A Graphic Method of Optical Imagery

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XV. *A Graphic Method of Optical Imagery.* By WILLIAM R. BOWER, B.Sc., A.R.C.S., Technical College, Huddersfield.

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1. The object of this Paper is to show how the important properties of an optical system may be readily deduced from diagrams drawn on the following simple principle: The vertex (called the object or radiant) of a small incident pencil of rays is considered to move along a line parallel to a principal axis of the system. This line is also regarded as the direction of the incident portion of a fixed ray. Then the directions of the ray after refraction or reflection at the several surfaces do not alter as the position of the object changes. These directions are therefore the fixed loci of the images or vertices of the refracted and reflected pencils. By the diagrams obtained, the path of any ray right through the system may be traced and the successive positions of the images resulting from a progressive movement of the object exhibited. Also the constructions are such that, in the case of a thick lens, the importance of the cardinal points and the planes associated with them is demonstrated. I ask especially for the criticism of teachers, to them and their less advanced pupils the method appeals, because it visualises what is usually accomplished.

by somewhat mechanical algebraic treatment. Graphic methods have previously been given by which the incident and emergent portions of a ray through an optical system can be identified. Such methods may be convenient to adopt after a time, but the beginner is like a young traveller who wants to know not only where to start and where to finish, but also something of what happens on the way.

Rule of Signs.—The one adopted is optically old-fashioned. The $+$ sense of a line is the reverse to that of the incident light: if this travels from right to left, then the $+$ sense of a line is from left to right. Also the $+$ sense of vertical lines is from below upwards.

I. SPHERICAL SURFACE.

2. *Refraction at a Spherical Surface* (Fig. 1).—Let AB represent the spherical refracting surface, centre C, radius r , sepa-

FIG. 1.—REFRACTION AT A SPHERICAL SURFACE.

rating the media M_1, M_2 , for which the index of refraction is μ . Assume $\mu = (\text{velocity of light in } M_1) / (\text{velocity of light in } M_2)$. Through any point O, and each of any two points B, B_1 , on the circle AB, draw BO, B_1O . Draw the circular arcs Bb, B_1b_1 , having their centre at O, and of radii BO, B_1O , respectively.

Suppose Bb, B_1b_1 represent two positions of a portion of a spherical wave. This undergoes refraction between B, B_1 . Let Bc, B_1c_1 be the corresponding positions of the refracted wave, and c_1BJ be the normal at c_1 to the refracted wave. Then $Bb_1 = \mu \cdot Bc_1$.

On BC as diameter describe a circle cutting BO in Y and BJ in Y' . Draw YY' , also CK perpendicular to YY' . Produce CK to cut BO in Z and BJ in Z' .

When B, B_1 are very close, the arcs BB_1, Bb, Bc are sensibly straight and perpendicular to the respective radii BC, BO, BJ . Then the triangles Bc_1b_1 and $CY'Y$ are similar. $\therefore CY = \mu \cdot CY'$. Then Y' is a fixed point when the line BY is fixed relatively to AB .

3. It is now convenient to deal with rays instead of waves. Let ϕ, ϕ' be the angles of incidence and refraction respectively.

(i.) Since $\mu = CY/CY'$, $\therefore \sin \phi / \sin \phi' = \sin CY'Y / \sin CYY' = \mu$. Thus the second law of refraction is deduced.

(ii.) The angles $CZ'B, CY'K$ are each equal to ϕ , and CZB, CYK to ϕ' .

Then, by similar triangles, $BZ/BZ' = CY/CY' = \mu$.

Also $BZ/CZ = BZ'/BC$; $\therefore CZ = \mu r$.

Also $CZ'/BZ' = BC/BZ$; $\therefore CZ' = r/\mu$.

Thus the locus of Z for any incident ray is the circle of radius μr and centre C . And that of Z' is the concentric circle of radius r/μ .

Z, Z' are the *aplanatic pair of points* for an incident pencil diverging from Z , whose chief ray is along CZ .

Draw AC parallel to the incident ray, cutting BZ' at G' and the circular loci of Z, Z' at T, T' . Also draw GC parallel to BG' , cutting OB in G . Then $BG'/CG' = CY/CY' = \mu$. And $GB = CG' = BG'/\mu$.

(iii.) As the angle of incidence CBZ diminishes, Z, Z' move along their circular loci towards T, T' respectively; K also approaches C , its locus being the circle on CY as diameter.

