INFINITE DEVELOPMENTS AND THE COMPOSITION PROPERTY $(K_{12}B_1)_*$ IN GENERAL ANALYSIS.

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PREFACE.

In his Introduction to a Form of General Analysis ¹⁾ E. H. MOORE has formulated a theory of developments Δ of a general class $\mathfrak{B}$. These developments Δ are generalizations on the one hand of the partition by sequential halving of the class $\mathfrak{B}$: $a \leq p \leq b$; and on the other of the denumeration ($p = 1, 2, 3, \dots$) of the class $\mathfrak{B}$: $p = 1, 2, 3, \dots$. A development Δ is there defined as a sequence of stages, each stage consisting of a finite system of subclasses of $\mathfrak{B}$.

Taking as a basis a general class $\mathfrak{B}$ admitting a development Δ , MOORE erects a theory of properties of classes of real valued functions defined on the class $\mathfrak{B}$. In terms of Δ he defines two properties K_1, K_2 of classes $\mathfrak{M}$ of functions on $\mathfrak{B}$. We may say for the special case of functions of a real variable that «the class $\mathfrak{M}$ has the property K_1 » corresponds roughly to «the functions of the class $\mathfrak{M}$ approach zero as the variable increases without limit»; and that «the class $\mathfrak{M}$ has the property K_2 » corresponds in similar fashion to «the functions of the class $\mathfrak{M}$ are uniformly continuous». Denoting by $\mathfrak{M}_*$ the class of all functions which are the limits of relatively uniformly convergent ²⁾ sequences of linear combinations of functions of the class $\mathfrak{M}$, the theory turns about the functional characterization of the class $\mathfrak{M}_*$. It is found that, under suitable hypotheses on the class $\mathfrak{M}$ of a very general character, the class $\mathfrak{M}_*$ is the class of all functions having the properties K_1, K_2 simultaneously which are bounded by constant multiples of functions of $\mathfrak{M}$. It may be mentioned that in the general case this characterization depends not only on the simultaneous presence of the properties K_1, K_2 but also on the notion of uniformity relative to a scale ³⁾.

The generality and field of application of this work are increased by considering the class $\mathfrak{B}$ as the multiplicative composite ⁴⁾ of two classes $\mathfrak{B}', \mathfrak{B}''$, of which $\mathfrak{B}'$

¹⁾ pp. 1-150 of the volume: The New Haven Mathematical Colloquium; Yale University Press, New Haven, 1910.

²⁾ Loc. cit. ¹⁾, Art. 6. See also article 3 of this paper.

³⁾ Relatively uniform convergence of series (Art. 3) is a type of such uniformity.

⁴⁾ Loc. cit. ¹⁾, Arts. 4, 50. See also this paper, Art. 2 a.

alone is assumed to admit a development. The variable p'' plays the role of an arbitrary parameter. It is clear that in the development of the theory distinct hypotheses will need to be made on the classes $\mathfrak{M}'$, $\mathfrak{M}''$. The result attained is a functional characterization of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ similar to that obtained for the class $\mathfrak{M}_*$ in the case of a single class $\mathfrak{P}$. Finally the effect of composition and extension of classes $\mathfrak{P}$ and $\mathfrak{M}$ on the various sets of properties occurring in the theory is considered.

To this brief resume of the work of MOORE it is only necessary to add in regard to the present paper that, following the trend of real analysis, with particular reference to the work of BOREL, LEBESGUE, and W. H. YOUNG in integration and the theory of sets of points, it extends the theory of MOORE to the case of developments which may have a denumerable infinity of subclasses in any stage. The point of view which this generalization necessitates has permitted the union in one theorem of results obtained separately by MOORE and Dr. A. D. PITCHER ⁵⁾ which appeared to have no direct connection. In passing we may remark that the theory of the present paper may be applied in the study of the properties of classes of functions which may be discontinuous on a nowhere dense set of points, and in fact took its origin in such a study.

We do not undertake an exact formulation of the elements of the theory of MOORE. Such preliminary explanations and definitions as seem essential to the later discussion are presented in the introductory section.

The advice and encouragement of Professor MOORE have proved very helpful in the preparation of this paper, and are gratefully acknowledged.

INTRODUCTION.

Functions on a General Range. Composition of Ranges and Classes of Functions.

Articles 1-2.

1. *Functions on a general range* ⁶⁾. — We are considering the theory of single-valued, real-valued functions μ and classes $\mathfrak{M}=[\mu]$ ⁷⁾ of functions μ whose *variable* p ranges

⁵⁾ The Interrelations of Eight Fundamental Properties of Classes of Functions, Kansas University Science Bulletin, Vol. VII, No 1, pp. 1-67, 1914.

⁶⁾ For a fuller treatment of the subject matter of this article see loc. cit. ¹⁾, Arts 1, 2, 4, 5.

⁷⁾ We shall consistently use the bracket to denote a class of elements, the type of the elements being indicated within the bracket.

over a *general class* $\mathfrak{P} = [p]$. We speak of functions μ , and classes $\mathfrak{M}$ of functions μ on the class $\mathfrak{P}$ to the class $\mathfrak{A} = [a]$ of real numbers a . In fact, the function μ is the *system of functional values* obtained by considering the value μ_p of the function at the place (or argument value) p for all elements p belonging to the class $\mathfrak{P}$ ⁸⁾.

