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XXII. *Correlated Averages.*

By Professor F. Y. EDGEWORTH, M.A., D.C.L.*

THE "correlation"† between the members of a system such as the limbs or other measurable attributes of an organism may in general be expressed by the formula

$$\Pi = J e^{-R} dx_1, dx_2, dx_3, \&c. ;$$

where

$$R = p_1(\bar{x}_1 - x_1)^2 + p_2(\bar{x}_2 - x_2)^2 + \&c., \\ + 2q_{12}(\bar{x}_1 - x_1)(\bar{x}_2 - x_2) + 2q_{13}(\bar{x}_1 - x_1)(\bar{x}_3 - x_3) + \&c.;$$

$\bar{x}_1, \bar{x}_2, \bar{x}_3$ &c. are the average values of the respective organs; x_1, x_2 , &c. are particular values of the same; $p_1, p_2 \dots q_{12}, q_{13}$ are constants to be obtained from observation; J is a constant deduced from the condition that the integral of Π between extreme limits should be unity. The expression Π represents the probability that any particular values of x_1, x_2 , &c. should concur. It enables us to answer the questions: What is the *most probable* value of one deviation x_r corresponding to *assigned* values x'_1, x'_2 &c. of the other variables? and What is the *dispersion* of the values of x_r about its mean (the other variables being assigned)?

This general formula for the concurrence of particular values of several organs is deducible from the proposition, proved by theory and observation, that each organ considered by itself assumes different values according to the exponential law of error. In a subsequent paper I hope to justify this principle; at present, assuming the propriety of the above-written formula, I propose to show how the constants $p_1, p_2 \dots q_{12}, q_{13} \dots$ are calculated. This problem has been solved by Mr. Galton for the case of two variables. The happy device of measuring each deviation by the corresponding quartile taken as unit enables him to express the sought quadratic in terms of a single parameter; as thus:—

$$R = \frac{x_1^2}{1-\rho^2} - \frac{2\rho x_1 x_2}{1-\rho^2} + \frac{x_2^2}{1-\rho^2};$$

where our ρ is Mr. Galton's r , and the $\bar{x}_1, \bar{x}_2$ of our general formula are zero. The parameter is found by observing the

* Communicated by the Author.

† See Galton, Proc. Roy. Soc. 1888, "Co-relations and their Measurement;" and Weldon, Proc. Roy. Soc. 1892, "Certain Correlated Variations in *Crangon vulgaris*."

value of x_1 , say ξ_1 , which most frequently corresponds to an assigned value of x_2 , say x_2' (or *vice versa*). From the equation $\frac{dR}{dx_1} = 0$ we have

$$\xi_1 = \rho x_2', \quad \rho = \xi_1 \div x_2'.$$

It should be observed that for the purpose of this calculation it is not necessary, as Mr. Galton has done, to pick out the values of x_1 corresponding to *each* value of x_2 . It is sufficient to take the sum, or the mean, of all or some of the *positive*, exclusive of negative, values [or *negative*, exclusive of positive] and the sum or mean of the corresponding values, not exclusive of negative [or positive]; and to equate

$$S\xi_1 = \rho Sx_2', \quad \rho = S\xi_1 \div Sx_2';$$

or

$$\frac{S\xi_1}{n} = \rho \frac{Sx_2'}{n}, \quad \rho = \frac{S\xi_1}{n} \div \frac{Sx_2'}{n};$$

omitting perhaps the extreme observations, with respect to which the law of error is liable to break down.

