The Problem of Two Electrified Spheres

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II. *The Problem of Two Electrified Spheres.* By ALEXANDER RUSSELL, M.A.,
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ABSTRACT.

It is shown that Poisson's method can be readily applied to get the complete solution of the problem of two electrified spherical conductors. The case of a spherical condenser is first considered and simplified methods of computing its capacity are given. Simple expressions for the potentials at the two inverse points due to the charges on each of the spheres are found. Poisson's method is then used to show that the capacity coefficients for two spheres external to one another can be very simply expressed in terms of the capacities of certain spherical condensers and can thus be readily computed in all cases. Formulæ for the attraction and repulsion of two electrified spheres are given. When the charges are of like sign the electrostatic energy of the system has a maximum value when the spheres are at a given distance apart. Similarly, when the potentials are of like sign and maintained constant, the electrostatic energy of the system has a minimum value for a definite distance between them. The particular case when the spheres are close together is considered in detail. The exactly analogous problems in thermal and electric conduction are also considered, and it is shown how their solutions can be written down at once. Finally, the formula for the maximum value of the electric stress in the medium surrounding the spheres is given and its practical importance is pointed out. For equal spherical electrodes at a fixed distance apart there is a definite size of radius which makes the disruptive potential difference a maximum. The distance between them where the attraction has a maximum value is also found.

THE problem of two electrified spherical conductors has attracted the study of physicists for over a hundred years. From the point of view of the mathematician it may be considered that the complete solution has been obtained. The solutions, however, are usually given in the form of series which are in general very laborious to compute numerically, and hence they are rarely used by the physicist. The author has therefore thought it advisable to give what he considers to be the simplest solution of the problem, and to show by numerical examples that in practically every case the numerical solution can be readily obtained. The methods of computing the capacity of a spherical condenser are first considered. It is then shown how the self and mutual capacity coefficients of external spheres can always be written down by simple formulæ in terms of the capacities of certain spherical condensers, and can thus be readily computed. Finally their attractions and repulsions, the maximum value of the potential gradient at their surfaces and the value of the potentials at all points are given.

In calculating the capacity of a spherical condenser* the author formerly used Kelvin's method of images. Difficulties arose as in some cases the Kelvin points were at infinity. In what follows he gives a modification of the method which Poisson† gave 112 years ago. He now thinks that this method is the simplest that has so far been devised.

* Proc. Roy. Soc., Vol. 94, p. 206.

† Mémoires de l'Institut (1812).

LEMMA.

Consider an electrified spherical conductor of radius a with its centre at O (Fig. 1). The charge q on it is distributed over its surface. Let the potential due to this charge at a point P distant x from O be denoted by $\varphi(x/a)$. Then if $OP \cdot OQ = a^2$, we have

$$\varphi\left(\frac{x}{a}\right) = \Sigma \frac{\delta q}{AP} = \frac{a}{OP} \Sigma \frac{\delta q}{AQ} = \frac{a}{x} \varphi\left(\frac{a}{x}\right). \quad (1)$$

This is true, whether x is greater or less than a . Notice also that $\varphi(0) = \frac{q}{a}$.

THE CAPACITY OF A SPHERICAL CONDENSER.

Let A (Fig. 2) be the centre of a spherical cavity in a conducting solid, and let B be the centre of a spherical conductor in this cavity. Let a and b be the radii of the two spheres, and let c be the distance AB between their centres. Let the inner sphere have a charge q . There will be a charge $-q$ over the surface of the cavity.

FIG. 1.

FIG. 2.

The potential at a point in the outer conductor at a great distance away will be zero, for its distances from the charges $+q$ and $-q$ will be practically the same. Hence every point on the outer conductor must be at zero potential. We shall suppose that the inner conductor is at potential V . Let P (Fig. 2) be any point on the line $RABQ$, inside the sphere B , and let $AP = x$. Let $f_1(x/a)$ be the potential at P produced by the charge $-q$, and let $f_2\{(x-c)/b\}$ be the potential at the same point produced by the charge $+q$. Since the outer conductor is at potential zero, we have for the potential at a point Q in it, on AB produced

$$f_1\left(\frac{AQ}{a}\right) + f_2\left(\frac{BQ}{b}\right) = 0$$

Hence if $AQ = \xi$ we have by (1)

$$\frac{a}{\xi} f_1\left(\frac{a}{\xi}\right) + \frac{b}{\xi - c} f_2\left(\frac{b}{\xi - c}\right) = 0 \quad (2)$$

where ξ lies outside the limits $+a$ and $-a$.

We have also at every point P on RQ inside the sphere B

$$f_1\left(\frac{x}{a}\right) + f_2\left(\frac{x-c}{b}\right) = V \quad (3)$$

where x lies in value between $c-b$ and $c+b$.

Now let $\frac{x}{a} = \frac{a}{\xi}$. This is always possible since a must be greater than $b+c$.

Hence eliminating $f_1\left(\frac{x}{a}\right)$ between (2) and (3) we get

$$f_2\left(\frac{x-c}{b}\right) - \frac{ab}{a^2-cx} f_2\left(\frac{bx}{a^2-cx}\right) = V. \quad (4)$$

If we write

$$\frac{x_n - c}{x_{n-1}} = \frac{b^2}{a^2 - cx_{n-1}} \quad (5)$$

we deduce by successive applications of (4) that

$$f_2\left(\frac{x_1 - c}{b}\right) = V \left\{ 1 + \frac{ab}{a^2 - cx_1} + \frac{a^2 b^2}{(a^2 - cx_1)(a^2 - cx_2)} + \dots \right\} \quad (6)$$

The formulæ (6) and (5) enable us to compute the potential due to the charge $+q$ at all points on the diameter of the inner sphere which passes through A and B .

It is convenient to express a , b and c in terms of certain functions α , β and r , which may be defined* by the equations

$$a = r/\sinh \alpha, \quad b = r/\sinh \beta, \quad c = r \sinh(\beta - \alpha)/(\sinh \alpha \sinh \beta) \quad (7)$$

It can be shown that r is the radius of the sphere which cuts the two spheres A and B orthogonally, and which has its centre on AB produced. If we denote $\beta - \alpha$ by ω , we have

$$\cosh \alpha = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cosh \beta = \frac{a^2 - b^2 - c^2}{2bc},$$

$$\cosh \omega = \frac{a^2 + b^2 - c^2}{2ab}, \quad \text{and} \quad \sinh \omega = \frac{cr}{ab}.$$

$$\text{Hence also } c = a \cosh \alpha - b \cosh \beta = a\varepsilon^\alpha - b\varepsilon^\beta = a\varepsilon^{-\alpha} - b\varepsilon^{-\beta} \quad (8)$$

The potential at B (Fig. 2) due to the charge q is obviously q/b , and this equals KV/b , where K is the capacity of the spherical condenser. Hence, putting $x_1 = c$, and noting that $f_2(0)$ is the potential at B , we get

$$\frac{K}{b} = 1 + \frac{ab}{a^2 - c^2} + \frac{a^2 b^2}{(a^2 - c^2)(a^2 - cx_2)} + \dots \quad (9)$$

where $x_2, x_3, \dots$ are found successively by (5).

$$\text{Now} \quad \frac{ab}{a^2 - cx_1} = \frac{\sinh \beta}{\sinh(\beta + \omega)}$$

$$\text{and} \quad \frac{ab}{a^2 - cx_2} = \frac{\sinh(\beta + \omega)}{\sinh(\beta + 2\omega)}, \text{ etc.}$$

$$\text{Hence} \quad K = \sum_{n=0}^{\infty} \frac{r}{\sinh(\beta + n\omega)} \quad (10)$$

This agrees with formula (5), given in the Proc. Roy. Soc. A., Vol. XCIV., p. 209,

* Russell, Alternating Currents, Vol. I, p. 166.

which was obtained by Kelvin's method of images. We see that (10) can be written in the form

$$K = \frac{r}{\sinh \beta} + K_1 \dots \dots \dots (11)$$

Where K_1 is the capacity of a spherical condenser for which the inner radius b_1 equals $r/\sinh (\beta + \omega)$, the outer radius a_1 equals $r/\sinh (\beta + \omega - \omega)$ and the distance between their centres c_1 equals $\frac{a_1 b_1}{r} \sinh \omega$.