Thus we are considering in this introductory section systems Σ composed of the fundamental class $\mathfrak{A}$ of real numbers, a general class $\mathfrak{P}$, and a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$, that is, the system

$$\Sigma: (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}).$$

2. Composition of ranges and classes of functions. — Since $\mathfrak{P}$ is a general class it is clear that the considerations of article 1 apply to the case in which $\mathfrak{P}$ is the composite class of a system of classes $(\mathfrak{P}', \mathfrak{P}'', \dots)$. It is sufficient for our present purposes to define the forms of composition to be considered in the case of the composition of two classes $\mathfrak{P}', \mathfrak{P}''$.

2 a. Multiplicational composition ⁹⁾. — We derive the class $\mathfrak{P}$ multiplicationally from the classes $\mathfrak{P}', \mathfrak{P}''$ by taking the elements p', p'' of $\mathfrak{P}', \mathfrak{P}''$ in pairs (p', p'') ; obtaining the bipartite element

$$p = (p', p'') = p' p'' \text{ }^{10)}.$$

The *multiplicational composite* $\mathfrak{P} = \mathfrak{P}' \mathfrak{P}''$ of the classes $\mathfrak{P}', \mathfrak{P}''$, is the class of all bipartite elements p of the form (p', p'') .

Similarly, from functions μ', μ'' on $\mathfrak{P}', \mathfrak{P}''$, respectively, we derive the *multiplicational composite function*

$$\mu = \mu' \mu''$$

whose value

$$\mu_p = \mu'_{p'} \mu''_{p''}$$

at the place p is the numerical product of the values of μ', μ'' at the corresponding places p', p'' ¹¹⁾.

The *multiplicational composite class* $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ of the classes $\mathfrak{M}' = [\mu']$, $\mathfrak{M}'' = [\mu'']$ on $\mathfrak{P}', \mathfrak{P}''$, respectively, is the class of all functions μ of the form

$$\mu = \mu' \mu''.$$

2 b. Adjunctive composition ¹²⁾. — We also consider composition of classes by

⁸⁾ Unless otherwise specified, we use the word function to mean a function on a general range $\mathfrak{P}$ to $\mathfrak{A}$. In general the explicit mention of the class $\mathfrak{A}$ of functional values will be omitted.

⁹⁾ Cf. loc. cit. ¹⁾, Arts. 4, 50, 51, 53, 54.

¹⁰⁾ The abbreviated notation can lead to no confusion, since the elements p', p'' , do not appear in a numerical sense.

¹¹⁾ In the case of a function φ on a composite range $\mathfrak{P} = \mathfrak{P}' \mathfrak{P}''$ the notations $\varphi_{p'}, \varphi_{p''}$ will be understood to mean the functions on $\mathfrak{P}'', \mathfrak{P}'$, respectively, obtained from φ by keeping the argument indicated fixed and allowing the other argument to vary.

¹²⁾ E. H. MOORE, *On the Foundations of the Theory of Linear Integral Equations* [Bulletin of the American Mathematical Society, April, 1912, p. 334].

adjunction. Two classes $\mathfrak{P}'$, $\mathfrak{P}''$ with *no common elements* from a composite sum class

$$\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}'',$$

the class of all elements p which are in one or the other of the two classes $\mathfrak{P}'$, $\mathfrak{P}''$; the least common superclass of the classes $\mathfrak{P}'$, $\mathfrak{P}''$.

The adjunctional composite function μ on $\mathfrak{P}$ of functions μ' , μ'' on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, is defined as the function on $\mathfrak{P}$ which is equal to μ' for p in $\mathfrak{P}'$, and to μ'' for p in $\mathfrak{P}''$; that is,

$$\mu_p = \mu'_p \quad (p = p'),$$

$$\mu_p = \mu''_p \quad (p = p'');$$

and we write ¹³⁾

$$\mu = \mu' + \mu''.$$

The *adjunctional composite* class $\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}''$ of functions $\mu = \mu' + \mu''$ on $\mathfrak{P}$, the composite class of classes $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$, is the class of all functions of the form

$$\mu = \mu' + \mu''.$$

From the definition of $\mu = \mu' + \mu''$ we have the proposition ¹⁴⁾

$$(1) \quad A\mu = A(\mu' + \mu'') = A\mu' + A\mu''.$$

Relatively Uniform Convergence. Extensions of a Class of Functions. Dominance and Closure Properties.

Articles 3-8.

3. Relatively uniform convergence ¹⁵⁾.—In various special fields of analysis the notion of a sequence of functions converging to a limit function uniformly with respect to the range of the variable is fundamental. For the purposes of General Analysis MOORE introduced a fundamental generalization of this notion in the way shown below. We first define *uniform convergence* of a sequence.

A sequence $\{\mu_n\}$ of functions μ_n converges to a limit μ uniformly on the range $\mathfrak{P}$ in case for every positive number ϵ there is an integer n_ϵ (in general dependent

¹³⁾ There need be no confusion arising from this notation, as the superscripts ', '' are used exclusively to denote the fact that the functions and classes of functions considered are on ranges $\mathfrak{P}'$, $\mathfrak{P}''$ respectively.

¹⁴⁾ When a relation between functions is written without the variable appearing explicitly, it is understood to hold for identically in the variable on the range considered. The letter A is used as an abbreviation for «absolute value of». $A\mu$ means the function whose values are the absolute of the values of μ .

