O on the sphere is a system of dipolar circles in a plane, as the circles of latitude on the stereographic representation of a hemisphere.

6. The line $O'T'$ from O' , the centre of the sphere on the diameter OD , makes a constant angle, c' , with $O'L'$, and the angle $DO'T'$ is double the angle DOT' ; so that if the arc DT in fig. 5 in the representation on a sphere, centre O , is produced to double length to V , OV will make a constant angle c' with OA , which is parallel to $O'L'$, and the arc $AV = c'$.

Fig. 5.

Then in fig. 5, by Spherical Trigonometry,

$$(1) \quad \cos AT \cdot \cos DT = \frac{1}{2}(\cos AD + \cos AV) = \frac{1}{2}(\cos c + \cos c'),$$

a constant; so that T describes another sphero-conic, interior

to that described by Q , and with the same cyclic arcs, so that its tangent QQ_1 cuts off from the cyclic arcs a triangle UU' of constant area, and UU' is bisected at T (Salmon, 'Solid Geometry,' §§247, 248).

But since, in figs. 2, 3,

$$(2) \quad \frac{QT}{TQ_1} = \frac{QL}{LQ_1} = \frac{QO}{OQ_1},$$

OT bisects the angle QQ_1O , and T in fig. 5 is the midpoint of QQ_1 , so that (Salmon, §252) QQ_1 cuts off a constant area from the outer sphero-conic.

With constant $u_1 - u = w$, the area QMM_1Q_1 is constant, so that the spherical quadrilateral QMM_1Q_1 is constant, and this implies that the sum of the angles DQ_1Q_1 , DQ_1Q or DQV is constant, and this is found in §8 to be $\text{am}(K - w) + \frac{1}{2}\pi$.

As before in fig. 4, XU' and YU are quadrants, $QY = Q_1X = p$ suppose, $Q_1Y = QX = q$ suppose ;

$$DX = YV, DX + AY = AV = DW = c'.$$

7. The angle $ADX = \phi + \psi$ in fig. 5, as in fig. 3 ; $QDX = \psi$, $Q_1DX = \phi$.

Similarly $DAY = \phi' + \psi'$, $Q_1AY = \phi'$, $QAY = \psi'$.

Then in the spherical triangle AQV in fig. 5,

$$(1) \quad \sin QV = \sin DQ_1 = \kappa \sin \psi', \quad \sin AQ = \kappa \sin \phi,$$

$$(2) \quad \frac{\sin QV}{\sin QAV} = \kappa = \frac{\sin AQ}{\sin QVA} = \frac{\kappa \sin \phi}{\sin QVA} = \frac{\sin AV}{\sin AQV},$$

so that $QVA = \phi$; this is seen also from the equality of the triangles QYV , Q_1DX , in which $QY = Q_1X$, $YV = DX$, so that $QVY = Q_1DX = \phi$; this is the equivalent of Mr. Rose-Innes's theorem (III.).

Also

$$(3) \quad \sin AQV = \frac{\sin c'}{\kappa}, \quad \sin AQV = \sin MQV = -\cos DQV,$$

since DQM is a right angle ; and so we put, as in (8) §8,

$$(4) \quad AQV = \text{am}(K + w), \quad MQV = \text{am}(K - w),$$

$$\sin c' = \kappa \text{sn}(K + w) = \kappa \text{sn}(K - w),$$

$$\cos c' = \text{dn}(K + w) = \text{dn}(K - w).$$

8. The addition formula of the elliptic function follows at once from a Legendre spherical triangle, in the manner employed by Mr. Kummell ('Analyst,' 1878); for in the spherical triangle AQV,

$$(1) \quad \sin p = \sin QY = \sin AQ \sin QAY = \kappa \sin \phi \sin \psi', \\ \cos p = \sqrt{(1 - \kappa^2 \sin^2 \phi \sin^2 \psi')},$$

$$(2) \quad \cos AY = \frac{\cos AQ}{\cos QY} = \frac{\Delta \phi}{\cos p}, \\ \sin AY = \frac{\sin AQ \cos QAY}{\cos QY} = \frac{\kappa \sin \phi \cos \psi'}{\cos p},$$

$$(3) \quad \cos YV = \frac{\cos QV}{\cos QY} = \frac{\Delta \psi'}{\cos p}, \\ \sin YV = \frac{\sin QV \cos QVY}{\cos QY} = \frac{\kappa \sin \psi' \cos \phi}{\cos p},$$

$$(4) \quad \cos c' = \cos AV = \cos (AY + YV) \\ = \frac{\Delta \phi \Delta \psi' - \kappa^2 \sin \phi \cos \phi \sin \psi' \cos \psi'}{1 - \kappa^2 \sin^2 \phi \sin^2 \psi'} \\ = \operatorname{dn}(u + K - u_1) = \operatorname{dn}(K - w) = \operatorname{dn}(K + w).$$