For example, let it be required to find the coefficient of correlation between the stature and left cubit of adult males, from the data utilized by Mr. Galton in Table II. of his paper on "Co-Relations" (Proc. Roy. Soc. 1888, p. 138), without the trouble of the detailed selection, the elaborate "*dépouillement*," which the construction of his table requires. Take the cubit as the independent variable, x_2 of the last paragraph, and write down all the deviations of the cubit (from its mean value) which are above zero and short of the extremity. There are ninety-three such instances—between 18.5 and 19.5 inches—among the materials which Mr. Galton has employed in constructing his Table II. (*op. cit.*). The sum of these deviations is 1762.75 inches; the sum of the ninety-three concurrent deviations of stature is 6422 inches. Each of these has to be divided by the corresponding quartile; .56 inch in the case of the cubit, 1.75 inch in the case of the statures. Thus

$$\rho = \frac{S\xi_1}{Sx_2} = \frac{1762.75}{.56} \div \frac{6422}{1.75} = .8.$$

Which is the value for the coefficient found by Mr. Galton.

In working this example I have taken the figures from the compartments of Mr. Galton's Table II. ; for instance, reckoning that there are 55 cubit-deviations "18.5 and under 19.0,"

i. e. 18·75 nearly, and similarly 38 cubit-deviations measuring on an average 19·25. But it is not necessary in general (the prevalence of the law of error being presumed) for the purpose of calculating the coefficient to work up the materials as in Mr. Galton's Table II., or Table III. It suffices to take the figures from the original observations pell-mell as they come; in fact it is better to proceed thus, as we avoid the inaccuracy involved in assuming that all the observations in each small compartment, *e. g.* between 18·5 inches and 19·0 inches, have the same measure—18·75 inches. I take an example from some of Mr. Galton's unmanipulated observations which he has kindly submitted to me. The raw material consists of a table like the following, each column giving the measurement of one organ, each row the measurements on one man,—in the random order in which the visitors to the Anthropometric laboratory presented themselves. Here is a specimen:—

TABLE I.

Statures.	Left Cubits.
72·4	19·3
71·9	18·7
71·5	18·8
70	18·4
66	18
70·3	18·6
72·2	19·6
69	18·4
68	18·3
64·4	18·2
65·5	16·8
66·1	17·1
71·6	19·3
68·4	18·2
65	17·3
63·7	16·8
62·3	16·5
⋮	⋮

(There are other columns with which we are not at present concerned.) From the right column I pick out all the figures which are above 18·05 (the mean value of the cubit) and write them down in a new column, as thus:—

TABLE II.

Left Cubit (above 18·05).	Corresponding Stature.
19·3	72·4
18·7	71·9
18·8	71·5
18·4	70
18·6	70·3
19·6	72·2
18·4	69
18·3	68
19·3	71·6
18·2	68·4
18·1	68·3
18·7	68·4
*18	70·2
19·4	71·1
18·8	68·3
15)280·6	15)1051·6
18·707	70·107

The mean deviation of the cubits, in absolute measure, is $\frac{18·707 - 18·05}{·56}$. The corresponding mean deviation of the

statures is $\frac{70·107 - 67·2}{1·75}$; assuming from Mr. Galton's tables in Proc. Roy. Soc. 1888 that 18·05 and 67·2 inches are the respective mean values of the cubit and stature, ·56 and 1·75 their respective quartiles. For the coefficient of correlation we have

$$\begin{aligned} & \frac{\text{mean deviation of cubit in absolute measure} \div \text{corresponding}}{\text{mean deviation of stature}} \\ & = 1·17 \div 1·66 = ·7 \text{ nearly.} \end{aligned}$$

The truer value is ·8, as determined by Mr. Galton from some hundreds of observations; but we must not expect an accurate result from a few observations here instanced by way of paradigm.

In the work of this example I have purposely allowed one mistake not largely affecting the result to stand by way of warning; the 18 marked with an asterisk in the second table which, being *below* the mean value of the cubit, has no

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business in a column furnishing *positive* values of the deviation x_2 . It is clear that, if we admitted such values indiscriminately, we should obtain an indeterminate result for ρ .

Of course, the more elaborate tabulation employed by Mr. Galton in the paper referred to and elsewhere is useful for the purpose of verifying that the law of error is indeed fulfilled, of testing up to what distance from the mean it holds good, and for other purposes.