4. Refraction at a Spherical Surface : Normal Incidence.

(iv.) At normal incidence, that is, when ϕ is very small, Y' sensibly lies on CY and K coincides with C . Also Z and Z' are close to T, T' , respectively; and, sensibly, $BZ = AT = (\mu + 1)r$; $BZ' = AT' = (\mu + 1)r/\mu$.

Suppose F and F' to be the respective positions of G, G' when ϕ is 0. Then $AF' = BF' = \mu \cdot CF' = \mu(AF' - AC) = \mu r/(\mu - 1)$. And $BF = F'C = -(AF' - AC) = -r/(\mu - 1)$.

F, F' are respectively the first and second principal foci of a pencil incident normally at A .

(v.) When the incidence is normal it is convenient to increase largely the scale of length perpendicular to the principal axis in comparison with that along the axis. This is legitimate in an elementary discussion restricted to formulæ

of the first order, provided that trigonometric relations are used cautiously. The difference of scale gives a distorted figure in which angles that are actually equal appear unequal. But the relative lengths of lines perpendicular to the axis will be maintained, and also that of lengths parallel to the principal axis.

(vi.) In Fig. 2, let AB represent the spherical surface and C its centre.

Draw any incident ray, BO, parallel to the principal axis, AC. Draw CY perpendicular to BO and cut off $CY' = CY/\mu$. Draw the refracted ray BY', cutting AC in F'.

FIG. 2.—REFRACTION AT A SPHERICAL SURFACE. NORMAL INCIDENCE.

When the object is in some position O, draw CO, cutting the refracted ray at I. Then I is the image of O.

When O is at infinity, I coincides with F'. Thus, F' is the *second principal focus*.

When O is at a point Q, such that QC is parallel to the refracted ray BF', then the image is at infinity. Draw FQ parallel to AB, cutting AC in F. Then F is the *first principal focus*.

$$\text{Then } CF' = FA; FC = AF'; AF' = AC + FA.$$

$$\text{Also } AF'/FA = FC/FA = CY/CY' = \mu.$$

$$\text{Also } FA/AC = AY/Y'Y = 1/(\mu - 1).$$

$$\text{Also } AF'/AC = AB/\gamma B = \mu/(\mu - 1).$$

Through I draw αI parallel to AC, cutting AB in α . Then O α F are collinear. $\therefore QO/BO=AB/\alpha B=AF'/\alpha I$.

$$\therefore (FA+BO)/BO=AF'/\alpha I.$$

5. It is convenient to express some of these results algebraically. Write r for the radius, u, v, f, f' for the respective distances of the object, image, first and second principal foci from the refracting surface, and regard them as $+$ when measured in the opposite sense to that of the light, and $-$ when in the same sense. Then

$$(i.) f' + \mu \cdot f = 0.$$

$$(ii.) f + f' = r.$$

$$(iii.) f = -r/(\mu - 1).$$

$$(iv.) f' = r \cdot \mu/(\mu - 1).$$

$$(v.) \frac{f}{u} + \frac{f'}{v} = 1.$$

$$(vi.) \frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

6. *Construction for the Principal Foci, F, F', when r and μ are given.*—Draw, as above, AC = r ; AB; BO at any distance; CY. Make CY' = CY/ μ . Draw BY'F'; then $\gamma Y'$ parallel to AC, cutting AB in γ ; then Y γ , cutting AC in F.

Further construction for the Aplanatic Points, Z, Z'.—Make CC' = r . Draw C' γ' perpendicular to CC', meeting $\gamma Y'$ in γ' . Draw C γ' , cutting BO in ζ and BF' in ζ' . Draw ζZ and $\zeta' Z'$ perpendicular to AF' and meeting it in Z, Z'.

II. THICK LENS.

7. *Thick Lens.*—In Fig. 3, suppose two spherical surfaces to separate media M_1, M_2, M_3 . Draw the principal axis through the respective centres C_1, C_2 , and cutting the surfaces at A_1, A_2 . Mark off F_1, F_1' , the first and second principal foci of the surface through A_1 , and similarly F_2, F_2' for the other surface.

Take any position, O, of the luminous point.

Draw OB₁ parallel to the principal axis. This represents the incident ray from O that is parallel to the principal axis. B₁F₁' is the corresponding refracted ray; this is incident upon the second refracting surface at B₂.

Draw C₁O. The point L where it cuts B₁F₁' is the image of O formed by the first surface. Draw F₂L cutting the surfaces in P₁, P₂ respectively. Draw P₂I parallel to the principal axis, also C₂LI meeting P₂I at I. Then I is the final image formed by the system.