We shall denote the capacity of a spherical condenser by $K(a, b, c)$. Hence

$$K(a, b, c) = b + K\left(b, \frac{ab^2}{a^2 - c^2}, \frac{cb^2}{a^2 - c^2}\right) = b + \frac{b^2}{a^2 - c^2} K\left(\frac{a^2 - c^2}{b}, a, c\right) \dots \dots (12)$$

for $K(ma, mb, mc) = mK(a, b, c)$.

As the condenser on the right-hand side of (12) is always more nearly concentric than the one on the left-hand side, (12) is very useful for computing purposes.

The author has previously shown (*l.c. ante*) that

$$K(a, b, c) = \frac{ab}{a-b} \left\{ 1 + A_2 \frac{c^2}{(a-b)^2} + A_2 A_4 \frac{c^4}{(a-b)^4} + \dots \right\}$$

where $A_2 = \frac{ab}{a^2 + b^2 + ab}$, and $A_2 A_4$ has its maximum value $\frac{1}{5}$ when $a = b$.

Hence, when $c^4/5(a-b)^4$ can be neglected compared with unity, we have

$$K(a, b, c) = \frac{ab}{a-b} \left\{ 1 + \frac{ab}{a^2 + b^2 + ab} \cdot \frac{c^2}{(a-b)^2} \right\} \dots \dots \dots (13)$$

with high accuracy.

When $c/(a-b)$ is large, it is better to use the formula (*l.c. ante*)

$$K(a, b, c) = \frac{b}{2} + \frac{r}{2\omega} \log_e \frac{\cosh \beta + 1}{\cosh \beta - 1} + \frac{r\omega}{12} \frac{\cosh \beta}{\sinh^2 \beta} \dots \dots \dots (14)$$

Writing $2s = a + b + c$, we get

$$cr = 2[s(a-s)(s-b)(s-c)]^{\frac{1}{2}}$$

$\cosh (\omega/2) = \{s(s-c)/ab\}^{\frac{1}{2}}$; $\sinh (\omega/2) = \{(a-s)(s-b)/ab\}^{\frac{1}{2}}$; $\omega/2 = \log_e \{\cosh (\omega/2) + \sinh (\omega/2)\}$; and $\cosh (\beta/2) = \{(s-b)(s-c)/bc\}^{\frac{1}{2}}$.

The formulæ give the capacities in electrostatic units (centimetres). To find the capacities in microfarads we divide by 900,000.

NUMERICAL EXAMPLES.

Using (12), we get

$$\begin{aligned} K(10, 1, 6) &= 1 + \frac{1}{64} K(64, 10, 6) \\ &= 1 + \frac{1}{32} K(32, 5, 3) \end{aligned}$$

Similarly

$$K(32, 5, 3) = 5 + \frac{5}{203} K(203, 32, 3)$$

We see by (13) that the error made in computing $K(203, 32, 3)$ as if it were the concentric spherical condenser $K(203, 32, 0)$ is very small. Writing, therefore,

$$K(203, 32, 3) = \frac{203 \cdot 32}{203 - 32}$$

and, substituting, we get

$$K(10, 1, 6) = 1.185490 \dots$$

The true value is $1.185491 \dots$

As a further example of the use of (12)

$$K(3, 1, 1) = 1 + \frac{1}{8} K(8, 3, 1)$$

$$K(8, 3, 1) = 3 + \frac{1}{7} K(21, 8, 1)$$

$$K(21, 8, 1) = 8 + \frac{8}{55} K(55, 21, 1)$$

$$\begin{aligned} \text{Thus} \quad K(3, 1, 1) &= 1 + \frac{3}{8} + \frac{1}{7} + \frac{1}{385} K(55, 21, 1) \\ &= 1.6061 \dots \text{approx.} \end{aligned}$$

Using (14), we get

$$\begin{aligned} K(10, 7, 1) &= 3.5 + 20.520 + 0.204 \\ &= 24.224 \dots \text{approx.} \end{aligned}$$

More simply, by (12) and (13)

$$\begin{aligned} K(10, 7, 1) &= 7 + \frac{7}{99} K(99, 70, 7) \\ &= 24.21 \dots \text{approx.} \end{aligned}$$

When $c/(a-b)$ is very nearly unity we can use the formula

$$K(a, b, c) = \frac{ab}{a-b} \left\{ \frac{1}{2} \log_e \frac{2ab}{(a-b)(a-b-c)} - \psi \left(\frac{a}{a-b} \right) \right\} \dots \quad (15)$$

where $\psi \left(\frac{a}{a-b} \right)$ is the logarithmic derivate of the gamma function, tables of the values of which are given in Russell's "Alternating Currents," Vol. I., p. 241.

For example, by (15)

$$K(8, 4, 3.99) = 27.82 \dots$$

It will be seen that the capacities of spherical condensers can be computed in all cases with little labour.

THE POTENTIALS AT THE INVERSE POINTS OF THE TWO SPHERES.

If P and Q be the inverse points of the two spheres in Fig. 2, then $AP \cdot AQ = a^2$ and $BP \cdot BQ = b^2$. It easily follows that $AP = a\epsilon^{-\alpha} = c + BP = c + b\epsilon^{-\beta}$. From equa-

tion (4) we can readily find simple expressions for the potentials at P and Q due to the charge q . Equation (4) may be written

$$f_2\left(\frac{x-c}{b}\right) - \frac{ab}{a^2-cx} f_2\left\{\frac{x-c}{b} + \frac{c(x-ae^{-a})(x-ae^a)}{b^2(a^2-cx)}\right\} = V.$$

Hence, if V_P be the potential at P due to the charge q , we have

$$V_P\left\{1 - \frac{ab}{a^2-cae^{-a}}\right\} = V,$$

and thus by (8)

$$V_P = V \frac{\varepsilon^a}{\varepsilon^a - 1} \quad \dots \dots \dots (16)$$

By (1) we get

$$V_Q = V \frac{\varepsilon^{-a}}{\varepsilon^a - 1} \quad \dots \dots \dots (17)$$