In a similar manner, with Spherical Trigonometry,

$$(5) \quad \cos AQY = \frac{\sin QAY \cos AQ}{\cos QY} = \frac{\sin \psi' \Delta \phi}{\cos p}, \\ \sin AQY = \frac{\cos QAY}{\cos QY} = \frac{\cos \psi'}{\cos p},$$

$$(6) \quad \cos VQY = \frac{\sin QVY \cos QV}{\cos QY} = \frac{\sin \phi \Delta \psi'}{\cos p}, \\ \sin VQY = \frac{\cos QVY}{\cos QY} = \frac{\cos \phi}{\cos \psi'},$$

$$(7) \quad \cos AQV = \cos (AQY + VQY) \\ = \frac{\sin \phi \Delta \phi \sin \psi' \Delta \psi' - \cos \phi \cos \psi'}{1 - \kappa^2 \sin^2 \phi \sin^2 \psi'} \\ = -\operatorname{cn}(u + K - u_1) = -\operatorname{cn}(K - w) = \operatorname{cn}(K + w),$$

$$(8) \quad AQV = \operatorname{am}(K + w), \quad DQV = \frac{1}{2}\pi + \operatorname{am}(K - w).$$

9. In the reciprocal diagram of fig. 4, drawn on the left hand, R is the pole of XY, and the perimeter of the triangle

ARD is π ; the tangent intercept ZZ' by the cyclic arcs is $\frac{1}{2}\pi$, so that QZZ' is a spherical triangular octant; RZ=ZK, RD=DK, RA=AK'; and if R, Z moves to r, z , as Q advances to q , and Dz crosses ZR in z' ,

$$(1) \quad \cos c \, du = \cos DQd\phi = \sin DZd\phi = Zz' \\ = \text{arc } Rr - rz + RZ,$$

$$(2) \quad u \cos c = \text{arc } ER - RZ,$$

thus representing the time by the difference of the arc ER and RZ.

In the reciprocal part of the diagram in fig. 5, where the tangents at R, R₁ intersect in S,

$$(3) \quad AS + DS = \pi - c',$$

$$(4) \quad (u_1 - u) \cos c = \text{arc } RR_1 - R_1Z_1 + RZ;$$

$$(5) \quad \text{arc } RR_1 = SR + SR_1 - \text{constant} \quad (\text{Salmon, } \S 252),$$

$$(6) \quad (u_1 - u) \cos c + \text{a constant} = SZ - SZ_1 = SZ + SZ_1' - \frac{1}{2}\pi.$$

The Spherical Trigonometry interpretation is the same as before in § 8; since

$$(7) \quad q = XQ = XSQ = \frac{1}{2}\pi - DSZ = \frac{1}{2}\pi - ASZ_1', \\ \sin q = \sin DQ \sin QDX = \kappa \sin \phi' \sin \psi,$$

$$(8) \quad \cos SZ = \frac{\cos ZDS}{\sin DSZ} = \frac{\cos \psi}{\cos q}, \\ \sin SZ = \frac{\sin ZDS \sin DZ}{\sin DSZ} = \frac{\sin \psi \Delta \phi'}{\cos q},$$

$$(9) \quad \cos SZ_1' = \frac{\cos Z_1'AS}{\sin ASZ_1'} = \frac{\cos \phi'}{\cos q}, \\ \sin SZ_1' = \frac{\sin Z_1'AS \sin AZ_1'}{\sin ASZ_1'} = \frac{\sin \phi' \Delta \psi}{\cos q},$$

$$(10) \quad \sin (SZ + SZ_1') = \frac{\sin \psi \cos \phi' \Delta \phi' + \sin \phi' \cos \psi \Delta \psi}{\cos^2 q} \\ = \text{sn } (K - u + u_1) = \text{sn } (K + w), \\ SZ + SZ_1' = \text{am } (K + w).$$