Draw B₂I crossing A₁A₂ at F'. This line is the direction of B₂B₁ after refraction at the second surface. Then the lines

B_1O , B_1B_2 , B_2I are the directions of what may be called the *second fixed ray* of the system, since their positions are unaltered, and also that of B' (the intersection of B_2F' and B_1O) for every point occupied by the object along B_1O .

Draw P_1O crossing A_1A_2 at F ; produce to meet P_2I at P . Through F' , B' , P and F draw parallels to A_1B_1 , and mark off the obvious crossing points P' , A' , A , B and Q .

8. In dealing with the loci of the images formed by the surfaces as the position of the object is systematically altered,

FIG. 3.—THICK LENS.

it is convenient to consider a radiant O , that is the vertex of a normally incident pencil, to pass along OB_1 from the extreme right to the extreme left. To show that—

(i.) F and F' are fixed points. All rays that emerge parallel to A_1A_2 diverge from F_2 . Now, by construction, one ray P_1F through F , after refraction at the first surface passes through F_2 ; therefore, since the incidence is normal, all rays through F in medium M_1 are refracted so that in M_2 they diverge from F_2 . F and F_2 are therefore in regard to the first surface related as object and image. Similarly, F_1' and F' in regard to the second surface. Then F and F' are fixed because F_2 and F_1' are fixed. Thus, for each position of the object the ray corresponding to P_1O passes through F .

(ii.) *The locus of P is parallel to A_1B_1 .*

$$\frac{AF}{A_1F} = \frac{AP}{A_1P_1} = \frac{A_2P_2}{A_1P_1} = \frac{A_2F_2}{A_1F_2} \therefore AF = \frac{A_2F_2}{A_1F_2} \times A_1F,$$

and is constant. Then, as F is fixed for all incident rays, the locus of P is the line (and plane) through A parallel to A_1B_1 .

(iii.) *The locus of B' is parallel to A_2B_2 .* Suppose a second incident ray ob_1 , parallel to OB_1 (not drawn in the figure: the points corresponding to those on the diagram are indicated by small letters of the same name). Also the subsequent directions, $F_1'b_1b_2$ and $b_2b'F'$. Then

$$\frac{d_2b'}{D_2B'} = \frac{d_2b'}{A_2F'} \times \frac{A_2F'}{D_2B'} = \frac{d_2b_2}{A_2b_2} \times \frac{A_2B_2}{D_2B_2} = \frac{A_2A_1}{A_2F_1'} \times \frac{A_2F_1'}{A_2A_1} = 1.$$

∴ $d_2b' = D_2B'$, and hence the locus of B' is the straight line (and plane) through A' parallel to A_2B_2 .

The locus of P is the first principal or unit plane, that of B' the second; A, A' are the first and second principal points, F, F' the first and second principal foci.

(iv.) As O approaches Q from the extreme right, the slope of OFP increases, and P and I (whose locus is B_2I) get further from A_1A_2 .

When the radiant is at Q , QF is parallel to B_1A_1 : then P and the image are at infinity and the emergent rays, and therefore C_2Q' (Q' is the intersection of C_1Q and B_1F_1') are parallel to $B'F'$.

As the radiant leaves Q , the image passes along and in the direction $B'F'$, towards F' . Also P moves downwards along BA towards A .

When the image is at B' so is P' ; at the same time the radiant and P are at B .

9. *The Principal Planes are planes of unit magnification.*—

(i.) B' is the image of B and BB' is parallel to AA' . Also, if another ray (not shown in the figure) parallel to the principal axis passes through the first and second principal planes at b, b' respectively, then b' will be the image of b . Hence an object, Bb , will have an equal image, $B'b'$.

(ii.) To prove that any incident ray, $O\beta_1$, and its emergent portion, β_2I , intersect the first and second principal planes respectively at points, β, β' , that are equidistant from the principal axis.

Since the principal planes are planes of unit magnification, an object at β on the first principal plane will have an image at, say, β'' on the second principal plane, and $A'\beta'' = A\beta$. Now $O\beta_1\beta$ could be a ray incident at β_1 , hence the emergent portion passes through β'' . But it passes through β' . Hence β' and β'' are coincident.

