Note on Riemann's paper „Versuch einer allgemeinen
Auffassung der Integration und Differentiation“
Werke pp. 331—344.

By

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The Editors of Riemann's works remark that the paper in question was contained in a M.S. of his student time (dated 14. Jan. 1847) and was probably never intended for publication: indeed that he would not in later years have recognised the validity of the principles upon which it is founded. The idea is however a noticeable one: Riemann considers z_{x+h} , a function of $x+h$, expanded in a doubly infinite, necessarily divergent, series of integer or fractional powers of h , according to the law

$$(2) \quad z_{x+h} = \sum_{v=-\infty}^{v=+\infty} k_v \partial_x^v z \cdot h^v,$$

where the meaning is explained to be that the exponents differ from each other by integer values, in effect that v has all the values $\alpha + p$, α a given integer or fractional value, and p any integer number from $-\infty$ to $+\infty$, zero included.

Riemann deduces a theory of fractional differentiation: but without considering the question which has always appeared to me to be the great difficulty in such a theory: what is the real meaning of a complementary function containing an infinity of arbitrary constants? or, in other words, what is the arbitrariness of the complementary function of this nature which presents itself in the theory?

I wish to point out the relation between the paper referred to, and a short paper of my own „On a doubly infinite Series“ Quart. Math. Jour. t. VI (1851) pp. 45—47: this commences with the remark „The following completely paradoxical investigation of the properties of the function Γ (which I have been in possession of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series.“ And I then give the expansion

$$C_n e^x = \sum_r [n-r]^r x^{n-1-r}$$

where n is any integer or fractional number whatever, and the summation extends to all positive and negative integer values (zero included) of r . And I remark that, n being an integer, we have $C_n = \Gamma(n)$, and hence that assuming that this is so in general, or writing

$$\Gamma(n) \cdot e^x = \sum_r [n-1]^r x^{n-1-r},$$

we have this equation as a definition of $\Gamma(n)$. The point of resemblance of course is that we have a doubly infinite expansion of e^x in a series of integer or fractional powers of x , corresponding to Riemann's like expansion of z_{x+h} in powers of h .

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