¹⁵⁾ Cf. loc. cit. ¹⁾, Arts. 6-14, 20.

upon e) such that for every n greater than n_e we have the inequality

$$(a) \quad A(\mu_{n_p} - \mathfrak{z}_p) \leq e$$

holding for every element p of $\mathfrak{P}$.

The uniformity lies in the fact that a single n_e is effective for all p .

We obtain the generalization mentioned on replacing the inequality (a) by

$$(b) \quad A(\mu_{n_p} - \mathfrak{z}_p) \leq e A\sigma_p,$$

the inequality holding for all p as before, where σ is a function on $\mathfrak{P}$ to $\mathfrak{A}$ which we term the *scale function of convergence*. This type of convergence is called *relatively uniform convergence as to a scale function σ* and indicated by:

$$\mathfrak{z} = \lim_{n \rightarrow \infty} \mu_n(\mathfrak{P}; \sigma) \text{ }^{16}.$$

We speak of convergence *relatively uniformly* as to a class $\mathfrak{S} = [\sigma]$ of functions σ , i.e.;

$$\lim_{n \rightarrow \infty} \mu_n = \mathfrak{z}(\mathfrak{P}; \mathfrak{S}),$$

in case there is some function σ of the class $\mathfrak{S}$ effective as a scale function of convergence ¹⁷).

Relative to a class $\mathfrak{M} = [\mu]$ of functions μ , the class $\mathfrak{M}_\sigma$ of all functions $\mathfrak{z}$ of the form

$$\mathfrak{z} = \lim_{n \rightarrow \infty} \mu_n(\mathfrak{P}; \sigma)$$

is called the *extension $\mathfrak{M}_\sigma$* of the class $\mathfrak{M}$ as to the scale function σ , that is,

$$\mathfrak{M}_\sigma = [\text{all } \mathfrak{z} = \lim_{n \rightarrow \infty} \mu_n(\mathfrak{P}; \sigma)].$$

¹⁶) Read: $\mathfrak{z}$ is the limit of a sequence of functions converging on $\mathfrak{P}$ relatively uniformly as to the scale function σ . This notation will be used frequently. It will also be used to represent relatively uniform convergence of series.

¹⁷) As an example of the character of results obtained by the introduction of relative uniformity we cite the following proposition from loc. cit. ¹), Art. 7f.

Let the range $\mathfrak{P}$ be the range of the positive integers ($p = 1, 2, 3, \dots$). We consider the class $\mathfrak{M}$ of all functions (sequences) on $\mathfrak{P}$ whose series converge absolutely, that is, for which the limit as k approaches infinity of

$$\sum_{p=1}^{p=k} A\mu_p$$

exists. Then every function $\mathfrak{z}$ on $\mathfrak{P}$ of the form:

$$\mathfrak{z} = \lim_{n \rightarrow \infty} \mu_n(\mathfrak{P}; \mathfrak{M})$$

belongs to the class $\mathfrak{M}$.

This proposition is not true in the case of *uniform convergence*. We note the analogy between this proposition and the theorem that the limit of a uniformly convergent sequence of continuous functions on the continuous interval $\mathfrak{P}: (0, 1)$ is a continuous function.

In the case cited scale convergence is more exacting than uniform convergence. It may also be less exacting. In case σ is the constant function 1, the two forms of convergence are the same, that is,

$$\lim_{n \rightarrow \infty} \mu_n = \mathfrak{z}(\mathfrak{P}; 1)$$

means that the sequence $\{\mu_n\}$ converges to the limit $\mathfrak{z}$ *uniformly* on $\mathfrak{P}$.

Similarly we have the extension $\mathfrak{M}_{\mathfrak{S}}$ ($\mathfrak{M}$ extended as to $\mathfrak{S}$):

$$\mathfrak{M}_{\mathfrak{S}} = [\text{all } \mathfrak{z} = \lim_{n \rightarrow \infty} \mu_n(\mathfrak{P}; \mathfrak{S})].$$

It is easy to see that the class $\mathfrak{M}_{\mathfrak{S}}$ is the least common superclass of the classes $\mathfrak{M}_{\sigma}$ ($\mathfrak{M}$ extended as to σ), obtained by considering all functions σ of $\mathfrak{S}$.

4. *The Linear and *-Extensions* ¹⁸). — From a class $\mathfrak{M}$ of functions we derive the linear extension $\mathfrak{M}_L$ of $\mathfrak{M}$, namely, the class of all functions which are linear combinations with constant coefficients of functions μ .

The **-extension* $\mathfrak{M}_*$ of a class $\mathfrak{M}$ is the extension as to the class $\mathfrak{M}$ (functioning as a scale class) of the class $\mathfrak{M}_L$ ($\mathfrak{M}$ extended to be linear), that is,

$$\mathfrak{M}_* = (\mathfrak{M}_L)_{\mathfrak{M}}.$$

It is clear that the definitions above apply to the composite class of classes $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively. For example

$$(\mathfrak{M}' \mathfrak{M}'')_* = ((\mathfrak{M}' \mathfrak{M}'')_L)_{(\mathfrak{M}' \mathfrak{M}'')},$$

$$(\mathfrak{M}' + \mathfrak{M}'')_* = ((\mathfrak{M}' + \mathfrak{M}'')_L)_{(\mathfrak{M}' + \mathfrak{M}')}.$$

We note the propositions:

(1) If a function φ belongs to the class $(\mathfrak{M}' \mathfrak{M}'')_*$, then for every p' the function $\varphi_{p'}$, ¹⁹) on $\mathfrak{P}''$ belongs to the class $\mathfrak{M}''_*$, and for every p'' the function $\varphi_{p''}$ on $\mathfrak{P}'$ belongs to the class $\mathfrak{M}'_*$.