If we denote the potentials at P and Q due to $-q$ by V'_P and V'_Q , we obviously have

$$V_P + V'_P = V, \quad \text{and} \quad V_Q + V'_Q = 0.$$

Thus

$$V'_P = -V \frac{1}{\varepsilon^a - 1} \quad \text{and} \quad V'_Q = -V \frac{\varepsilon^{-a}}{\varepsilon^a - 1} \quad \dots \dots (18)$$

Note also that

$$\varepsilon^{\frac{\beta}{2}} = \left\{ \frac{(s-b)(s-c)}{bc} \right\}^{\frac{1}{2}} + \left\{ \frac{s(a-s)}{bc} \right\}^{\frac{1}{2}} \quad \dots \dots \dots (19)$$

and

$$2 \frac{\varepsilon^a}{\varepsilon^a - 1} = 1 + \left\{ \frac{s(s-c)}{(a-s)(s-b)} \right\}^{\frac{1}{2}} \quad \dots \dots \dots (20)$$

For instance, in the condenser $K(5, 2, 2)$, $BP = be^{-\beta} = \frac{1}{2}$, $BQ = 8$, and if the potential of the inner sphere be V , $V_P = 2V$, $V_Q = V/2$, $V'_P = -V$, and $V'_Q = -V/2$. By (12) also the charges on the inner and outer conductors are $3.892 V$ and $-3.892 V$ approximately.

RELATIONS BETWEEN THE CAPACITIES OF SPHERICAL CONDENSERS.

There are an infinite number of algebraical relations connecting the capacities of various condensers with one another. For example, provided that c is greater than 2, we can show by (9) and (7) that

$$K(c^2-1, c, 1) - cK(c^2-1, 1, c) = \frac{c^2-1}{c^2-2} K(c^2-2, 1, 1). \quad \dots \dots \dots (21)$$

Putting $c=4$, we get

$$K(15, 4, 1) - 4K(15, 1, 4) = \frac{15}{14} K(14, 1, 1) \text{ exactly} = 1.154 \text{ (approx.)}.$$

THE FORCE ON THE INNER SPHERE OF A SPHERICAL CONDENSER.

If W denote the electrostatic energy stored in the condenser we have

$$W = q^2/2K = \frac{1}{2}KV^2.$$

As K always increases as the distance c between the centres of the spheres increases, we see that if the charge q be maintained constant, W diminishes as c increases, but that if V be maintained constant, W increases as c increases. We have now to find out whether the mutual force tends to increase or diminish c . When the charge is maintained constant it follows from the conservation of energy that the work done by the mutual force F must be supplied from the electrostatic energy. Hence F must act so that W diminishes and consequently c increases. When c is zero the equilibrium is unstable. It follows also that when the P.D. V is maintained constant c increases, and the electrostatic energy increases. Since

$$W_2 - W_1 = \int_1^2 F \delta c,$$

we see that when the P.D. is kept constant, the increase in the electrostatic energy $W_2 - W_1$ equals the concurrent work done $\int_1^2 F \delta c$. The battery which maintains the P.D. constant has therefore to supply an amount $2(W_2 - W_1)$ of energy.

If F denote the force tending to increase c we have $F = \frac{V^2}{2} \frac{\delta K}{\delta c}$, and thus by (10),

$$\begin{aligned} \frac{2F}{V^2} = & \sum \frac{n \cosh(\beta + n\omega)}{\sinh^2(\beta + n\omega)} \\ & - \frac{\cosh a \cosh \beta}{\sinh \omega} \sum \frac{1}{\sinh(\beta + n\omega)} \\ & + \frac{\sinh \beta \cosh a}{\sinh \omega} \sum \frac{\cosh(\beta + n\omega)}{\sinh^2(\beta + n\omega)} \quad \dots \dots \dots (22) \end{aligned}$$

the summation being taken from $n=1$ to $n=\infty$.

It will be noticed that so long as V is constant the attractive force is the same whether the geometrical constants are a , b and c , or ma , mb and mc .

By substituting q/K for V in (22) we see that the force when q remains constant varies inversely as K^2 , when the relative values of a , b and c , and therefore a and β , remain constant. Hence, for a given value of q the force when the constants are a , b and c would be m^2 times the force when the constants are ma , mb and mc .

When $c^2/(a-b)^2$ can be neglected compared with unity, we get, by (13)

$$F = \frac{\delta W}{\delta c} = \frac{a^2 b^2 c}{(a^3 - b^3)(a-b)^2} V^2 = \frac{c}{a^3 - b^3} q^2. \quad \dots \dots \dots (23)$$

Hence in this case the attractive force varies as the distance between the centres.

When the distance x ($= a-b-c$) between the spheres is very small, we get (*l.c. ante*)

$$F = \frac{ab}{4(a-b)x} V^2. \quad \dots \dots \dots (24)$$

In this case, therefore, the force varies inversely as the distance between the spheres. If V be measured in electrostatic units (one electrostatic unit = 300 volts), F will be given in dynes. By means of (15) we can readily express F in terms of q^2 , but the formula is not so simple.

EXTERNAL SPHERES.

Let us consider two conducting spheres (Fig. 3) with their centres at A and B . Let their radii be a and b , and let c be the distance between their centres. We shall denote their charges by Q_1 and Q_2 , and their potentials by v_1 and v_2 respectively. We have, therefore,

$$Q_1 = k_{11}v_1 + k_{12}v_2 \text{ and } Q_2 = k_{22}v_2 + k_{12}v_1,$$

where k_{11} and k_{22} are the self capacity coefficients and k_{12} is the mutual capacity coefficient.

We shall first show how these coefficients can be computed in terms of a , b and c . To shorten the formulæ, we shall commence with the assumption that $v_1 = 1$ and $v_2 = 0$.

Let $f_1(x/a)$, where $AP = x$, denote the potential due to Q_1 at any point P on AB

FIG. 3.

which is inside the sphere A . Similarly, let $f_2\{(c-x)/b\}$ be the potential due to Q_2 at the point P .

Then, if x lies between $+a$ and $-a$, we have

$$f_1(x/a) + \{b/(c-x)\}f_2\{b/(c-x)\} = 1 \quad \dots \quad (25)$$

Similarly, if x_1 lies between $c+b$ and $c-b$, we have

$$(a/x_1)f_1(a/x_1) + f_2\{(c-x_1)/b\} = 0 \quad \dots \quad (26)$$

Now, writing

$$\frac{c-x_1}{b} = \frac{b}{c-x}$$

so that

$$\frac{a}{x_1} = \frac{a(c-x)}{c^2 - b^2 - cx}$$

and eliminating the second function between (25) and (26), we get

$$f_1\left(\frac{x}{a}\right) = 1 + \frac{ab}{c^2 - b^2 - cx} f_1\left\{\frac{a(c-x)}{c^2 - b^2 - cx}\right\} \quad \dots \quad (27)$$

At A , where $x=0$, we obviously have

$$f_1(0) = \frac{Q_1}{a} = \frac{k_{11}}{a}.$$

Thus

$$\frac{k_{11}}{a} = 1 + \frac{ab}{c^2 - b^2} f_1\left(\frac{ac}{c^2 - b^2}\right) \quad \dots \quad (28)$$