Alternative proof—

$$\left. \begin{array}{l} \frac{\beta P}{\beta_1 P_1} = \frac{BO}{B_1 O} = \frac{BP}{B_1 P_1} \\ \frac{\beta_1 P_1}{\beta_2 P_2} = \frac{La_1}{La_2} = \frac{B_1 P_1}{B_2 P_2} \\ \frac{\beta_2 P_2}{\beta' P'} = \frac{P_2 I}{P' I} = \frac{B_2 P_2}{B' P'} \end{array} \right\} \begin{array}{l} \text{Multiply the columns together.} \\ \therefore \frac{\beta P}{\beta' P'} = \frac{BP}{B' P'} = 1. \\ \therefore \beta P = \beta' P'. \\ \therefore \beta' A' = \beta A. \end{array}$$

10. *The Nodal Points.*—(Fig. 3.)—Suppose the object to be at Q in the first focal plane. Draw an incident ray, QC, parallel to the fixed emergent ray, $B_2 F'$, cutting $A_1 A_2$ in C, also $A_1 B_1$ in η_1 and AB in η . Draw QC_1 cutting $B_1 F'_1$ in Q' . Draw $Q'\eta_1$ cutting $A_2 B_2$ in η_2 . Draw $\eta_2 C'$ parallel to $B_2 F'$, cutting $A_1 A_2$ in C' , also $A' B'$ in η' .

$Q\eta_1$, $\eta_1\eta_2$, and $\eta_2\eta'$ are the three portions of a ray through the system. C, C' are the *first* and *second nodal points*. Also C and C' are related as object and image points.

11. *Properties of the Nodal Points.*—(i.) The distance between the nodal points is equal to the distance between the principal points.

Since $A'\eta' = A\eta$, the similar triangles $AC\eta$, $A'C'\eta'$ are equal. Then $A'C' = AC$ and $\therefore CC' = AA'$.

(ii.) When the incident portion of a ray passes through the first nodal point, its emergent portion passes through the second, and is parallel to the incident portion.

Take any position, O, of the radiant. Draw OC cutting AB in θ . Draw $\theta\theta'$ parallel to $A_1 A_2$. The emergent ray goes through θ' . It also goes through C' , because C' is the image of C. It also passes through I. Thus I, θ' and C' are collinear. Then, since $A'C' = AC$ and $A'\theta' = A\theta$, the triangles $A'C'\theta'$ and $AC\theta$ are equal. $\therefore C'I$ is parallel to CO.

12. *Construction for the Six Cardinal Points and Planes* (Fig. 4).—Draw the principal axis of the system and mark off A_1 , A_2 , C_1 , C_2 , F_1' , F_2 . Draw any parallel to the principal axis OB_1 , cutting the surface through A_1 in B_1 , and the surface through A_2 in D_2 . Draw lines through B_1 , F_1' and D_2 , F_2 intersecting at S. Draw lines through S, C_1 and S, C_2 intersecting OB_1 at B, B' respectively. Then B' is the final image of B, and the object, AB, has an equal image, $A'B'$. Thus, B, B' are points on the first and second principal planes respectively.

Let D_1 be the point of intersection of A_1B_1 and D_2F_2 . Join BD_1 and produce to cut A_2A_1 in F . Also draw B_2B' , cutting A_1A_2 in F' . Then F, F' are the *first and second principal focal points*.

Draw FQ parallel to A_1B_1 , meeting B_1D_2 in Q . Draw QC parallel to B_2B' . Make CC' equal to AA' . Then C, C' are the *first and second nodal points*.

Complete the three directions through B_1B, B_1B_2, B_2B' , o the second fixed ray. For any position, O , of the object on the incident direction through B_1B , the first image, L , is on the intermediate direction through B_1B_2 , and is found at the intersection, L , of B_1B_2 with OC_1 . The final image, I , is on the

FIG. 4.—CARDINAL POINTS OF A THICK LENS.

emergent direction through B_2B' , and is at the point where LC_2 cuts B_2B' . The final images are real when I is on the actual emergent ray, and virtual when on its backward continuation.

Complete the three directions D_2B' , D_2D_1 , BD_1 of what may be called the *first fixed ray* through the system. The positions of images for any position of the object is found as in the previous case.

If the direction of the light is reversed, so that it is first in the medium, M_3 , and incident on A_2B_2 , then the positions of the cardinal points are unaltered. The names, however, must be interchanged, a first point in the former case becoming a second in this, and conversely. To demonstrate these matters in regard to the principal and focal points, consider a point object to move along $B'B$ from the extreme left. To show that

the nodal points are unaltered draw $F'q$ parallel to A_2D_2 , meeting D_2B' in q . Draw qC' . Then $F'C' = F'A' - C'A' = CF - CA = AF$. $\therefore qC'$ and BF are parallel.