And with

$$(11) \quad p = XQ_1 = XSQ_1 = \frac{1}{2}\pi - DSZ_1 = \frac{1}{2}\pi - ASZ',$$

$$(12) \quad \sin SZ = \cos SZ' = \frac{\cos SAZ'}{\sin ASZ'} = \frac{\cos \psi'}{\cos p},$$

$$\cos SZ = \frac{\sin SAZ' \sin AZ'}{\sin ASZ'} = \frac{\sin \psi' \Delta \phi}{\cos p},$$

$$(13) \quad \cos SZ_1 = \frac{\cos SDZ_1}{\sin DSZ_1} = \frac{\cos \phi}{\cos p},$$

$$\sin SZ_1 = \frac{\sin SDZ_1 \sin DZ_1}{\sin DSZ_1} = \frac{\sin \phi \Delta \psi'}{\cos p},$$

$$(14) \quad \sin (SZ - SZ_1) = \frac{\cos \phi \cos \psi' - \sin \phi \sin \psi' \Delta \phi \Delta \psi'}{\cos^2 p} \\ = \operatorname{cn} (u + K - u_1) = \operatorname{cn} (K - w),$$

$$(15) \quad SZ - SZ_1 = \frac{1}{2}\pi - \operatorname{am} (K - w),$$

$$Z_1 Z_1' = \frac{1}{2}\pi = \operatorname{am} (K + w) + \operatorname{am} (K - w) - \frac{1}{2}\pi,$$

$$(16) \quad SZ + SZ_1' = \operatorname{am} (K + w).$$

Thus the constant in (6) is $\operatorname{am} (K + w) - \frac{1}{2}\pi - \kappa' w$.

The sphero-conic ERE' of fig. 4 is the projection on the sphere of the polar reciprocal of the circle AQD of fig. 1 with respect to D , and this is the parabola ER ; while the sphero-conic S in fig. 5 will be the projection of the hyperbola of S in fig. 3, polar reciprocal of the circle aTd .

10. The motion of P at the same level as Q in fig. 1, oscillating on the arc BAB' of the circle on the diameter AE , will represent the associated motion of a pendulum, swinging through an angle $4c$, and then if θ denotes the inclination of the pendulum OP to the vertical

$$(1) \quad \sin^2 \frac{1}{2}\theta = \frac{AN}{AE} = \frac{AD}{AE} \cdot \frac{AN}{AD} = \kappa^2 \sin^2 \phi,$$

$$\cos \frac{1}{2}\theta = \Delta \phi = \cos DQ, \quad \frac{1}{2}\theta = DQ, \quad \text{in fig. 4.}$$

Draw a circle through B and B' , centre o , in fig. 6; draw PB , and PpB' crossing this circle at p ; and draw PH and Oq perpendicular to BB' and PB' .

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$$(2) \quad \frac{Pp}{Oo} = \frac{PB}{OB} = \frac{PB}{PO} = \frac{PH}{Pq} = \frac{2PH}{PB'},$$
$$(3) \quad PR^2 = P_p \cdot PB' = 2O_o \cdot PH, \quad P_1R^2 = 2O_o \cdot P_1H_1.$$
$$(4) \quad \frac{\text{vel. of P}}{\text{vel. of P}_1} = \sqrt{\frac{PH}{P_1H_1}} = \frac{PR}{P_1R},$$

11. It is desirable to have a geometrical interpretation of the addition formula of (4) and (7), § 8, on the plane in figs. 3 and 6; this is seen by drawing AT, AR to meet DX and EX₁ produced, drawn perpendicular to QQ₁ and PP₁, in W and W₁.

Then if P_1 and Q_1 come to P_0 and Q_0 when P and Q are starting together from A , when

$$u=0, u_1=w, \text{ and } ADQ_0=\gamma=\text{am } w, \quad \cos AEP_0=\Delta\gamma,$$

$$(1) \quad \frac{DW}{DA} = \frac{cT}{cA} = \frac{cT_0}{cA} = \frac{DQ_0}{DA} = \cos ADQ_0 = \cos \gamma, \quad DW=DQ_0;$$

$$(2) \quad \frac{EW_1}{EA} = \frac{oR}{oA} = \frac{oR_0}{oA} = \frac{EP_0}{EA} = \cos AEP_0 = \Delta\gamma, \quad EW_1=EP_0.$$

$$(3) \quad AQ=AD \sin \phi, \quad DQ=AD \cos \phi, \\ AP=AE \cdot \kappa \sin \phi = AB \sin \phi, \quad EP=AE \cdot \Delta\phi,$$

so that, by Euclid VI. C,

$$(4) \quad DX = \frac{DQ \cdot DQ_1}{AD} = AD \cos \phi \cos \psi,$$

$$AY = \frac{AQ \cdot AQ_1}{AD} = AD \sin \phi \sin \psi,$$

$$(5) \quad AY_1 = \frac{AP \cdot AP_1}{AE} = AE \cdot \kappa^2 \sin \phi \sin \psi \\ = AD \sin \phi \sin \psi = AY,$$