It may be observed that the calculation of the coefficients of correlation is not affected if we take as the unit, not the quartile, but the same $\div 476 \dots$, the *modulus* of each set of observations. In what follows it is to be understood, when the contrary is not expressed, that the variables x_1, x_2 , &c.—representing deviations of different organs or attributes from their mean value—are thus measured.

To extend Mr. Galton's method to the case of three variables x_1, x_2, x_3 ; first determine the coefficients of correlation for each pair $(x_1 x_2), (x_1 x_3), (x_2 x_3)$; say, $\rho_{12}, \rho_{13}, \rho_{14}$. Thus the probability of any particular x_1 and x_2 concurring is $K e^{-S} dx_1 dx_2$, where K is a properly taken coefficient, and

$$S = \frac{x_1^2}{1 - \rho_{12}^2} - \frac{2x_1 x_2 \rho_{12}}{1 - \rho_{12}^2} + \frac{x_2^2}{1 - \rho_{12}^2}.$$

(See above, p. 190, and Mr. Galton's work there referred to.) Now the expression above written must be derived from the sought expression $J e^{-R} dx_1 dx_2 dx_3$, where

$$R = p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + 2q_{12} x_1 x_2 + 2q_{13} x_1 x_3 + 2q_{23} x_2 x_3,$$

by integration with respect to x_3 between extreme limits $+\infty$ and $-\infty$. If we watch the process of integration, we shall find that the coefficient of x_1^2 (in the exponent of the integral) is

$$\frac{p_2 p_3 - q_{13}^2}{p_3} :$$

the coefficient of $x_1 x_2$ is

$$2 \frac{q_{12} p_3 - q_{13} q_{23}}{p_3} ;$$

and the coefficient of x_2^2 is

$$\frac{p_1 p_3 - q_{23}^2}{p_3}.$$

These coefficients are to be equated respectively to

$$\frac{1}{1 - \rho_{12}^2}, \quad \frac{-2\rho_{12}}{1 - \rho_{12}^2}, \quad \text{and} \quad \frac{1}{1 - \rho_{12}^2}.$$

the coefficients of x_1^2 , $x_1 x_2$, and x_2^2 , in the quadratic above written considered as derived directly from the Galtonian coefficient ρ_{12} . Employing the quadratic to determine the most probable x_1 corresponding to any assigned x_2 (cf. above, p. 191) we have

$$\rho_{12} = \frac{q_{13} q_{23} - q_{12} p_3}{p_2 p_3 - q_{13}^2}.$$

This may be written

$$\rho_{12} = \frac{Q_{12}}{P_1},$$

where Q_{12} and P_1 are certain minors of the determinant Δ , which forms the discriminant of R ;

$$\Delta = \begin{vmatrix} p_1 & q_{12} & q_{13} \\ q_{12} & p_2 & q_{23} \\ q_{13} & q_{23} & p_3 \end{vmatrix}$$

If we carry the integration one step further, integrating Ke^{-s} with respect to x_2 between extreme limits, we shall find for the probability of any particular x_1 the expression

$$Le^{-x_1^2 \frac{\Delta}{P_1}} dx_1.$$

But by convention the *modulus* of the probability-curve under which the values of x_1 range is unity (above, p. 194). Therefore $\Delta = P_1$.

By parity of reasoning,

$$\Delta = P_3 = P_1.$$

$$\rho_{13} = \frac{Q_{13}}{P_1} = \frac{Q_{13}}{P_3},$$

$$\rho_{23} = \frac{Q_{23}}{P_2} = \frac{Q_{23}}{P_3}.$$

Therefore

$$Q_{12} = \Delta \rho_{12}; \quad Q_{13} = \Delta \rho_{13}; \quad Q_{23} = \Delta \rho_{23}.$$

Thus we may write the reciprocal Δ' of the determinant Δ :—

$$\Delta' = \begin{vmatrix} \Delta & \Delta \rho_{12} & \Delta \rho_{13} \\ \Delta \rho_{12} & \Delta & \Delta \rho_{23} \\ \Delta \rho_{13} & \Delta \rho_{23} & \Delta \end{vmatrix}$$

But, by a well-known theorem, each first minor of $\Delta' = \Delta$ multiplied by the corresponding constituent of Δ . Thus

$$(\Delta^2 - \Delta^2 \rho_{23}^2) = \Delta p_1; \quad (\Delta^2 \rho_{23} \rho_{13} - \Delta^2 \rho_{12}) = \Delta q_{12}.$$

Whence

$$p_1 = \Delta(1 - \rho_{23}^2)$$

$$q_{12} = \Delta(\rho_{23}\rho_{13} - \rho_{12}) ;$$

with corresponding values for the other coefficients.