13. (i.) *Relations between the Focal Lengths, $A'F'$ and AF (Fig. 4).—*

$$\frac{AF}{BB_1} = \frac{AB}{D_1B_1} = \frac{A_2F_2}{A_2A_1}; \quad \frac{F'A'}{B'D_2} = \frac{A'B'}{D_2B_2} = \frac{A_1F_1'}{A_2A_1}; \quad \frac{BB_1}{B'D_2} = \frac{C_1F_1'}{C_2F_2}.$$

$$\therefore \frac{AF}{F'A'} = \frac{C_1F_1'}{C_2F_2} \cdot \frac{A_2F_2}{A_1F_1'} = \frac{1}{\mu_{12} \cdot \mu_{23}} = \mu_{31}.$$

$$\therefore A'F' + \mu_{13} \cdot AF = 0.$$

(ii.) *Distances of the Nodal Points, AC and $A'C'$.—*

$$AC = FC - FA = A'F' + AF = f + f'.$$

$$A'C' = A'F' - C'F' = A'F' + AF = f + f'.$$

(iii.) When the media M_1 , M_3 are the same, $\mu_{13} = 1$, then $f' = -f$, and the nodal points C , C' coincide with the principal points A , A' .

(iv.) *Standard Definitions of the Focal Lengths.—*

$$\tan FBQ = F'q/AF, \quad \tan F'B'q = F'A'/FQ.$$

$\therefore f = F'q/\tan FBQ$, or the first principal focal length is the ratio of the length of the image formed in the second focal plane to the apparent or angular magnitude of the corresponding infinitely distant object.

Also $f' = FQ/\tan F'B'q$, or the second principal focal length is the ratio of the length of the object when in the first focal plane to the apparent or angular magnitude of the corresponding infinitely distant image.

These are the Gauss and Abbe definitions of the focal lengths of an optical system.

14. *Focal Distances, &c.*—Suppose an object, OU , and the corresponding image, VI , obtained by drawing OFP ; then $PP'I$ parallel to AA' , and meeting $B'F'$ in I . Then $VF'/F'A' = PA/AB = AF/FU$.

Write $AU = u$, $A'V = v$, $AF = f$, $A'F' = f'$.

$$\therefore \frac{v - f'}{f'} = \frac{f}{u - f} \quad \therefore \frac{f}{u} + \frac{f'}{v} = 1.$$

$$\text{Also } f'/f = -\mu_{13}. \quad \therefore \mu_{13}/v - 1/u = -1/f = \mu_{13}/f'.$$

15. *Magnification*.—(i.) *Lateral Magnification* (m) or (length of image)/(length of object).

$$m = -IV/F'q = -VF'/F'A' = (f' - v)/f' = f/(f - u).$$

(ii.) *Axial or Depth Magnification* (d) or (axial displacement of the image)/(very small axial displacement of the object producing it).—Let $I'V'$ be the position of the image when the object is at $U'O'$. Then $VV'/OO' = VF'/QO'$.

When O' is very close to O , VV'/OO' is the depth magnification. Also $QO' = QO$ sensibly.

$$\therefore d = -(f' - v)/(f - u) = -ff'/(f - u)^2 = \mu_{13} \cdot m^2.$$

(iii.) *Angular Magnification or Convergence-ratio* (w) or (angle between two emergent rays through an image point) $\div$ (angle between the two corresponding incident rays).

Then

$$w = \frac{\text{angle } B'IP'}{\text{angle } POB} = \frac{P'B'}{IP'} \cdot \frac{BO}{BO} = \frac{BO}{IP'} = -\frac{u}{v} = \frac{f - u}{f'} = \frac{f}{f' - v},$$

$$\text{or } w = \frac{\text{angle } C'IF'}{\text{angle } COB} = \frac{\text{angle } C'IF'}{\text{angle } IC'F'} = \frac{C'F'}{IF'} = \frac{-AF}{VF'} = \frac{f}{f' - v}.$$

FIG. 5.—COMPOSITION OF CARDINAL POINTS.

16. *Extension to Three Coaxial Refracting Surfaces* (Fig. 5).—Suppose a third surface through A_3 , centre C_3 , first principal focus F_3 . For the surfaces through A_1, A_2 find B, B', F, F', C, C' .