Let us introduce, as before, functions α , β and r , where r is the radius of the sphere which cuts A and B orthogonally, and

$$\sinh \alpha = \frac{r}{a}, \quad \sinh \beta = \frac{r}{b}. \quad \text{We write also}$$

$$\sinh \omega = \sinh (\alpha + \beta) = \frac{cr}{ab}, \quad \text{and}$$

$$c = a \cosh \alpha + b \cosh \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

We find that (28) becomes

$$k_{11} \frac{\sinh \alpha}{r} = 1 + \frac{\sinh \alpha}{\sinh (\alpha + \omega)} f_1 \left\{ \frac{\sinh \omega}{\sinh (\alpha + \omega)} \right\}.$$

And by (27)
$$f_1 \left\{ \frac{\sinh \omega}{\sinh (\alpha + \omega)} \right\} = 1 + \frac{\sinh (\alpha + \omega)}{\sinh (\alpha + 2\omega)} f_1 \left\{ \frac{\sinh 2\omega}{\sinh (\alpha + 2\omega)} \right\}$$

Proceeding in this way, we get by substitution that

$$k_{11} = \sum_0^{\infty} \frac{r}{\sinh (\alpha + n\omega)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence, from symmetry

$$k_{22} = \sum_0^{\infty} \frac{r}{\sinh (\beta + n\omega)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

At the centre of the sphere B , the potential due to the charge Q_2 is Q_2/b , that is, k_{12}/b . Hence from (26), putting $x=c$, we get

$$f_1(0) = \frac{k_{12}}{b} = -\frac{a}{c} f_1 \left(\frac{a}{c} \right)$$

Therefore

$$-k_{12} = \frac{r}{\sinh \omega} f_1 \left(\frac{\sinh \beta}{\sinh \omega} \right)$$

And from (27)

$$\begin{aligned} f_1 \left(\frac{\sinh \beta}{\sinh \omega} \right) &= 1 + \frac{\sinh \omega}{\sinh 2\omega} f_1 \left\{ \frac{\sinh (\beta + \omega)}{\sinh 2\omega} \right\} \\ f_1 \left\{ \frac{\sinh (\beta + \omega)}{\sinh 2\omega} \right\} &= 1 + \frac{\sinh 2\omega}{\sinh 3\omega} f_1 \left\{ \frac{\sinh (\beta + 2\omega)}{\sinh 3\omega} \right\} \end{aligned}$$

and so on. Hence

$$-k_{12} = \sum_1^{\infty} \frac{r}{\sinh n\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Comparing (30) with (10) we get at once

$$k_{11} = \frac{r}{\sinh \alpha} + K \left\{ \frac{r}{\sinh \alpha}, \frac{r}{\sinh (\alpha + \omega)}, \frac{r \sinh \omega}{\sinh \alpha \sinh (\alpha + \omega)} \right\}. \quad (33)$$

This may be written

$$k_{11}(a, b, c) = a + \frac{a^2}{c^2 - b^2} K \left\{ \frac{c^2 - b^2}{a}, b, c \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Similarly we find that

$$-k_{12}(a, b, c) = \frac{ab}{c} + \frac{a^2b^2}{c(c^2 - a^2 - b^2)} K \left\{ \frac{c^2 - a^2 - b^2}{ab}, 1, 1 \right\} \dots \dots \dots (35)$$

$$k_{22}(b, a, c) = b + \frac{b^2}{c^2 - a^2} K \left\{ \frac{c^2 - a^2}{b}, a, c \right\} \dots \dots \dots (36)$$

The computation of the capacity coefficients, therefore, reduces to calculating the capacities of spherical condensers and so we can calculate them by the formulæ given above.

NUMERICAL EXAMPLES.

Poisson (*l.c. ante*) considers the case for which $a=1$, $b=3$ and $c=5$. By (34) we get

$$k_{11}(1, 3, 5) = 1 + \frac{1}{5^2 - 3^2} K(16, 3, 5)$$

and by (12)
$$K(16, 3, 5) = 3 + \frac{3}{77} K(77, 16, 5)$$

If we only want a four-figure accuracy for k_{11} we can assume that

$$K(77, 16, 5) = K(77, 16, 0) = \frac{77.16}{77 - 16}$$

Hence
$$k_{11} = 1 + \frac{3}{16} + \frac{3}{61} = 1.2367$$

Similarly, by (35),
$$\begin{aligned} -k_{12} &= \frac{3}{5} + \frac{3}{25} K(5, 1, 1) \\ &= \frac{3}{5} + \frac{3}{25} + \frac{1}{200} K(24, 5, 1) \\ &= \frac{3}{5} + \frac{3}{25} + \frac{3}{85} = 0.7516. \end{aligned}$$

These numbers agree exactly with Poisson's calculations.

Similarly
$$k_{22} = 3 + \frac{3}{8} K(8, 1, 5)$$

Proceeding as before, we get $k_{22} = 3.4719$.

As another example, take the system (1, 7, 10).

In this case
$$k_{11} = 1 + \frac{1}{51} K(51, 7, 10) = \dots$$

Thus
$$k_{11} = 1.160 \ 112 \ 4 \dots$$

$$k_{22} = 7.576 \ 596 \ 7 \dots$$

and
$$-k_{12} = 0.814 \ 326 \ 5 \dots$$

THE CAPACITY BETWEEN TWO SPHERES.

When the charges on the two spheres are equal but opposite in sign, the ratio of the positive charge to the potential difference between the spheres is called the

capacity between the spheres. We shall denote it by $C(a, b, c)$ or by C . It is easy to show that

$$C(a, b, c) = \frac{k_{11}k_{22} - k_{12}^2}{k_{11} + k_{22} + 2k_{12}} \quad \dots \quad (37)$$

As a rule C is computed by first calculating the values of k_{11} , k_{22} and k_{12} and then substituting in (37). When, however, the spheres are equal to one another $k_{11} = k_{22}$ and (37) becomes

$$C(a, a, c) = \frac{1}{2}(k_{11} - k_{12}) \quad \dots \quad (38)$$

It readily follows from (34) and (35) that

$$C(1, 1, c) = \frac{1}{2} + \frac{1}{2c} K(c, 1, 1) \quad \dots \quad (39)$$

For example, take the case of two spheres, the radius of each being 1 centimetre and the distance between their centres being 4 cm. We have

$$\begin{aligned} C(1, 1, 4) &= \frac{1}{2} + \frac{1}{8} K(4, 1, 1) \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{120} K(15, 4, 1) \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{30} + \frac{1}{1680} K(56, 15, 1) \\ &= 0.670 \, 529 \, 88 \dots \end{aligned}$$

SPHERES CLOSE TOGETHER.

When the spheres are so close together that it is laborious to apply (34) and (35) we can use the formulæ.*

$$k_{11} = \frac{r}{\omega} \left\{ -\psi\left(\frac{\alpha}{\omega}\right) - \log_e \frac{\omega}{2} + \frac{\omega^2}{72} + \frac{7\omega^4}{43200} + \dots - \frac{\alpha\beta}{12} - \frac{7\alpha^2\beta^2}{1440} - \dots \right\} \quad (40)$$

$$\text{and} \quad -k_{12} = \frac{r}{\omega} \left\{ \gamma - \log_e \frac{\omega}{2} + \frac{\omega^2}{72} + \frac{7\omega^4}{43200} + \dots \right\} \quad (41)$$

where $\psi(x)$ denotes the logarithmic derivate of the gamma function and γ is Euler's constant (0.577216). We have

$$\psi(x) + \gamma = -\frac{1}{x} + S_2x - S_3x^2 + S_4x^3 - \dots$$

where $S_m = \sum \frac{1}{n^m}$, the summation being taken from $n=1$ to $n=\infty$.