It remains only to determine Δ . This is effected by the equation $\Delta = P_1$. Employing the values of p_1, p_2 &c., q_{12} , &c., which have just been found,

$$P_1 = \begin{vmatrix} \Delta(1 - \rho_{13}^2) & \Delta(\rho_{12}\rho_{23} - \rho_{13}) \\ \Delta(\rho_{12}\rho_{23} - \rho_{13}) & \Delta(1 - \rho_{12}^2) \end{vmatrix} ;$$

whence

$$\Delta = \Delta^2 \{ (1 - \rho_{13}^2)(1 - \rho_{12}^2) - (\rho_{12}\rho_{23} - \rho_{13})^2 \} ;$$

and Δ is determined in terms of the ρ 's, of which the numerical values are supposed to have been ascertained. The whole system of coefficients is therefore determined numerically.

Example.—Let x_1, x_2, x_3 respectively represent deviations of stature, cubit, and height of knee. The coefficients of correlation for each pair are $\rho_{12} = .8, \rho_{13} = .9, \rho_{23} = .8$; as ascertained by Mr. Galton (Proc. Roy. Soc. 1888, *Co-relations*, Table V.). To find the coefficients of the expression

$$p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + 2q_{12} x_1 x_2 + 2q_{13} x_1 x_3 + 2q_{23} x_2 x_3,$$

which is the exponent of the expression for the probability that any particular values of x_1, x_2, x_3 should concur; or, in other words, the equation of the ellipsoid of equal probability (the final constant being omitted).

Here for the reciprocal of the discriminant Δ we have

$$\Delta' = \begin{vmatrix} \Delta, & \Delta \cdot 8, & \Delta \cdot 9 \\ \Delta \cdot 8, & \Delta, & \Delta \cdot 8 \\ \Delta \cdot 9, & \Delta \cdot 8, & \Delta \end{vmatrix}.$$

Whence

$$p_1 = \Delta \begin{vmatrix} 1, & .8 \\ .8, & 1 \end{vmatrix} = \Delta(1 - .64) = \Delta \cdot 36,$$

$$q_{12} = \Delta \begin{vmatrix} .8, & .8 \\ 1, & .9 \end{vmatrix} = \Delta(.72 - .8) = -\Delta \cdot 08.$$

By parity $p_2 = \Delta \cdot 19, p_3 = \Delta \cdot 36, q_{13} = -\Delta \cdot 26, q_{23} = -\Delta \cdot 08$.

$$\text{Thus } \Delta = \begin{vmatrix} \Delta \cdot 36, & -\Delta \cdot 08, & -\Delta \cdot 26 \\ -\Delta \cdot 08, & \Delta \cdot 19, & -\Delta \cdot 08 \\ -\Delta \cdot 26, & -\Delta \cdot 08, & \Delta \cdot 36 \end{vmatrix}.$$

And to determine Δ we have

$$\Delta = \Delta^2 \begin{vmatrix} \cdot 19, & -\cdot 08 \\ -\cdot 08, & \cdot 36 \end{vmatrix} = \Delta^2 (\cdot 36 \times \cdot 19 - \cdot 08^2)$$

$= \Delta^2 \cdot 0620$ (a result which may be verified by observing that the other two principal minors afford, as they ought, the same equation for Δ). Whence $\Delta = 16 \cdot 129$. Substituting this value in the values of the coefficients above found, we have for the sought expression

$$5 \cdot 806x_1^2 + 3 \cdot 064x_2^2 + 5 \cdot 806x_3^2 - 2 \times 1 \cdot 290x_1x_3 \\ - 2 \times 4 \cdot 194x_1x_2 - 2 \times 1 \cdot 290x_2x_3.$$