Let QB pierce the surface A_3 at H_3 , and $H_3 F_3$ intersect $B'F'$ at S' . Draw $S'C_3$ to cut QB in B''' . Then B''' is on the *second principal plane* of the system. Let $B''' B_3$ intersect the principal axis at F''' . Then F''' is the *second principal focus* of the system.

Draw CB'' parallel to $S'C'$, intersecting QB in B'' ; then B'' is a point on the first principal plane of the system.

Let $H_3 F_3$ pierce $A'B'$ at H' . Make $HA = H'A'$. Then F'' , where $B''H$ cuts the principal axis, is the *first principal focus* of the system.

From Q'' , where QB pierces the focal plane through F'' , draw $Q''C''$ parallel to $F''B_3$ and cutting the principal axis at C'' . Make $C''C'' = A''A''$. Then C'' , C''' are the *first and second nodal points* of the system.

Thus the six cardinal points of a system are compounded with those of a refracting surface, and the whole reduced to six cardinal points. These may again be compounded with another refracting system and so on.

III. SPHERICAL SURFACE (*continued*).

17. (i.) *Young's construction for the Refracted Ray* (Fig. 6).—From C, the centre of the spherical surface (radius r), describe circles of radii $r\mu$ and r/μ . Let the incident ray, BO, cut the former circle at Z; join CZ cutting the latter circle at Z'. Then the direction of the refracted ray is through BZ' . Z, Z' are the aplanatic pair of points for the ray ZB, Z'B. (See § 3.)

(ii.) *Alternative construction*.—On BC as diameter draw a circle cutting BO in Y. Draw the chord $CY' = CY/\mu$. Then the refracted ray passes through B, Y'.

18. *Meridian Rays*.—(iii.) YY' crosses CZ at right angles at the point K (called, by Cornu, the *junction point*).

(iv.) Let Z, Z', Z_1 , Z_1' be the aplanatic pairs of points for two parallel rays incident at B, B_1 , respectively. Let the corresponding refracted rays intersect at E'.

Through Z_1' draw a parallel to BZ' , cutting the chord B_1B at b_2 . Draw Z_1b_2e , cutting ZB in e. Now,

$$\frac{eB}{B_1Z_1} = \frac{b_2B}{B_1b_2} = \frac{Z_1'E'}{B_1Z_1'} \quad \therefore \frac{eB}{Z_1'E'} = \frac{B_1Z_1}{B_1Z_1'} = \mu.]$$

(v.) Draw CY_1Y perpendicular to the parallels B_1Z_1 , BZ ; also YK , Y_1K_1 perpendicular to CZ, CZ_1 respectively. Through

K draw KF' , KF parallels to BZ , BZ' respectively; similarly $K_1F'_1$. Then $FB = KF' = \mu \cdot Z'F'$.

(vi.) When B_1 is very close to B , Y_1 , whose locus is CY , is very close to Y and K_1 to K . Then sensibly F'_1 , and E' coincide with F' . Also e coincides with F , for in all cases $eB = \mu \cdot Z_1'E'$.

Also the chord BB_1 very closely coincides with the arc BB_1 , and the geometrical relationships deduced from the finite chord become optical relationships for rays of light refracted at the infinitesimal arc. The above relations are therefore approximately true for a small arc.

(vii.) Suppose OO_1 is a position of a very small object moving between BZ , B_3Z_3 . Join FO_1 cutting the chord and

FIG. 6.—REFRACTION AT A SPHERICAL SURFACE. MERIDIAN RAYS.

arc, B_3B , at B_1 . Draw B_1I_1 parallel to BF' , cutting B_3F' at I_1 . Then I_1 is the image of O_1 .

Join I_1O_1 , also KI_1 . Then

$$\frac{B_3O_1}{FB} = \frac{B_3B_1}{B_1B} = \frac{B_3I_1}{I_1F'} \quad \therefore \frac{B_3O_1}{B_3I_1} = \frac{KF'}{I_1F'}$$

Then, since KF' is parallel to B_3O_1 , the points O_1 , I_1 , K are collinear.

When the object is very small, O_1 coincides with O , and I_1 with I . Hence the position of the image on the refracted ray BZ' , corresponding to any position O of the object on the incident ray, BZ , is at the point I where the line through OK cuts the refracted ray BZ' .

Then, considering the object to move from the extreme right

to the extreme left, F, F' are identified as the first and second *principal focal points* for narrow meridional pencils having BZ for their chief ray.