Let C_a and C_b denote the capacities of the two spheres when touching one another. In this case we have

$$C_a = k_{11} + k_{12} = -\frac{ab}{a+b} \left\{ \psi\left(\frac{b}{a+b}\right) + \gamma \right\} = a \left\{ 1 - \left(\frac{b}{a+b}\right)^2 S_2 + \left(\frac{b}{a+b}\right)^3 S_3 - \dots \right\}$$

$$\text{and} \quad C_b = b \frac{ab}{(a+b)^2} \left\{ S_2 + \frac{b}{a+b} S_3 + \frac{b^2}{(a+b)^2} S_4 + \dots \right\}^\dagger$$

* Russell, *Alternating Currents*, Vol. 1, p. 240.

† J. J. Thomson, *Electricity and Magnetism*, Chap. 5.

It is to be noticed that when $b^3/(a+b)^3$ can be neglected compared with unity, we can write $C_a + C_b = a$. When the spheres are touching the following formulæ are exact :—

$$a=b, \quad C_a = a \log 2 = 0.6931a = C_b$$

$$a=2b, \quad C_a = \frac{a}{6} \left(3 \log 3 + \frac{\pi}{\sqrt{3}} \right); \quad C_b = \frac{a}{6} \left(3 \log 3 - \frac{\pi}{\sqrt{3}} \right)$$

$$= 0.8516a \qquad = 0.2470a$$

$$a=3b, \quad C_a = \frac{a}{4} \left(3 \log 2 + \frac{\pi}{2} \right); \quad C_b = \frac{a}{4} \left(3 \log 2 - \frac{\pi}{2} \right)$$

$$= 0.9126a \qquad = 0.1272a$$

$$a=5b, \quad C_a = \frac{a}{12} (\log 2 + 3 \log 6 + \pi \sqrt{3}), \quad C_b = \frac{a}{12} (\log 2 + 3 \log 6 - \pi \sqrt{3})$$

$$= 0.9592a \qquad = 0.0523a$$

When x/a and x/b can be neglected compared with unity, where $x=c-a-b$, we can show that

$$C_a + C_b = -\frac{ab}{a+b} \left\{ \psi\left(\frac{a}{a+b}\right) + \psi\left(\frac{b}{a+b}\right) + 2\gamma \right\} \quad \dots \quad (42)$$

and

$$C = \frac{r}{\omega} \log_e \frac{2}{\omega} + \frac{\gamma^2}{\omega^2} \frac{\psi\left(\frac{\alpha}{\omega}\right) \psi\left(\frac{\beta}{\omega}\right) - \gamma^2}{C_a + C_b} \quad \dots \quad (43)$$

$$= \frac{ab}{2(a+b)} \log_e \frac{2ab}{(a+b)x} + \frac{a^2b^2}{(a+b)^2} \cdot \frac{\psi\left(\frac{a}{a+b}\right) \psi\left(\frac{b}{a+b}\right) - \gamma^2}{C_a + C_b} \quad \dots \quad (44)$$

where C is the capacity between the spheres.

For example, when $a=b$, we have, since $\psi(1/2) = -\gamma - 2 \log 2$ and $C_a + C_b = 2a \log 2$

$$C = \frac{a}{4} \log \frac{a}{x} + \frac{a}{4} (2\gamma + 2 \log 2)$$

$$= \frac{a}{4} \left\{ \log \frac{a}{x} + 2.540726 \dots \right\} \quad \dots \quad (45)$$

When b/a is very small (44) becomes—

$$C = \frac{b}{2} \log \frac{2b}{x} + \gamma b. \quad \dots \quad (46)$$

This formula gives the capacity between a sphere of radius b and an infinite plane.

Hence, since $-k_{12} = C$ in this case, we get by (35)

$$K \left\{ b, \frac{b^2}{2(b+x)}, \frac{b^2}{2(b+x)} \right\} = \frac{b}{2} \log \frac{2b}{x} + (\gamma - 1)b, \quad \dots \quad (47)$$

when x is very small.

THE POTENTIALS AT THE INVERSE POINTS DUE TO Q_1 AND Q_2 RESPECTIVELY.

Looking back at (27) we see that the potential at the centre of the sphere A , where $x=0$, equals 1+the potential at a point $x=a_1$, multiplied by $\sinh a/\sinh(a+\omega)$. Similarly the potential at the point $x=a_{n-1}$ equals 1+the potential at the point $x=a_n$ multiplied by $\sinh(a+n-1\omega)/\sinh(a+n\omega)$. We also see that $a_n=a \sinh n\omega/\sinh(a+n\omega)$.

Hence, when n is very great

$$a_{n-1}=a_n=a\epsilon^{-\alpha}=AP,$$

where P is the inverse point of the two spheres which is inside A . Hence, if V_P be the potential at this point due to Q_1 , we have

$$V_P=1+\epsilon^{-\omega}V_P, \text{ and so } V_P=\epsilon^{\omega}/(\epsilon^{\omega}-1).$$

Hence, also $V_Q=\epsilon^{\beta}/(\epsilon^{\omega}-1)$. If V'_P, V'_Q be the potentials due to Q_2 , we see that

$$V'_P=-1/(\epsilon^{\omega}-1) \text{ and } V'_Q=-\epsilon^{\beta}/(\epsilon^{\omega}-1).$$

We see, therefore, by superposition, that when the potentials of the spheres are V_1 and V_2 , we have

$$\begin{aligned} (\epsilon^{\omega}-1)V_P &= \epsilon^{\alpha}(V_1\epsilon^{\beta}-V_2); \quad (\epsilon^{\omega}-1)V_Q = V_1\epsilon^{\beta}-V_2 \\ (\epsilon^{\omega}-1)V'_P &= V_2\epsilon^{\alpha}-V_1; \quad (\epsilon^{\omega}-1)V'_Q = \epsilon^{\beta}(V_2\epsilon^{\alpha}-V_1) \end{aligned}$$

Hence, if $V_1/V_2=\epsilon^{-\beta}$, $V_P=V_Q=0$; and $V'_P=V_1$, $V'_Q=V_2$.

ELECTRICAL IMAGES OF SPHERES.

If the surfaces of two spheres A and B are reflecting there will be a series of images formed inside A , and a similar series formed inside B . From this

FIG. 4.

optical analogy Kelvin called the inverse spheres $A_1, A_2, \dots$ and $B_1, B_2, \dots$ where A and B are the centres of inversion (Fig. 4), the electrical images of the two spheres. It has to be noted that these electrical images do not coincide with the optical images. A_1 is the electrical image of B formed by A , and B_1 is the image of A formed by B . Similarly A_2 is the image of B_1 formed by A , and B_2 is the image of A_1 formed by B . In general A_n is the image of B_{n-1} , and B_n is the image of A_{n-1} .