$$EX_1 = \frac{EP \cdot EP_1}{AE} = AE \cdot \Delta\phi\Delta\psi.$$

$$(6) \quad \cos \gamma = \frac{DW}{AD} = \frac{DX}{AD} + \frac{XW}{AD} = \frac{DX}{AD} + \frac{AY}{AD} \cdot \frac{XW}{AY},$$

$$\frac{XW}{AY} = \frac{XT}{TY} = \frac{Dc}{cA} = \frac{Q_0T_0}{T_0A} = \frac{Q_0L}{LA} = \Delta\gamma,$$

$$(7) \quad \cos \gamma = \cos \phi \cos \psi + \sin \phi \sin \psi \Delta\gamma.$$

$$(8) \quad \Delta\gamma = \frac{EW_1}{AE} = \frac{EX_1}{AE} + \frac{X_1W_1}{AE} = \frac{EX_1}{AE} + \frac{AY_1}{AE} \cdot \frac{X_1W_1}{AY_1},$$

$$(9) \quad \frac{X_1W_1}{AY_1} = \frac{X_1R}{RY_1} = \frac{Eo}{oA} = \frac{P_0R_0}{R_0A} = \sqrt{\frac{DN_0}{DA}} = \cos \gamma,$$

$$(10) \quad \Delta\gamma = \Delta\phi\Delta\psi + \kappa^2 \sin \phi \sin \psi \cos \gamma.$$

Here (7) and (10) are the well-known formulas of Legendre; and thence, as before in (7) and (4), § 8,

$$(11) \quad \cos \gamma = \cos \phi \cos \psi + \sin \phi \sin \psi (\Delta\phi\Delta\psi + \kappa^2 \sin \phi \sin \psi \cos \gamma) \\ \cos \gamma = \frac{\cos \phi \cos \psi + \sin \phi \sin \psi \Delta\phi\Delta\psi}{1 - \kappa^2 \sin^2 \phi \sin^2 \psi};$$

$$(12) \quad \Delta\gamma = \Delta\phi\Delta\psi + \kappa^2 \sin \phi \sin \psi (\cos \phi \cos \psi + \sin \phi \sin \psi \Delta\gamma), \\ \Delta\gamma = \frac{\Delta\phi\Delta\psi + \kappa^2 \sin \phi \sin \psi \cos \phi \cos \psi}{1 - \kappa^2 \sin^2 \phi \sin^2 \psi}.$$

Or, geometrically,

$$(13) \quad \frac{\cos \gamma - \cos \phi \cos \psi}{\sin \phi \sin \psi} = \frac{XW}{AY} = \frac{EW_1}{AE} = \frac{EX_1}{AE} + \frac{X_1W_1}{AE} \\ = \Delta\phi\Delta\psi + \kappa^2 \sin \phi \sin \psi \cos \gamma,$$

$$(14) \quad \frac{\Delta\gamma - \Delta\phi\Delta\psi}{\kappa^2 \sin \phi \sin \psi} = \frac{X_1W_1}{AW_1} = \frac{DW}{AD} = \frac{DX}{AD} + \frac{XW}{AD} \\ = \cos \phi \cos \psi + \sin \phi \sin \psi \Delta\gamma,$$

equivalent to (11) and (12) above.

So also the formula for $\sin \gamma = \sin w$ can be interpreted.

LXXXII. *The Optical Determination of Stress.* By E. G. COKER, M.A., D.Sc., *Professor of Mechanical Engineering, City and Guilds of London Technical College, Finsbury* *.

THE principal advances in our experimental knowledge of the strength and properties of materials have been made by the use of mechanical apparatus for applying stress and measuring strain, and instruments of this class are now in general use possessing a sufficient degree of accuracy for the most refined measurements, but whatever applications purely mechanical methods may have, they possess a common characteristic feature that measurements must be taken over a definite length, area, or volume maintained in a standard condition throughout in order that the state of stress or strain may be referred to some standard measure possessed by the instrument, or by which it is calibrated. Whatever the arrangement may be, it is not in general possible to measure the stress or strain at a point, if the body is subjected to stress varying from point to point.

This defect in purely mechanical devices is one which from the nature of the case is hardly likely to be overcome entirely, yet in the great majority of the problems which arise in practice the stresses change very rapidly from point to point, and experimental information, if it exists, has almost invariably been obtained by using mechanical apparatus incapable of determining the stress at a point. Mathematical researches of the state of stress and strain in bodies give exact solutions of a variety of complicated problems; but some of the simplest forms of practical construction offer problems of the greatest difficulty, as for example the determination of the stresses in hooks, chain links, and rivetted

* Communicated by the Author: read in abstract at the British Association, Sheffield.