(viii.) Since (Fig. 7)

$$\frac{FB}{BO} = \frac{KF'}{BO} = \frac{IF'}{BI} = \frac{BF' - BI}{BI}, \quad \therefore \frac{BF}{BO} + \frac{BF'}{BI} = 1.$$

$$(ix.) \quad \frac{KF'}{BY} = \frac{KY'}{Y'Y}, \quad \frac{FK}{BY'} = \frac{KY}{Y'Y}, \quad \therefore \frac{KF'}{FK} = \frac{BY}{BY'} \cdot \frac{KY'}{KY}.$$

FIG. 7.—REFRACTION AT A SPHERICAL SURFACE.

Since the triangles $KYC, Y'BC$ are similar, $KY/CY = BY'/BC$. Also the triangles $KY'C, CYB$ are similar. $\therefore KY'/CY' = BY/BC$.

$$\therefore \frac{KY'}{KY} \cdot \frac{CY}{CY'} = \frac{BY}{BY'} \quad \therefore \frac{KY'}{KY} = \frac{1}{\mu} \cdot \frac{BY}{BY'}$$

$$\text{Also } \frac{FB}{BF'} = \frac{KF'}{FK} = \frac{BY}{BY'} \cdot \frac{KY'}{KY} \quad \therefore \frac{BF}{BF'} = \frac{1}{\mu} \cdot \frac{BY^2}{BY'^2}$$

$$(x.) \quad \frac{KF'}{BF'} = \frac{NZ'}{BZ'} = \frac{NZ'}{\mu \cdot TZ'} \quad \therefore \frac{NZ'}{TZ'} = \mu \cdot \frac{FB}{BF'} = \frac{BY^2}{BY'^2}$$

(xi.) For BF, BF', BO, BI , write f, f', u, v_1 respectively. And for BY put y or $r \cdot \cos \phi$, and for BY' substitute y' or $r \cdot \cos \phi'$.

$$\frac{1}{KF'} + \frac{1}{y} = \frac{1}{DY'} = \frac{1}{HY'} \cdot \frac{y'^2}{y^2} = \frac{\mu y'}{y^2}$$

$$\therefore \frac{1}{BF} = \frac{y - \mu \cdot y'}{y^2} \quad \therefore \frac{1}{f} = \frac{\cos \phi - \mu \cdot \cos \phi'}{r \cdot \cos^2 \phi}$$

$$\frac{1}{BF'} = -\frac{1}{BF} \cdot \frac{1}{\mu} \cdot \frac{y^2}{y'^2} = \frac{\mu \cdot y' - y}{\mu \cdot y'^2} \quad \therefore \frac{1}{f'} = \frac{\mu \cdot \cos \phi' - \cos \phi}{\mu \cdot r \cdot \cos^2 \phi'}$$

(xiv.) When B_6 is very close to B , the points B_6, B_7, b_7 sensibly coincide. Also the chord B_6B very nearly coincides with the arc B_6B , and the geometrical relationships deduced from the finite chord become optical relationships for rays of light refracted at the infinitesimal arc. The above relations are therefore approximately true for a small arc.

(xv.) Suppose OO' is a position of a very small object moving between BZ and B_9Z_9 . Suppose these lines to be two incident sagittal rays and BG', B_9G' the respective refracted directions. Join GO' , cutting the chord (and arc) B_9B at B_{10} . Draw $B_{10}J'$ parallel to BG' , cutting B_9G' at J' . Then J' is the image of O' .

Join CJ' , also $J'O'$. Then

$$B_9O'/GB = B_9B_{10}/B_{10}B = B_9J'/J'G'.$$

Also CG' is parallel to B_9Z_9 and equal to GB . Therefore, $CJ'O'$ are collinear.

Then for any position of the object along BZ the image is at the point J , where OC crosses the direction of the refracted ray BG' .

Consider the object to move from the extreme right to the extreme left. Then G, G' are identified as the first and second principal foci for narrow sagittal pencils having BZ for their chief ray.

(xvi.) In Fig. 7, $\frac{GB}{BO} = \frac{CG'}{BO} = \frac{JG'}{BJ} = \frac{BG' - BJ}{BJ}$. Write v_2 for BJ and g, g' for BG, BG' respectively. Therefore, $g/u + g'/v_2 = 1$.

(xvii.) Let BZ cut the circle of radius r at B and β .

Then $BZ = r(\mu \cdot \cos \phi' + \cos \phi)$ and $\beta Z = r(\mu \cdot \cos \phi' - \cos \phi)$.