(2) The class $\mathfrak{M}$ of functions on $\mathfrak{P}$ belongs to the classes $\mathfrak{M}_L$, $\mathfrak{M}_{\mathfrak{M}}$, and $\mathfrak{M}\mathfrak{M} = [\text{all } a\mu]$; and these in turn belong to the class $\mathfrak{M}_*$.

(3) Relative to classes $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively,

$$(\mathfrak{M}' \mathfrak{M}'')_L = (\mathfrak{M}'_L \mathfrak{M}''_L).$$

5. *The dominance properties* ²⁰). — A function σ is *dominated by a function* τ in case

$$A\sigma \leq A\tau,$$

and by a *class* $\mathfrak{T} = [\tau]$ in case there is a function τ_{σ} (dependent upon σ) in the class $\mathfrak{T}$ by which σ is dominated.

A class $\mathfrak{S} = [\sigma]$ is *dominated by a function* τ or a *class* $\mathfrak{T}$ in case its every function σ is dominated by τ or by $\mathfrak{T}$.

It is convenient to introduce the abbreviation B_i for «is dominated by», and to write $\mathfrak{S}^{B_i \mathfrak{T}}$, for: The class $\mathfrak{S}$ is dominated by the class $\mathfrak{T}$ ²¹).

A class $\mathfrak{M}$ has the dominance property D (in notation, $\mathfrak{M}^D$) in case for every sequence $\{\mu_n\}$ of functions of the class there is a sequence $\{a_n\}$ of constants a_n and a

¹⁸) Cf. loc. cit. ¹), Arts. 27, 28, 44. (Read: *star-extension*).

¹⁹) Cf. footnote ¹¹).

²⁰) Cf. loc. cit. ¹), Arts. 9, 18, 19, 21-26.

²¹) This abbreviation is part of a systematic notation introduced by MOORE in loc. cit. ¹) which will be used frequently in the present paper.

function μ of the class such that for every n

$$A\mu_n \leq Aa_n\mu.$$

A class $\mathfrak{M}$ has the dominance property $D_i(\mathfrak{M}^{D_i})$ in case for every two functions μ_1, μ_2 of $\mathfrak{M}$ there is a function μ and a constant a such that the function $a\mu$ dominates both μ_1 and μ_2 .

It is clear that if a class $\mathfrak{M}$ has the dominance property D it also has the property D_i .

The dominance properties are fundamental. They enter the theory chiefly as properties of classes functioning as scales of uniformity.

6. Propositions involving the dominance properties and relatively uniform convergence ²²). — The following propositions are quite easily proved by consideration of the various definitions involved.

(1) Uniformity of convergence as to σ implies uniformity as to every function dominating σ .

(2) Uniformity as to σ and uniformity as to $a\sigma$ ($a \neq 0$) are equivalent.

(3) If a class $\mathfrak{S}$ with the dominance property D is the scale class of each of a sequence of instances of relatively uniform convergence, then the corresponding sequence $\{\sigma_n\}$ of scale functions may be replaced by a single scale function effective for each of the instances.

(4) If a class $\mathfrak{S}$ with the dominance property D_i is the scale class of each of a finite number of instances of relatively uniform convergence, then the corresponding finite set of scale functions may be replaced by a single scale function effective for each of the instances.

(5) If a class $\mathfrak{M}$ has the dominance property D its extension $\mathfrak{M}_{\mathfrak{M}}$ as to itself is invariant under extension as to $\mathfrak{M}$, that is

$$\mathfrak{M}_{\mathfrak{M}} = (\mathfrak{M}_{\mathfrak{M}})_{\mathfrak{M}}.$$

Relative to classes $\mathfrak{M}', \mathfrak{M}''$ of functions on $\mathfrak{P}', \mathfrak{P}''$ respectively, and composition by multiplication (Art. 2 a), we have:

(6) If the classes $\mathfrak{M}', \mathfrak{M}''$ respectively, have the property D , resp. D_i then the classes $\mathfrak{M}'\mathfrak{M}'', (\mathfrak{M}'\mathfrak{M}'')_L, (\mathfrak{M}'\mathfrak{M}'')_*$ all have the property D , resp. D_i ; and, in case D , $(\mathfrak{M}'\mathfrak{M}'')_* = (\mathfrak{M}'_*\mathfrak{M}''_*)$.

(7) If the classes $\mathfrak{M}', \mathfrak{M}''$ respectively, have the property D_i then

$$\sum_n \mu'_n = \mathfrak{z}' (\mathfrak{P}'; \mathfrak{M}'); \quad \sum_n \mu''_n = \mathfrak{z}'' (\mathfrak{P}''; \mathfrak{M}'')$$

implies

$$\sum_n \mu'_n \mu''_n = \mathfrak{z}' \mathfrak{z}'' (\mathfrak{P}'\mathfrak{P}''; \mathfrak{M}'\mathfrak{M}'').$$

²²) For propositions 1, 2, see loc. cit. ¹), Art. 9 a 1, 2; for 3, loc. cit. ¹), Art. 25, Theorem II; 4, loc. cit. ⁵), Art. 7, Theorem II; 5, loc. cit. ¹), Art. 26, 18; 6, loc. cit. ⁵), Art. 12 a 1, 2. Proposition 8 is the analogue for adjunctional composition of 6. Propositions 9 and 10 are, respectively, from loc. cit. ⁵), Arts. 25.1, 2; 32.5. For proposition 7 see loc. cit. ¹), Art. 13.6, Art. 24.7.