It is not difficult to show that A_2 is the image (inverse sphere) of A formed by A_1 , and in general that A_n is the image of A_{n-2} formed by A_{n-1} . If R_n be the radius of A_n and a_n the distance of its centre from A , it can be shown that

$$R_n=r/\sinh(a+n\omega) \text{ and } a_n=a \sinh n\omega/\sinh(a+n\omega).$$

Hence, when n is very large $a_n = ae^{-n}$ and the n th image practically coincides with the inverse point P .

By formula (33) we get

$$k_{11}(a, b, c) = a + K(a, R_1, a_1),$$

and by (12)

$$K(a, R_1, a_1) = R_1 + K(R_1, R_2, a_2 - a_1),$$

and

$$K(R_1, R_2, a_2 - a_1) = R_2 + K(R_2, R_3, a_3 - a_2), \text{ etc.}$$

Thus

$$k_{11}(a, b, c) = a + R_1 + R_2 + \dots \quad (48)$$

which agrees with (30).

Interpreting the equations we see that k_{11} equals the radius of A , together with the capacity of the spherical condenser formed by A and A_1 . The capacity of this condenser equals the radius of A_1 plus the capacity of the condenser formed by A_1 and A_2 , and so on. Hence the images divide the sphere A into an infinite number of spherical condensers, the capacity between any consecutive pair A_{n-1} and A_n being equal to the radius of A_n plus the capacity between the next pair A_n and A_{n+1} .

In the corresponding case of two cylinders it will be found that the capacities between the consecutive cylindrical images (inverse cylinders) are all equal to one another, and they also equal the capacity between the two cylinders.

It is interesting to note that the centres of the sphere A and of the images $A_1, A_2, \dots$ coincide with the points where we have to place Kelvin's charges $q_0, q_1, q_2, \dots$ so that these charges may produce a field external to the sphere which is identical with the field produced by the charge on A when its potential is unity, and that of B is zero. Kelvin's charges* also are given by $q_0 = a, q_1 = R_1, q_2 = R_2, \dots$

Both Poisson's and Kelvin's methods give the magnitudes and positions of the electrical image spheres.

ATTRACTIONS AND REPULSIONS OF TWO ELECTRIFIED SPHERES.

Let W denote the electrostatic energy of the system. Then

$$W = \frac{1}{2}k_{11}v_1^2 + \frac{1}{2}k_{22}v_2^2 + k_{12}v_1v_2 \quad (49)$$

This can also be written in the form

$$W = \frac{1}{2}p_{11}q_1^2 + \frac{1}{2}p_{22}q_2^2 + p_{12}q_1q_2 \quad (50)$$

where $p_{11} = k_{22}/\Delta^2, p_{22} = k_{11}/\Delta^2, p_{12} = -k_{12}/\Delta^2$ and $\Delta^2 = k_{11}k_{22} - k_{12}^2$. (51)

If we suppose the spheres to move, the rate at which the potential energy increases is given by

$$\frac{\delta W}{\delta c} = \frac{1}{2} \frac{\delta k_{11}}{\delta c} v_1^2 + \frac{1}{2} \frac{\delta k_{22}}{\delta c} v_2^2 + \frac{\delta k_{12}}{\delta c} v_1v_2 = n v_1v_2 - \frac{1}{2} l v_1^2 - \frac{1}{2} m v_2^2 \quad (52)$$

Where $l = -\delta k_{11}/\delta c, m = -\delta k_{22}/\delta c$ and $n = \delta k_{12}/\delta c$, and l, m and n are positive quantities. The value of $\delta W/\delta c$ gives the force acting between the two spheres.

Similarly we find from (50) that when q_1 and q_2 are kept constant,

$$\frac{\delta W}{\delta c} = \frac{1}{2} \frac{\delta p_{11}}{\delta c} q_1^2 + \frac{1}{2} \frac{\delta p_{22}}{\delta c} q_2^2 + \frac{\delta p_{12}}{\delta c} q_1q_2.$$

* Russell, Alternating Currents, Vol. 1, p. 238.

Substituting for p_{11} , p_{22} and p_{12} their values from (51), and writing for q_1 and q_2 , $k_{11}v_1 + k_{12}v_2$ and $k_{22}v_2 + k_{12}v_1$, we readily find that

$$\frac{\delta W}{\delta c} = -(n v_1 v_2 - \frac{1}{2} l v_1^2 - \frac{1}{2} m v_2^2) \dots \dots \dots (53)$$

Hence
$$\frac{\delta W}{\delta c} (v_1, v_2 \text{ constant}) = - \frac{\delta W}{\delta c} (q_1, q_2 \text{ constant}) \dots \dots \dots (54)$$

By the conservation of energy we see that when q_1 and q_2 are constant, the work done by the mutual force between them in moving the spheres must be taken from the electrostatic energy. Hence the spheres move so as to diminish the electrostatic energy. It follows also from (54) that when their potentials are maintained constant the spheres move so that the electrostatic energy is increased.

We see from (52) and (53) that $\delta W / \delta c$ vanishes when

$$\frac{v_1}{v_2} = \frac{n}{l} \pm \frac{\sqrt{n^2 - lm}}{l}$$

and
$$\frac{q_1}{q_2} = \frac{k_{11}(v_1/v_2) + k_{12}}{k_{22} + k_{12}(v_1/v_2)} \dots \dots \dots (55)$$

When for a given value of c either of these conditions is satisfied, the electrostatic energy at this distance is a maximum when the charges are kept constant, and a minimum when the potentials are kept constant. In order that (55) be satisfied it is necessary that v_1 and v_2 have the same sign. If they have opposite signs the electrostatic energy continually diminishes as the spheres approach one another when the charges are kept constant and continually increases when the potentials are maintained constant.

When the charges and therefore the potentials are of unlike sign, the mutual force is always attractive. When, however, they are of like sign the force is in general attractive when they are close together and repulsive when they are far apart. In an intermediate position determined by (55) the mutual force is zero.

SPHERES CLOSE TOGETHER.

When the spheres are at potentials v_1 and v_2 , and the distance x between them is small compared with the radius of either, the force F between them is given by

$$\begin{aligned} F = & - \frac{(v_1 - v_2)^2 ab}{4x(a+b)} \left\{ 1 + \frac{(a^2 + b^2 - ab)x}{3ab(a+b)} \log \frac{(a+b)x}{2ab} + \frac{(2a^2 + 2b^2 + ab)x}{18ab(a+b)} \right\} \\ & - \frac{a^2 + b^2 - ab}{6(a+b)^2} \left\{ \psi \left(\frac{b}{a+b} \right) v_1^2 + \psi \left(\frac{a}{a+b} \right) v_2^2 + 2\gamma v_1 v_2 \right\} \\ & - \frac{ab(a-b)}{6(a+b)^3} \left\{ \psi' \left(\frac{b}{a+b} \right) v_1^2 - \psi' \left(\frac{a}{a+b} \right) v_2^2 \right\} \\ & - \frac{ab}{6(a+b)^2} v_1 v_2 \dots \dots \dots (56) \end{aligned}$$

When F is negative, the force is attractive, and when F is positive it is repulsive. It will be seen that when x is very small we only have repulsion when v_1 is very nearly equal to v_2 . In all other cases the force is attractive.