Thus we see that the dispersion of x_1 corresponding to assigned values of x_2, x_3 has for modulus $\frac{1}{\sqrt{5 \cdot 806}}$. The

most probable deviation of one organ, *e.g.* the cubit, corresponding to assigned deviations of the two other organs is found by differentiating with respect to x_3 the expression above written and equating to zero. Thus, if x_1', x_3' be the assigned deviations of stature and height of knee, and ξ_2 the most probable corresponding deviation of cubit, $3 \cdot 064\xi_2 = 1 \cdot 290x_1' + 1 \cdot 290x_3'$. $\xi_2 = \frac{1 \cdot 290(x_1' + x_3')}{3 \cdot 064} = \cdot 42(x_1' + x_3')$.

I have verified this deduction by actually observing the value of cubit-deviation which on an average corresponds to assigned values of the other deviations. For this purpose I employ a table such as Table I. at p. 192, with an additional column for *height of knee*. This material, consisting of unmanipulated observations on the stature, cubit, and knee-height of three hundred men, was kindly furnished to me by Mr. Galton. Out of these three hundred triplets I pick out ninety-six which have the stature and knee-height above the respective means of those organs; and proceed in a manner analogous to the simpler calculation discussed at p. 193. The columns for stature and cubit will be nearly the same as those of Table II. in that passage; but not quite the same. For negative values are admissible there for the deviation of stature which is there the dependent variable, but not here, where both stature and knee-height are treated as independent variables. On the other hand, negative deviations of cubit are admissible here, but not there. The rationale of this distinction is sufficiently explained by the remark at p. 194 (top).

Here is a specimen of the calculation, the columns for

cubit and stature being here, as *for the most part* they must be, identical with those used for the simpler calculation (at p. 193).

TABLE III.

Height of knee (above 20·5).	Statures. (above 67·2).	Corresponding cubit.
22·9	72·4	19·3
21·4	71·9	18·7
21·6	71·5	18·8
21	70	18·4
21·7	70·3	18·6
22	72·2	19·6
21·1	69	18·4
20·6	68	18·3
21·6	71·6	19·3
21·4	68·4	18·2
20·6	68·3	18·1
20·8	68·4	18·7
21	70·2	18
22·3	71·1	19·4
21·4	68·3	18·8
321·4	1051·6	280·6

The mean deviations of cubit and of stature are as before (in units of the quartiles) 1·17 and 1·66. The mean deviation of knee-height is $\left(\frac{321·4}{15} - 20·5\right) \div \cdot 8$ (20·5 in. being the mean and ·8 in. the quartile for the height of knee ; Galton, Proc. Roy. Soc. 1888, Co-relations, Table I.) = 1·16. According to theory the sum of the mean positive deviations of the stature and knee-height divided into the corresponding mean deviation of cubit should yield a quotient ·42. In fact

$$1·17 \div (1·16 + 1·66) = \cdot 415.$$

This consilience between fact and theory will appear all the more striking when it is mentioned that the anthropometrical observations from which Mr. Galton deduced his coefficients ·8, ·9, and ·8 were not coincident with those by which I have verified the theory built upon those coefficients.

This theory is readily extended to the case of many variables. One way of looking at the whole matter is as follows :— Beginning with the case of three variables—the familiar x, y, z —let us suppose that w , the probability of particular values of x, y, z concurring, $= J e^{-R} dx dy dz$, where $R = ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$; and let us inquire how,

given the coefficients in this expression, we could deduce the coefficient of correlation between any two of the variables, x and y . The sought coefficient, say r , is such that to any assigned value of x , e. g. x' , there corresponds rx' as the most probable value of y : that is to say, the y which in the long run—the long run whose stages are different values of z —most frequently occurs. The direct method is to substitute x' for x in w , integrate between extreme limits with regard to z , differentiate with regard to y , and equate to zero. Geometrically we may imagine a surface in the fourth dimension of space at a distance w_1 from the plane $x=x'$; where w_1 is what w becomes when x' is substituted for x . The annexed diagram is intended to assist the imagination by representing the curves of probability projected on the plane yz . The point O_1 is the centre of that system of ellipses which is formed by

the section of the plane $x=x'$ with the ellipsoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = \text{const.}$$