Relative to composition by adjunction:

(8) If the classes $\mathfrak{M}'$, $\mathfrak{M}''$ have the property D , resp. D_1 , the classes $(\mathfrak{M}' + \mathfrak{M}'')$, $(\mathfrak{M}' + \mathfrak{M}'')_L$, $(\mathfrak{M}' + \mathfrak{M}'')_*$ all have the property D , resp. D_1 .

(9) If a class $\mathfrak{M}$ is dominated by the class of constant functions and contains a function bounded from zero it has the property D .

(10) The hypotheses $\mathfrak{M}^{B, \mathfrak{M}} \text{ } ^{23}$, $\mathfrak{M}^{B, \mathfrak{M}}$ imply that $\mathfrak{M}^D$ is equivalent to $\mathfrak{M}^D$, and $\mathfrak{M}^{D_1}$ is equivalent to $\mathfrak{M}^{D_1}$.

7. *The closure properties A, L, C ²⁴*. — We define three properties of classes of functions on a general range which are fundamental in the theory of systems ($\mathfrak{A}$; $\mathfrak{B}$; $\mathfrak{M}$).

$\mathfrak{M}^A$ (read: $\mathfrak{M}$ absolute, or $\mathfrak{M}$ with the property of being absolute) is defined to be a class $\mathfrak{M}$ which contains the absolute function $\mathfrak{A}\mu$ of every function μ belonging to $\mathfrak{M}$.

$\mathfrak{M}^L$ (read: $\mathfrak{M}$ linear) is defined to be a class $\mathfrak{M}$ which contains every linear combination with constant coefficients of functions μ of $\mathfrak{M}$. It is evident that for every $\mathfrak{M}$ the class $\mathfrak{M}_L$ (Art. 4) is linear.

$\mathfrak{M}^C$ (read: $\mathfrak{M}$ closed) is a class $\mathfrak{M}$ containing every function of the form

$$\lim_{n \rightarrow \infty} \mu_n \quad (\mathfrak{B}; \mathfrak{M}).$$

Classes having the composite property ALC are for instance the class $\mathfrak{M}$ of all continuous functions on a continuous interval; the class $\mathfrak{M}$ of all functions on any range.

8. *Propositions involving the closure properties*. — The following important propositions are readily proved by consideration of the definitions involved.

$$(1) \quad \mathfrak{M}^{AL} \cdot \supset \cdot \mathfrak{M}^{D_1} \text{ } ^{25},$$

$$(2) \quad \mathfrak{M}^L \cdot \supset \cdot \mathfrak{M} = \mathfrak{M}_L \cdot \mathfrak{M}_* = \mathfrak{M}_{\mathfrak{M}} \cdot \mathfrak{M}^{LC} \cdot \supset \cdot \mathfrak{M} = \mathfrak{M}_*.$$

(3) $\mathfrak{M}^D$ implies $\mathfrak{M}_{\mathfrak{M}}$, $\mathfrak{M}_L$, $\mathfrak{M}_*$, $\mathfrak{A}\mathfrak{M}$ have the property D and are dominated by the class $\mathfrak{A}\mathfrak{M}$; $\mathfrak{M}_*$ is linear and closed; furthermore, if $\mathfrak{M}^{D^A}$, then $\mathfrak{M}_{\mathfrak{M}}$, $\mathfrak{M}_L$, $\mathfrak{M}_*$, and $\mathfrak{A}\mathfrak{M}$ are absolute and have the property D . [Cf., loc. cit. ¹), Art. 44 a].

(4) $\mathfrak{M}^{D_1}$ implies $\mathfrak{M}_{\mathfrak{M}}$, $\mathfrak{M}_L$, $\mathfrak{M}_*$, $\mathfrak{A}\mathfrak{M}$ have the property D_1 , and are dominated by the class $\mathfrak{A}\mathfrak{M}$; the class $\mathfrak{M}_*$ is linear. [Cf., loc. cit. ¹), Art. 44 a].

(5) Relative to multiplicative and adjunctional composition of classes P' , P'' , the $*$ -extension of the composite class $\mathfrak{M}$ of classes $\mathfrak{M}'$, $\mathfrak{M}''$, having the dominance property D , resp. D_1 , is linear, has the property D , resp. D_1 , is dominated by the class $\mathfrak{A}\mathfrak{M}$; and, in case D , is closed. [Cf., loc. cit. ¹), Art. 54, 55].

²³) Read: The class $\mathfrak{M}$ is dominated by the class $\mathfrak{A}\mathfrak{M}$, etc. Every function of the class $\mathfrak{M}$ has a dominant function of the form $a\nu$, where ν belongs to $\mathfrak{M}$.

²⁴) Loc. cit. ¹), Arts. 15, 16, 17.

²⁵) In the statement of short propositions it is convenient to adopt PEANO's implication sign and MOORE's property notation. Proposition (1) is equivalent to: If $\mathfrak{M}$ has the properties absolute and linear it has the dominance property D_1 .

GENERAL DEVELOPMENTAL THEORY. SYSTEMS $(\mathfrak{P}; \Delta)$.

**Definition of Development. Composition of Developments.
Properties of Developments.**

Articles 9-13.