If $v_1=v_2=v$, the force is repulsive. When $a=b$, $F=0.07386v^2$; when $a=3b$, $F=0.0430v^2$; and when a is very large compared with b , the force vanishes.

When $a=b$, (56) reduces to

$$F = -\frac{a(v_1-v_2)^2}{8x} \left\{ 1 - \frac{12\gamma+1}{36} \cdot \frac{x}{a} - \frac{1}{6} \frac{x}{a} \log \frac{a}{x} \right\} + \frac{\log 2 - \frac{1}{2}}{12} (v_1^2 + v_2^2) \quad (57)*$$

When $v_1=v_2=v$, we get

$$F = 0.073858v^2 \quad (58)†$$

a formula that was given by Kelvin.

We see, however, that when x is very small the slightest disparity between the values of v_1 and v_2 will make the force attractive.

EQUAL SPHERES.

When the distance between two equal spheres is greater than the radius of either, the force F between them is given very approximately by

$$F = \left\{ 1 + \frac{14}{y^3} + \frac{54}{y^5} \right\} \frac{q_1 q_2}{c^2} - \left\{ \frac{2}{y^3} + \frac{3}{y^5} + \frac{4}{y^7} + \frac{25}{y^3(y^2-1)^3} + \frac{75}{y^5(y^2-1)^3} \right\} \frac{q_1^2 + q_2^2}{c^2} \quad (59)‡$$

where $y=c/a$.

It is also given by

$$F = \left\{ \frac{1}{y^2-4} - \frac{1}{y^4} - \frac{1}{y^6} - \frac{1}{y^8-4y^4} \right\} v_1 v_2 - \left\{ \frac{1}{y(y^2-4)} - \frac{1}{y(y^3-2y)} \right\} (v_1^2 + v_2^2) \quad (60)‡$$

When F is negative the force is attractive, and when F is positive it is repulsive. For example, when $c=4a$, the formulæ become

$$F = 1.004 \frac{q_1 q_2}{c^2} - 0.03456 \frac{(q_1^2 + q_2^2)}{c^2}$$

and

$$F = 0.0792 v_1 v_2 - 0.0208 (v_1^2 + v_2^2).$$

In this case F vanishes when $v_1/v_2=0.283$, which is the same condition as $q_1/q_2=$

0.0344. When c/a is large, $F = \frac{q_1 q_2}{c^2} - \frac{2a^3}{c^3} (q_1^2 + q_2^2)$, and $F = \frac{a^2}{c^2} v_1 v_2 - \frac{a^3}{c^3} (v_1^2 + v_2^2)$.

In using the formulæ given above it has to be remembered that when the pressure v and the charge q are given in electrostatic units the force F is given in dynes.

The electrostatic unit of pressure equals 300 volts and the electrostatic unit of charge (quantity) equals $1/(3 \times 10^9)$ coulombs.

ANALOGOUS PROBLEMS.

The formulæ given above can be usefully applied in the theory of heat, in electric and magnetic conduction, and in the theory of hydrodynamics. Consider, for instance, two spheres deeply embedded in a thermally homogeneous medium and

* Proc. Roy. Soc., Vol. 82, p. 531.

† Reprint, p. 83.

‡ Journal Inst. of El. Eng., Vol. 48, p. 237

maintained at constant temperatures θ_1 and θ_2 . Let Q_1 and Q_2 denote the flow of heat per second from each sphere. When the flow has become steady, we get

$$Q_1 = \frac{4\pi}{\sigma} (k_{11}\theta_1 + k_{12}\theta_2) \text{ and } Q_2 = \frac{4\pi}{\sigma} (k_{22}\theta_2 + k_{12}\theta_1),$$

where σ is the thermal resistivity, and the values of k_{11} , k_{22} and k_{12} can be computed from the formulæ given above. In the particular case when $Q_1 = -Q_2 = Q$, we have

$$\frac{Q}{\theta_1 - \theta_2} = \frac{4\pi}{\sigma} K \quad \dots \dots \dots (61)$$

where K is the electrostatic capacity between the two spheres.

Similarly, in electric conduction, we have, with the usual notation

$$I_1 = \frac{4\pi}{\rho} (k_{11}V_1 + k_{12}V_2); \quad I_2 = \frac{4\pi}{\rho} (k_{22}V_1 + k_{12}V_2)$$

and when $I_1 = -I_2 = I$ we get

$$\frac{I}{V_1 - V_2} = \frac{4\pi}{\rho} K \quad \dots \dots \dots (62)^*$$

where ρ is the electric resistivity.

We see that when V_1 and V_2 are maintained constant and are of the same

FIG. 5.

sign, the power supplied to the medium in which the spheres are immersed has, in general, a minimum value for a definite value of c .

THE POTENTIAL AT ANY EXTERNAL POINT.

Let the point P (Fig. 5) be at distances r_1 and r_2 from A and B and let the angles PAB , PBA be θ_1 and θ_2 respectively. Let the potentials of the spheres be V_1 and 0 respectively. So far as the electric field outside the two spheres is concerned, we can replace the charges on the spheres by point charges $q_0, q_1, q_2, \dots$ at $A, A_1, A_2, \dots$ and $q'_1, q'_2, \dots$ at $B_1, B_2, \dots$ where

$$q_n = aV_1 \frac{\sinh \alpha}{\sinh (n\omega + \alpha)}, \quad q'_n = -aV_1 \frac{\sinh \alpha}{\sinh n\omega}$$

$$AA_n = a_n = a \frac{\sinh n\omega}{\sinh (n\omega + \alpha)} \text{ and } BB_n = b_n = b \frac{\sinh \{(n-1)\omega + \alpha\}}{\sinh n\omega}$$

* Jeans, Electricity and Magnetism, p. 351.

Thus the potential V at the point P due to the given charges is got by

$$\begin{aligned}
 V = & \frac{q_0}{r_1} + \frac{q_1}{(r_1^2 - 2r_1a_1 \cos \theta_1 + a_1^2)^{\frac{1}{2}}} \\
 & + \frac{q_2}{(r_1^2 - 2r_1a_2 \cos \theta_1 + a_2^2)^{\frac{1}{2}}} + \dots \\
 & + \frac{q'_1}{(r_2^2 - 2r_2b_1 \cos \theta_2 + b_1^2)^{\frac{1}{2}}} + \dots \\
 = & \frac{k_{11}V_1}{r_1} + \frac{k_{12}V_1}{r_2} \\
 & + \frac{a^2V_1P_1(\cos \theta_1)}{r_1^2} \sum_1 \frac{\sinh \alpha \sinh n\omega}{\sinh^2(n\omega + \alpha)} \\
 & + \frac{a^3V_1P_2(\cos \theta_1)}{r_1^3} \sum_1 \frac{\sinh \alpha \sinh^2 n\omega}{\sinh^2(n\omega + \alpha)} + \dots \\
 & - \frac{b^2V_1P_1(\cos \theta_2)}{r_2^2} \sum_1 \frac{\sinh \alpha \sinh (n-1\omega + \alpha)}{\sinh^2 n\omega} - \dots \quad (61)
 \end{aligned}$$

where $P_1(\cos \theta_1)$, $P_2(\cos \theta_2)$. . . are zonal harmonics.

If the second sphere be at potential V_2 , we have to add to the series (61) a similar series which can be written down by symmetry at once.