The surface w_1 is evidently symmetrical on either side of planes parallel to the axes y and z through the point O_1 . Thus the sought y to which corresponds the greatest number of instances in the long run, the y for which the strip

$\int_{-\infty}^{+\infty} w_1 dz$ is greatest, is the ordinate of O_1 ; which is also

the y for which the corresponding w -ordinate is greatest. It is given therefore by the equations

$$\frac{dw_1}{dy} = 0, \quad \frac{dw_1}{dz} = 0$$

w_1 being what w becomes when x' is substituted for x) ; or by the equations

$$\frac{dR_1}{dy} = 0, \quad \frac{dR_1}{dz} = 0 ;$$

that is

$$hx' + by + fz = 0,$$

$$gx' + fy + cz = 0.$$

Whence

$$y = \frac{fg - ch}{bc - f^2} x' = \frac{H}{A} x',$$

where H and A are minors of the discriminant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Thus $r = \frac{H}{A}$ as before.

By parity of reasoning the most probable x corresponding to an assigned y' is $\frac{H}{B}$. Thus $A=B$. And by parity the coefficients of correlation between the other pairs xz and yz are respectively proportional to G and F ; and $C=B=A$.

From the same point of view it appears that the coefficient of x^2 in the exponent of the expression $\iint w dy dz$ is $\frac{\Delta}{A}$. For suppose for a moment that there were added to w a term $-2\mathbf{x}x$; the centre of the surface being thus transferred from zero to $\mathbf{x}$. The most probable x is found by solving the system

$$ax + hy + gz = \mathbf{x},$$

$$hx + by + fz = 0,$$

$$gx + fy + cz = 0.$$

Solving for x , we have $Ax = \Delta \mathbf{x}$, $x = \frac{\Delta}{A} \mathbf{x}$; and accordingly

$$\iint w dy dz \text{ is of the form } J e^{-\frac{\Delta}{A}(x-\mathbf{x})^2}.$$

But, since by convention the modulus of the variable is unity, $\Delta = A (=B=C)$; a result which continues to hold as $\mathbf{x}$ is supposed to vanish.

This reasoning is quite general; and accordingly, replacing the symbols x_1, x_2, x_3 , we may extend to four and higher numbers of variables the solution which has been given above for the case of three variables. In the case of four variables $\rho_{12}, \rho_{13}, \&c. \rho_{24} \dots$ being as before the coefficients of correlation for each pair, the reciprocal of the discriminant

$$\Delta' = \begin{vmatrix} \Delta, & \Delta\rho_{12}, & \Delta\rho_{13}, & \Delta\rho_{14}. \\ \Delta\rho_{12}, & \Delta, & \Delta\rho_{23}, & \Delta\rho_{24}. \\ \Delta\rho_{13}, & \Delta\rho_{23}, & \Delta, & \Delta\rho_{34}. \\ \Delta\rho_{14}, & \Delta\rho_{24}, & \Delta\rho_{34}, & \Delta. \end{vmatrix}$$

We have merely to border with a new row and column the determinant used for the case of three variables. But in forming the first minors of the reciprocal in the case of four (and similarly for any *even* number) it must be observed that, according to the rule of signs*, the minor of the reciprocal which is equated to $\Delta^2 q_{12}$ (in general $\Delta^{n-2} q_{12}$) is not now $\Delta^3(\rho_{23} \rho_{34} \rho_{14})$, but $-\Delta^3(\rho_{23} \rho_{34} \rho_{14})$; or, as it might be more elegant to write, $-\Delta^3(\rho_{23} \rho_{34} \rho_{41})$. By the same rule,

$$\begin{aligned} \Delta\rho_{13} &= +\Delta^3(\rho_{24} \rho_{31} \rho_{42}), \\ \Delta\rho_{14} &= -\Delta^3(\rho_{21} \rho_{32} \rho_{43}), \\ \vdots & \quad \quad \quad \vdots \\ \vdots & \quad \quad \quad \vdots \end{aligned}$$