9. Definition of general development ²⁶⁾. — A development Δ of a class $\mathfrak{P}$ of elements p is a sequence $\{\Delta^m\}$ of systems Δ^m of subclasses of $\mathfrak{P}$. The system Δ^m is stage m of the development Δ .

There is no question of order, but there may be repetitions amongst the classes of stage m , and it is to provide for this possibility of repetition that we say system Δ^m instead of class Δ^m .

As the theory is concerned with ranges $\mathfrak{P}$ and developments Δ we speak of the systems:

$$\Sigma: (\mathfrak{P}; \Delta).$$

With this understanding we introduce an index l^m of stage m , ranging over a class $\mathfrak{L}^m = [l^m]$ belonging to stage m of the development Δ . The superscript m of the index l^m is omitted when, in connection with the index m of the stage, it occurs in the bipartite index ml . We denote the classes of stage m by the notations

$$\mathfrak{P}^{ml}; [p^{ml}]$$

where l is understood to belong to the index class $\mathfrak{L}^m$ of stage m .

Thus stage m of the development Δ is the system:

$$\Delta^m = (\mathfrak{P}^{ml})$$

of all classes $\mathfrak{P}^{ml}$ of stage m , and the development Δ is the system:

$$\Delta = ((\mathfrak{P}^{ml}))$$

of subclasses $\mathfrak{P}^{ml}$ of the class $\mathfrak{P}$.

A *representative* system

$$\mathfrak{R} = ((r^{ml}))$$

of a development Δ is a system of elements r^{ml} of $\mathfrak{P}$, the element r^{ml} belonging to the class $\mathfrak{P}^{ml}$ of the development Δ . The element r^{ml} is said to represent the class $\mathfrak{P}^{ml}$ and its elements p^{ml} . It is understood however that only the existent (non-null) classes $\mathfrak{P}^{ml}$ have representative elements r^{ml} .

10. Various classes derived from a development Δ ²⁷⁾. — For a development Δ ele-

²⁶⁾ Cf. loc. cit. ¹⁾, Art. 75, which this follows, except for the introduction of the general index classes. MOORE considers only the case in which the index classes are finite.

²⁷⁾ Cf. loc. cit. ¹⁾, Art. 76.

ments p of $\mathfrak{P}$ are *developed* or *undeveloped* of stage m according as p does or does not belong to some class $\mathfrak{P}^{m_l}$ of stage m of the development Δ .

The class $\mathfrak{P}^m = [p^m]$ is the class of all elements (in notation p^m) *developed in stage* m , and the class $\mathfrak{P}^\Delta$ is the class of all elements (in notation p^Δ) *developed in the development* Δ , the least common superclass of the classes $\mathfrak{P}^{m_l}$ of Δ .

The class $\mathfrak{P}^{-m} = [p^{-m}] = \mathfrak{P} - \mathfrak{P}^m$ is the class of all elements (in notation p^{-m}) *not developed in stage* m , and the class $\mathfrak{P}_m$ is the class of all elements each for some stage $m_0 \geq m$ *not developed*, that is, the least common superclass of the classes $\mathfrak{P}^{-m_0}$ ($m_0 \geq m$).

As to an element p the indices l of existent classes of stage m are designated by the notations:

$$g^p; \quad h^p,$$

according as the class $\mathfrak{P}^{m_l}$ contains or does not contain the element p . The class $\mathfrak{P}^{p^m}$ is the least common superclass of the classes of stage m which contain p , the class of all elements, each of which is by stage m *directly connected* with the element p , in that they belong to the same class of stage m .

In applications of developmental theory various other derived classes need to be considered, these will be introduced as needed.

11. Developments of special ranges. The systems I-IV ²⁸). — We cite the following examples of developments of special ranges:

I. Development of $\mathfrak{P} = \mathfrak{P}^I$, the range with the single element unity. The *ordinary development* Δ^I of $\mathfrak{P}^I$ has for stages m the single class $\mathfrak{P} = \mathfrak{P}^I$.

II. Development of $\mathfrak{P} = \mathfrak{P}^{II_n}$, the range of the positive integers from one to n . The *ordinary development* Δ^{II_n} has stages m each consisting of m or n classes $\mathfrak{P}^{m_l} = [l]$ ($l = 1, 2, \dots$) according as $m \leq n$ or $m > n$.

III. Developments of $\mathfrak{P} = \mathfrak{P}^{III}$, the range of the totality of positive integers. The *ordinary development* Δ^{III} has stages m each consisting of m classes $\mathfrak{P}^{m_l} = [l]$, ($l = 1, 2, \dots, m$). This development differs in type from the preceding, in that not all the elements p are developed at any one stage, though every element p is developed at *some* stage m . We note that n undeveloped at stage m corresponds to $n > m$, from which it is apparent that convergence conditions can be stated in terms of this development of $\mathfrak{P}^{III}$.

Another type of development of $\mathfrak{P}^{III}$ possible under the present formulation has stages m whose index l ranges over the whole class of positive integers, and each class $\mathfrak{P}^{m_l}$ consists of the single element l . This development is of the type first presented, in that every element is developed at every stage m .

IV. Development of $\mathfrak{P} = \mathfrak{P}^{IV}$, the range of the real numbers p , $0 \leq p \leq 1$. The *ordinary development* Δ^{IV} has stages m , each consisting of $m + 1$ classes $\mathfrak{P}^{m_l}$ ($l = 1, 2, \dots, m, m + 1$) each class $\mathfrak{P}^{m_l}$ being the interval $\left(\frac{l-2}{m}, \frac{l}{m}\right)$, or (for $l = 1, m + 1$) that part of the interval contained in the interval $(0, 1)$.