THE MAXIMUM VALUE OF THE ELECTRIC STRESS.

It is important in practice to know the maximum values of the electric stress at points on the surface of the sphere. These values obviously occur at the points Q and S (Fig. 5). Denoting the maximum value of the stress at Q by $R_{\max}$, we get*

$$\begin{aligned}
 R_{\max} = & \frac{\cosh^2(\alpha/2)}{a \sinh(\alpha/2)} \left[V_1 \left\{ \frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(\alpha/2 + \omega)}{\cosh^2(\alpha/2 + \omega)} + \dots \right\} \right. \\
 & \left. - V_2 \left\{ \frac{\sinh(\omega - \alpha/2)}{\cosh^2(\omega - \alpha/2)} + \frac{\sinh(2\omega - \alpha/2)}{\cosh^2(2\omega - \alpha/2)} + \dots \right\} \right] \quad (62)
 \end{aligned}$$

The value of the stress at S is got by writing V_2 for V_1 , b for a and β for α in (62).

When the spheres are equal, the maximum value of the electric stress between them is given by

$$R_{\max} = \frac{V_1 - V_2}{x} f_1 + \frac{2V_2}{x} (f_1 - f) \quad (63)$$

$$\begin{aligned}
 \text{where} \quad f = & \sinh \alpha \cosh(\alpha/2) \left\{ \frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(3\alpha/2)}{\cosh^2(3\alpha/2)} + \dots \right\} \\
 f_1 = & 2 \sinh \alpha \cosh(\alpha/2) \left\{ \frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(5\alpha/2)}{\cosh^2(5\alpha/2)} + \dots \right\}
 \end{aligned}$$

and $\cosh \alpha = 1 + x/(2a)$.

Simplified formulae for computing f and f_1 are given in the Journ. of the Inst. of El. Engin., Vol. LVII., p. 223.

* Russell, Alternating Currents, Vol. 1, p. 253.

It is also shown that for a given pair of spherical electrodes the value of $R_{\max.}$ at which a disruptive discharge takes place is constant for distances between the spheres varying from about 1 mm. up to a distance equal to three times the diameter of either electrode. The author has shown that the value of $R_{\max.}$ at 25°C. and 76 cm. pressure for electrodes the diameters of which vary in size from half a centimetre to 50 cm. is given by

$$R_{\max.} = 27.4 + \frac{14.1}{\sqrt{a}} \quad \dots \dots \dots (64)$$

when the potentials of the electrodes are equal and opposite at the instant of discharge; a is the radius of an electrode in centimetres and $R_{\max.}$ is given in kilovolts per centimetre. The maximum inaccuracy of (64) over the range mentioned is approximately 1 per cent.

When $V_1 = -V_2 = E/2$, formula (63) becomes

$$R_{\max.} = \frac{E}{x} f$$

and thus

$$E = \left(27.4 + \frac{14.1}{\sqrt{a}} \right) \frac{x}{f} \quad \dots \dots \dots (65)$$

If we fix the value of x it is easy to see from this formula that in certain cases E has a maximum value for a given value of a . For instance, if x is 1 cm., then E has the greatest value when $a = 2.53$ cm., that is, approximately 1 in. Taking the experimental values of E from the Paper referred to above, we get:—

Radius of each electrode in cm.	0.25	0.5	1	3.125	6.25	12.5	25
Disruptive pressure in kilovolts	20.8	26.7	30.5	31.9	31.5	30.6	29.8

The results of the first three cases were given by A. Heydweiller* and the last four are taken from the standardisation Rules of the American Institution of Electrical Engineers.

THE MAXIMUM POSSIBLE ATTRACTION BETWEEN SPHERES IN AIR.

Taking Heydweiller's results for spheres 1 cm. in diameter, we readily find that the attractions between the electrodes at the instant of discharge are given by the following table:—

x in cm.	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
E in kilovolts...	8.37	11.37	14.55	17.31	19.92	22.05	24.09	25.59	27.00
F in dynes	193.1	219.0	250.0	264.6	274.5	271.7	268.7	256.3	244.4

The maximum attraction occurs when x equals $1.2a$ approximately.

Let a pair of spherical electrodes each have a radius a and be at a distance x apart. Let F_1 be the attraction between them when the disruptive voltage is E_1 . The potentials are supposed equal and opposite at the instant of discharge. Consider also a pair of electrodes the radii of which are each equal to na , and let the

* Entladungspotentiale Ann. der Phys. und Chemie, Vol. 48, p. 235.

distance between them be $n\kappa$. Let F_2 be the force between them at the instant of the disruptive discharge. Then $F_1 = aE_1^2$ and $E_1 = (27.4 + 14.1/\sqrt{a})(\kappa/f)$, where a and f depend only on the ratio κ/a . It readily follows that

$$\frac{F_2}{F_1} = \left\{ n \frac{27.4 + 14.1/\sqrt{na}}{27.4 + 14.1/\sqrt{a}} \right\}^2 \cdot \cdot \cdot \cdot \cdot \cdot \quad (66)$$

Hence this ratio is independent of κ and so if κ' be the distance between the first pair of electrodes which gives to F_1 a maximum value, F_2 will have a maximum value when the distance between the spheres in the second case is $n\kappa'$.

We saw above that when $a = \frac{1}{2}$ cm. approx. F has its maximum value when $\kappa' = 0.6$ cm. Let us take $n = 10$, then if F_2 be the attraction for electrodes of 5 cm. radius at the instant of discharge $F_2 = 50.8F_1$, and so the maximum value of F_2 is nearly 13940 dynes.

An electrostatic balance therefore having spherical electrodes each ten centimetres in diameter and 5 centimetres apart would be very suitable for measuring potential differences up to 100,000 volts.

DISCUSSION.

Sir OLIVER LODGE (communicated) : I feel sure that Dr. Russell's Paper is a valuable contribution to knowledge ; and I admire the way in which he has tackled the problem of the two electrified spheres. An electrostatic balance with spherical electrodes, instead of flat plates, would have many advantages, provided always that the surroundings did not exercise too much disturbance, and provided dust particles or irregularities on the spheres did not promote an undesired kind of brush discharge accompanied by air currents.

Mr. T. SMITH congratulated the President on a valuable piece of work. Having cleared up a subject at which physicists had been working for so long, the author would have been entitled to employ as his unit of length a characteristic wave-length of the most widely distributed element ; which, as Clerk Maxwell had remarked, was the only unit fitted for those investigators whose work might outlast the earth ! In regard to the use of "image" in an optical sense, in the course of the Paper, this hardly seemed the correct word to employ.

Dr. D. OWEN said that in investigating the attraction between conductors in the case where the disruption point was being reached, it had to be remembered that ions were present, which would interfere with the distribution of the electric field. This lent some uncertainty to practical methods based on this method for measuring voltages ; the time factor entered into this question. The same remark applied to the estimation of voltage in terms of the distance between spheres at which sparking just occurred.

The AUTHOR, in reply, said that it might be more correct to speak of the "inverse" of one sphere with respect to the other. The equation (64) to which Dr. Owen referred was an empirical one. Possibly the presence of ions might account for this formula, as theoretically the value of $R_{\max}$ should be constant.