Example.—To exhibit the correlation between four quantities, of which the first, x_1 , is formed by taking at random the sum of ten digits (say from a page of mathematical tables); the second, x_2 , is formed by adding to the first x_1 another random decade; $x_3 = x_2 +$ another random decade; $x_4 = x_3 +$ another random decade.

The coefficients of correlation between the pairs, which have usually to be ascertained by observation, are here deducible *a priori*. *E. g.* $\rho_{12} = \sqrt{\frac{1}{2}}$. For, putting ξ_1 and ξ_2 for the

actual deviations of the first and second quantities (the sum of ten and the sum of twenty digits), we may regard ξ_1 and $(\xi_2 - \xi_1)$ as fluctuating independently, according to a modulus which is the same for both, say c ; being that which apper-

* Cf. Salmon's 'Higher Algebra,' chap. 1, arts. 6 and 8.

tains to a sum of ten random digits (it is $\sqrt{165}$). Accordingly the equiprobable curves are

$$\frac{\xi_1}{c^2} + \frac{(\xi_1 - \xi_2)^2}{c^2} = \text{const.};$$

or

$$\frac{2\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2}{c^2} = \text{const.}$$

Reducing to units of modulus by putting

$$x_1 = \frac{\xi_1}{c}, \quad x_2 = \frac{\xi_2}{\sqrt{2}c},$$

we have

$$2x_1^2 - 2\sqrt{2}x_1x_2 + 2x_2^2;$$

whence $\rho_{12} = \frac{1}{\sqrt{2}}$. In like manner it can be shown that

$$\rho_{13} = \sqrt{\frac{1}{3}}, \quad \rho_{14} = \sqrt{\frac{1}{4}}, \quad \rho_{23} = \sqrt{\frac{2}{3}}, \quad \rho_{24} = \sqrt{\frac{2}{4}}, \quad \rho_{34} = \sqrt{\frac{3}{4}}.$$

Thus, for the determinant which is the reciprocal of the discriminant, we have

$$\Delta' = \begin{vmatrix} \Delta, & \Delta\sqrt{\frac{1}{2}}, & \Delta\sqrt{\frac{1}{3}}, & \Delta\sqrt{\frac{1}{4}} \\ \Delta\sqrt{\frac{1}{2}}, & \Delta, & \Delta\sqrt{\frac{2}{3}}, & \Delta\sqrt{\frac{2}{4}} \\ \Delta\sqrt{\frac{1}{3}}, & \Delta\sqrt{\frac{2}{3}}, & \Delta, & \Delta\sqrt{\frac{3}{4}} \\ \Delta\sqrt{\frac{1}{4}}, & \Delta\sqrt{\frac{2}{4}}, & \Delta\sqrt{\frac{3}{4}}, & \Delta \end{vmatrix}$$

$$\rho_{11} = \Delta \begin{vmatrix} 1, & \sqrt{\frac{2}{3}}, & \sqrt{\frac{2}{4}} \\ \sqrt{\frac{2}{3}}, & 1, & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{2}{4}}, & \sqrt{\frac{3}{4}}, & 1 \end{vmatrix} = \Delta \frac{1}{12}.$$

$$p_{12} = -\Delta \begin{vmatrix} \sqrt{\frac{2}{3}}, & \sqrt{\frac{2}{4}}, & \sqrt{\frac{1}{2}} \\ 1, & \sqrt{\frac{3}{4}}, & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{3}{4}}, & 1, & \sqrt{\frac{1}{4}} \end{vmatrix} = -\Delta \frac{1}{12} \frac{1}{\sqrt{2}}.$$

$$p_{13} = +\Delta \begin{vmatrix} \sqrt{\frac{2}{4}}, & \sqrt{\frac{1}{2}}, & 1 \\ \sqrt{\frac{3}{4}}, & \sqrt{\frac{1}{3}}, & \sqrt{\frac{2}{3}} \\ 1, & \sqrt{\frac{1}{4}}, & \sqrt{\frac{2}{4}} \end{vmatrix} = 0.$$

&c.