²⁸) Cf. loc. cit. ¹), Arts. 66a, 67a stet loc. cit. ⁵), Art. 15.

The development Δ^{IV} corresponds to the development with decreasing norm usually employed in the definition of the definite integral of a function on $\mathfrak{P}^{IV}$.

In connection with the ranges I-IV we define classes I-IV ²⁹⁾ of functions as follows: $\mathfrak{M}^I$, $\mathfrak{M}^{II}$, $\mathfrak{M}^{III}$, are the classes of all functions on $\mathfrak{P}^I$, $\mathfrak{P}^{II}$, $\mathfrak{P}^{III}$, respectively; $\mathfrak{M}^{IV}$ is the class of all continuous functions on $\mathfrak{P}^{IV}$. The classes $\mathfrak{M}^{I-IV}$ are all linear, closed, and have the dominance property D .

We speak of the systems I-IV, where Σ^I is the system

$$\Sigma^I: (\mathfrak{A}; \mathfrak{P}^I; \Delta^I; \mathfrak{M}^I),$$

and the other three systems are similarly constituted.

12. Composition of developments. — We have formulated the theory of developments in terms of a general class $\mathfrak{P}$. The class $\mathfrak{P}$ may be given as the composite of classes $\mathfrak{P}$ (s ranging over a general class $\mathfrak{S}$) for which developments

$$\Delta^s = ((\mathfrak{P}^{sm^s}))$$

are also given. We proceed to define explicitly for the case of composition of two classes $\mathfrak{P}'$, $\mathfrak{P}''$ the composite development of the composite class $\mathfrak{P}$ in terms of the developments Δ' , Δ'' for the respective classes $\mathfrak{P}'$, $\mathfrak{P}''$.

12 a. Multiplicational Composition ³⁰⁾. — The class $\mathfrak{P}$ the composite class of the classes $\mathfrak{P}'$, $\mathfrak{P}''$ is defined in article 2 a. By multiplicational composition of the classes $\mathfrak{P}'^{ml'}$ of stage m of the development Δ' of $\mathfrak{P}'$ with the classes $\mathfrak{P}''^{m'l''}$ of stage m of the development Δ'' of $\mathfrak{P}''$, we obtain the classes

$$\mathfrak{P}^{ml} = \mathfrak{P}'^{ml'} \mathfrak{P}''^{m'l''}$$

of stage m of the composite development Δ of the developments Δ' , Δ'' . It is convenient to exhibit the composite character of the index l by writing $l'l''$, for example,

$$\mathfrak{P}^{ml} = \mathfrak{P}'^{ml'} \mathfrak{P}''^{m'l''} = \mathfrak{P}'^{ml'} \mathfrak{P}''^{m'l''}.$$

The *multiplicational composite development* $\Delta = \Delta' \Delta''$ of the developments Δ' , Δ'' is the system of classes

$$\Delta = ((\mathfrak{P}^{ml})) = ((\mathfrak{P}'^{ml'} \mathfrak{P}''^{m'l''})).$$

Similarly, from two representative systems;

$$\mathfrak{R}' = ((r'^{ml'})); \quad \mathfrak{R}'' = ((r''^{m'l''}))$$

of Δ' , Δ'' , respectively, we obtain a *composite system*

$$\mathfrak{R} = ((r^{ml})) = ((r'^{ml'} r''^{m'l''}))$$

where

$$r^{ml} = r'^{ml'} r''^{m'l''} = r'^{ml'} r''^{m'l''} = (r'^{ml'}, r''^{m'l''}),$$

in fact, a representative system of the composite development, the composite representative system $\mathfrak{R}$ of the representative systems $\mathfrak{R}'$, $\mathfrak{R}''$.

²⁹⁾ loc. cit. ¹⁾, Arts. 5, 23.

³⁰⁾ loc. cit. ¹⁾, Art. 83.

By employing the notion of product class of a system of classes we may extend the definition of composition by multiplication of classes to the general case.

12 b. Adjunctional composition. — As in Art. 2 b, the classes $\mathfrak{P}'$, $\mathfrak{P}''$ are assumed to have no common elements. The adjunctional composite class $\mathfrak{P}$ of the classes $\mathfrak{P}'$, $\mathfrak{P}''$ has already been defined.

From stage m of a development Δ' of $\mathfrak{P}'$, and stage m of a development Δ'' of $\mathfrak{P}''$, we form stage m of a development $\Delta = \Delta' + \Delta''$ of $\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}''$, namely, the system $(\mathfrak{P}^{m'})$, where each class $\mathfrak{P}^{m'}$ is either a class of stage m of Δ' , or a class of stage m of Δ'' . The correspondence is to be such that every index l' of stage m of Δ' corresponds uniquely to an index l of stage m of Δ , and every index l'' of stage m of Δ'' also corresponds uniquely to an index l of stage m of Δ .

The adjunctional composite representative system $\mathfrak{R}$ of representative systems $\mathfrak{R}'$ of Δ' and $\mathfrak{R}''$ of Δ'' is obtained by defining each element $r^{m'}$ of the system $\mathfrak{R}$ as the element representative of the class $\mathfrak{P}^{m'}$ before composition.