Also $-g = GB = CG' = \mu \cdot Z'G' = \mu \cdot g' - \mu \cdot BZ' = \mu \cdot g' - BZ$.

$\therefore g + \mu \cdot g' = BZ = r(\cos \phi + \mu \cdot \cos \phi')$.

The triangles $BCG', Z\beta C$ are similar. Therefore,

$CG'/BC = C\beta/\beta Z$. $\therefore -g = r^2/\beta Z = r/(\mu \cdot \cos \phi' - \cos \phi)$.

And $BG'/BC = CZ/\beta Z$. $\therefore g' = \mu \cdot r/(\mu \cdot \cos \phi' - \cos \phi)$.

Write q for QJ . Then $1/q = -1/g + 1/u$.

$\therefore \mu/v_2 - 1/u = -1/g = (\mu \cdot \cos \phi' - \cos \phi)/r$.

20. *To find points on the Caustic formed by Refraction at a Spherical Surface.* (No figure.)—From C draw concentric circles of radii r (representing the surface), μr , $\mu r/2$, r/μ ,

$r/2\mu$. (For convenience, keep one pair of compasses adjusted to radius $\mu r/2$, another pair to $r/2\mu$.)

From the radiant O draw any incident ray giving B, Z; draw CZ, giving Z', also W, W' where CZ cuts the circles of radii $\mu r/2$ and $r/2\mu$ respectively. With W as centre, radius $\mu r/2$, draw a circle cutting the incident ray BZ in Y; similarly, with W' as centre, radius $r/2\mu$, find Y' on the refracted ray BZ'. (Since $WY = CW = WZ$. $\therefore$ CYZ is the angle in a semicircle. Similarly, CY' is perpendicular to BZ'.) Draw YY', obtain K. Join KO, obtain I. Join CO, obtain J.

Draw other incident rays and repeat the construction.

ABSTRACT.

The Paper contains a development of optical imagery based on elementary geometry, including limiting positions, but excluding cross-ratios, centres of perspective, &c. The method adopted is useful for teaching the properties of optical systems to those who are not essentially students of pure mathematics, and can be very satisfactorily used by those capable of draughtsmanship with mathematical instruments.

The principle of the method is as follows: The vertex (object) of a small incident pencil of rays is considered to move along a line parallel to the principal axis of the optical system. This line is also regarded as the incident portion of a fixed ray. Then the directions of the ray after reflection or refraction at the several surfaces do not alter as the position of the object changes. They are therefore the fixed loci of the images or vertices of the reflected and refracted pencils. Then the successive positions of the image resulting from a progressive movement of the object are obtained. Also the constructions are such that in the case of a thick lens the importance of the cardinal points and planes is demonstrated.

First, considering refraction at a spherical surface and assuming that μ is the relative velocity of light in the two media, a construction for the refracted portion of any incident ray is obtained from the wave principle; then the sine law of refraction and the position of the aplanatic pair of points are deduced.

When the incidence is normal it is convenient in drawing to increase largely the scale of length perpendicular to the principal axis in comparison with that along the principal axis. A distorted figure is obtained, but the relative lengths of lines perpendicular to the axis or of lines parallel to the axis will be maintained.

The principal foci for normal incidence on a single spherical refracting surface are obtained, and then the usual algebraical expressions are deduced from the figure. A convenient construction for finding the principal foci and aplanatic pair of points when μ and r are given is then shown.

For a thick lens it is most convenient to assume that the two surfaces separate three different media. After obtaining the refracted portions of a ray—called the second fixed ray—incident

parallel to the principal axis of the system, the positions of the cardinal and nodal points and planes are indicated and their more important properties deduced. A convenient construction is shown for finding the cardinal and nodal points, when r_1, r_2, μ, μ' are given. At the same time the course of the first fixed ray is determined. The first fixed ray is defined as the ray which emerges from the system in the same line as the incident portion of the second fixed ray obtained above. There is also a visualisation of the Gauss and Abbe definitions of focal length. The usual algebraic relations are deduced. After a discussion of magnification—lateral, axial and angular—it is shown that when the six cardinal points of one optical system are combined with those of another a resulting set of six cardinal points is obtained, and thus a composition of the cardinal points of various optical systems may be effected.

Returning to refraction at a single spherical surface the cases of meridian and sagittal rays are separately considered. For the former the position of Cornu's junction point is obtained in a simple manner. Finally, the usual algebraic relationships are deduced and a graphic construction for points on a caustic is given.