&c.

&c.

The determinant Δ , thus determined in terms of itself, may be written :—

$$\Delta = \begin{vmatrix} \frac{1}{12} \Delta, & -\frac{1}{12} \frac{1}{\sqrt{2}} \Delta, & 0, & 0 \\ -\frac{1}{12} \frac{1}{\sqrt{2}} \Delta, & \frac{1}{6} \Delta, & -\frac{1}{4} \frac{1}{\sqrt{6}} \Delta, & 0 \\ 0, & -\frac{1}{4} \frac{1}{\sqrt{6}} \Delta, & \frac{1}{4} \Delta, & -\frac{1}{4} \frac{1}{\sqrt{3}} \Delta \\ 0, & 0, & -\frac{1}{4} \frac{1}{\sqrt{3}} \Delta, & \frac{1}{6} \Delta. \end{vmatrix}$$

To determine Δ we have

$$\Delta = \Delta^3 \times \begin{vmatrix} \frac{1}{6}, & -\frac{1}{4} \frac{1}{\sqrt{6}}, & 0 \\ -\frac{1}{4} \frac{1}{\sqrt{6}}, & \frac{1}{4}, & -\frac{1}{4} \frac{1}{\sqrt{3}} \\ 0, & -\frac{1}{4} \frac{1}{\sqrt{3}}, & \frac{1}{6} \end{vmatrix} = \Delta^3 \frac{1}{576}.$$

When $\Delta^2=576$, $\Delta=24$. Substituting the value of Δ in the expressions for the coefficients, we find for the sought quantic R,

$$2x_1^2 + 4x_2^2 + 6x_3^2 + 4x_4^2 - 2\sqrt{2}x_1x_2 - 2\sqrt{6}x_2x_3 - 2\sqrt{12}x_3x_4.$$

To verify and illustrate this result, let it be employed to answer the question, What is the most probable value of x_1 corresponding to assigned values of x_2, x_3, x_4 . Differentiating R with respect to x_1 and equating to zero, we have $2x_1 - \sqrt{2}x_2 = 0$; importing that to any assigned deviation of the second variable the most probable corresponding deviation of the first variable is $\sqrt{2}$ times less; each deviation being reckoned in units of its own modulus.

The truth of this follows at once from the datum that $\rho_{12} = \frac{1}{\sqrt{2}}$. To give another proof of that proposition:—the

mean positive error of the *actual* deviations of the first decade of digits from its mean value 45 is the modulus for such a decade, say c , $\div \sqrt{\pi}$. Now let there be added a second random decade. Its deviations being in the long run as often positive as negative, the sum of the two decades is the same as that of the first (the deviations of the first being exclusively

positive). Thus to an actual mean positive deviation $\frac{c}{\sqrt{\pi}}$ of the first quantity corresponds the same actual mean deviation

$\frac{c}{\sqrt{\pi}}$ of the second quantity. But the modulus of the second

quantity (the sum of two decades) is $\sqrt{2}$ times that of the first. Therefore, in units of modulus, to x_1 assigned corre-

sponds, as the most probable value of x_2 , a deviation less in the ratio $\frac{1}{\sqrt{2}}$. Whence, by the Galtonian theorem, to x_2

assigned corresponds as the most probable value of x_1 a deviation less in the ratio $\frac{1}{\sqrt{2}}$. Which is the proposition here

deduced from our formula.

The case of *five* variables follows in respect of the signs of the first minors the analogy of the case of *three* above illustrated; the case of six is analogous to that of four. And so on.