NOTE ON THE DYNAMICS OF CAPILLARY FLOW.

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SYNOPSIS.

Theory of the movement of moisture in soil.—Attention is called to experimental data which show that there usually exist in the soil appreciable moisture gradients which indicate that it cannot be regarded as a bundle of capillary tubes and that therefore *Washburn's theory of capillary flow* does not rigorously apply in this case. However by using Poiseuille's equation modified by the assumption that the variation of moisture content does not appreciably change the kinematical resisting force, together with empirical equations derived from soil moisture studies connecting the capillary pressure and vapor pressure with the moisture content, an *equation for horizontal flow* is derived, $V = k(1 - e^{-Bt})$, which is in qualitative agreement with experimental results. A more extended treatment of the subject is given in an article by the author in Soil Science.

IN the March number of the PHYSICAL REVIEW there appears an article by Washburn on "The Dynamics of Capillary Flow." In part 7 of this article he discusses the rate of penetration of a porous body by a liquid, arriving at a tentative relation:

$$V = k'\left(\frac{\gamma}{\eta}\right)^{1/2} t^{1/2},$$

where V expresses the volume of liquid absorbed, t the time, γ the surface tension, η the coefficient of viscosity, and k' a constant. He finds, however, that for large values of V this relation does not hold in the case of the absorption of water by charcoal.

In this connection, it may be of interest to note that a considerable amount of data are available from soils literature[1] showing that the volume of water absorbed by a soil is not always proportional to the pore space, as his equation (15), viz.,

$$V = \pi r^2 l$$

would indicate. On the other hand, the liquid "density" ρ (*i.e.*, the mass of liquid per aggregate unit volume of soil) is a function of the distance x from the source. Uniformity of soil moisture content is exceptional, whereas moisture gradients are frequently observed in moisture studies, although their magnitudes are usually small except at the im-

[1] See, for example, "Soils," by Hilgard, p. 209; "Physics of Agriculture," by King, p. 134; "Studies of the Movement of Soil Moisture," by Buckingham, *in* U. S. Dept. of Agr. Bur. of Soils Bul. 38; also "The Movement of Soil Moisture" by Gardner and Widtsoe, *in* Soil Science, Vol. XI., No. 3, March, 1921, Fig. 3, p. 224.

mediate water front where an abrupt change is often observed. This fact would seem to invalidate also the application of Washburn's equation (8) wherein he has made use of the reciprocal of r (*i.e.*, the radius of the equivalent cylindrical tube) as a measure of the curvature of the liquid surface. It is clear of course that his equations have been derived on the assumption that the porous body should approximate in character a bundle of capillary tubes and an attempt to apply the equations to a mass of soil is perhaps somewhat beyond his expectations. If, however, we regard the curvature factor of his equations (8) and (9) as a function of the liquid "density," which in turn depends upon the distance x from the source, these equations will apply provided Poiseuille's equation holds for this case.

In the March number of *Soil Science* the writer has made use of Poiseuille's equation modified for variable "density" by introducing the assumption that the kinematical resisting force is independent of the "density." Or, what amounts to the same thing, that the average velocity v of the moisture is proportional to the pressure gradient and independent of the moisture content,

$$v = K_1 \nabla p. \tag{1}$$

The capillary pressure p is determined by the curvature of the air-water surface and this in turn is determined by the moisture content. Due to the complexity of this surface configuration, however, we are forced for the present to rely upon experiment for this functional relation. The experimental data of Briggs[1] on the moisture content as a function of the reciprocal of the "centrifugal" force to which the moist soil is subjected leads to an equation which holds over a wide range of moisture contents of the form,

$$p = \frac{c}{\rho} + b. \tag{2}$$

This involves the assumption that the capillary pressure is proportional to the equilibrium "centrifugal" force. Thomas's[2] vapor pressure measurements for varying moisture content lead also to an equation of the form,

$$\pi = \pi_0 - \frac{c_1}{\rho}, \tag{3}$$

where π is the vapor pressure and c_1 is a small constant, this equation representing the experimental facts over a considerable range of moisture contents.

[1] U. S. Dept. of Agr. Bur. of Soils, Bul. 45.

[2] Soil Science, Vol. XI., No. 6, June, 1921.

By a familiar method it may be shown that the vapor pressure and the capillary pressure are related as follows:

$$p = \pi - nRTln \frac{\pi}{\pi_0} \tag{4}$$

and a simple algebraic substitution leads to a converging series which, for finite values of π differing only slightly from the average vapor pressure π_0 of water, gives a relation also of the form,

$$p = \frac{c_2}{\rho} + b, \tag{5}$$

from which,

$$\nabla p = \frac{c_2}{\rho} \nabla \rho. \tag{5'}$$

Combining (1) and (5′), we obtain,

$$v = K_2 \frac{\nabla \rho}{\rho}. \tag{6}$$

And combining equation (6) with the equation of continuity,

$$\frac{\partial \rho}{\partial t} = - K_3 \nabla(\rho v) \tag{7}$$

we obtain,

$$\frac{\partial \rho}{\partial t} = - K_4 \nabla^2 \rho. \tag{8}$$

From (8), we may obtain a solution of the form,

$$\nabla \rho = K_5 l^{-\alpha x} l^{-\beta t}. \tag{9}$$

From equation (9), equation (6) may be reduced, for one-dimensional flow, to the following form,

$$v_l = \frac{dl}{dt} = \frac{K_2 l^{-\alpha x} l^{-\beta t}}{\rho_l} \tag{10}$$

where l is the coördinate of the water front, α and β are constants. If α (which is a measure of the moisture gradient) is small, ρ_l and the factor $K_2 l^{-\alpha x}$ remain practically constant, giving an equation of the form,

$$\frac{dl}{dt} = K_6 l^{-\beta t}, \tag{11}$$

which becomes when integrated

$$l = K_6(1 - e^{-\beta t}) \tag{12}$$

and the approximate value of V

$$V = K_7(1 - e^{-\beta t}), \tag{13}$$

which is in qualitative conformity with the experimental data of Cude and Hulett as quoted by Washburn. Equation (8) and a somewhat more general equation taking account of gravitational and hydrostatic potentials have been solved for special cases with qualitative experimental confirmation, as reported more in detail in the article noted.

It is true, of course, that these equations rest upon empirical data, together with the assumption which has been made with reference to the application of Poseuille's equation, but a considerable amount of material has been presented in the article cited indicating that they are not far from correct.

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