This method of composition may readily be extended to the general case.

13. Properties of Developments derived from properties of stages. — The character of a development Δ of a general range $\mathfrak{P}$ is determined by properties of the stages Δ^m , which in turn depend upon the character of the classes $\mathfrak{P}^{m'}$, and the index class $\mathfrak{L}^m$.

With respect to the character of the classes $\mathfrak{P}^{m'}$ of stage Δ^m we consider the properties:

V (*vacuous*), if every $\mathfrak{P}^{m'}$ is the null class;

F (*fine*), if no two elements of $\mathfrak{P}$ belong to the same class $\mathfrak{P}^{m'}$.

C (*complete*), if every element p of $\mathfrak{P}$ is developed at stage m .

With respect to the character of the index classes $\mathfrak{L}^m$ a stage Δ^m is:

(*finite*), if $\mathfrak{L}^m$ contains only a finite number of elements;

(*infinite*), if $\mathfrak{L}^m$ contains more than a finite number of elements.

A property P of stage Δ^m gives rise to certain derived properties of the development Δ . We shall need to consider but one such derived property, in notation,

Δ^P : The development Δ has the property P in case for every m the stage Δ^m has the property P .

Thus we have the properties V (*vacuous*), F (*fine*), etc., of developments Δ derived from the properties of stages Δ^m . The developments I-III of article 11 are F (*fine*) and *finite*. Δ^{IV} is C (*complete*) and *finite*.

One readily sees that many other properties of developments would be called for in their applications. We are at present concerned only with those defined above ³¹).

We note that the properties P of developments Δ considered in this article are individually invariant under both multiplicative and adjunctional composition.

³¹) This article follows closely loc. cit. ¹), Art. 67 b, Art. 75, loc. cit. ⁵), Art. 16. Dr. PITCHER makes a more detailed study of *finite* developments and obtains a large body of theorems in terms of conditioned developments.

PROPERTIES OF CLASSES OF FUNCTIONS DERIVED
FROM A DEVELOPMENT. SYSTEMS $(\mathfrak{A}; \mathfrak{B}; \Delta; \mathfrak{M})$.

Relations K_i . Properties K_i , K_i of Classes $\mathfrak{M}$.

Articles 14–15.

14. *The Relations K_1 , K_2 , K_{12} derived from a development* ³²⁾. — In his Introduction to a Form of General Analysis MOORE has developed for a class $\mathfrak{B}$ the general theory of relations K_1 , K_2 . The notations

$$\text{ad } K_1:K_{pm}; \quad \text{ad } K_2:K_{p_1p_2m}$$

denote pm , p_1p_2m fulfilling certain respective conditions.

In terms of a development Δ of $\mathfrak{B}$ we formulate conditions on pm , p_1p_2m as follows:

K_{pm} denotes an element p for some stage $m_i \geq m$ not developed.

$K_{p_1p_2m}$ denotes two elements p_1 , p_2 (not necessarily distinct) belonging to the same class of some stage $m_i \geq m$.

It is clear that for K_1 , K_2 as defined the following propositions hold:

- (1) ³³⁾ $K_{pm} \cdot \supset \cdot K_{pm_0} (m_0 \leq m); \quad K_{p_1p_2m} \cdot \supset \cdot K_{p_1p_2m_0} (m_0 \leq m),$
 (2) ³⁴⁾ $p \cdot m \cdot \supset \cdot K_{pm} \quad \text{or} \quad K_{ppm},$
 (3) $K_{p_1p_2m} \cdot \supset \cdot K_{p_2p_1m}.$

The relations K_1 , K_2 depending on the development are present simultaneously and we speak of the relation K_{12} . When we wish to consider the three relations at one time it is convenient to write K_i ($i = 1, 2, 12$) ³⁵⁾.

In terms of a development Δ , the composite development of a composite class $\mathfrak{B}$ of classes $\mathfrak{B}'$, $\mathfrak{B}''$, we obtain the *composite relation* K_i of the relations K'_i , K''_i associated with the developments Δ' , Δ'' respectively of $\mathfrak{B}'$, $\mathfrak{B}''$.

³²⁾ Cf. loc. cit. ¹⁾, Arts. 69, 70, 77.

³³⁾ If p and m are in the relation K_{pm} , then for every $m_0 \leq m$ the relation K_{pm_0} holds; etc.

³⁴⁾ For every p and m either K_{pm} or K_{ppm} holds.

³⁵⁾ In the ordinary development of $\mathfrak{B}^{III}$ the relation K_{pm} is equivalent to the relation $p \geq m$; and in the ordinary development of $\mathfrak{B}^{IV}$ the relation $K_{p_1p_2m}$ implies $A(p_1 - p_2) \leq 2/m$ while $A(p_1 - p_2) \leq 1/m$ implies $K_{p_1p_2m}$, and so in effect the relation $K_{p_1p_2m}$ is equivalent to the relation $A(p_1 - p_2) \leq 1/m$. In the first case K_{pm} is a relation of *infinitary* nature, in that it holds for large values of p , that is, as p tends towards ∞ . The relation $K_{p_1p_2m}$ is *infinitesimal* in nature, in that it holds for values of p_1 , p_2 distinct and close together. Cf. loc. cit. ¹⁾, Art. 67 b.

15. The properties K_1 , K_2 , K_{12} of functions and classes of functions. The extension properties of classes of functions ³⁶). — Relative to relations K_i ($i = 1, 2, 12$) on $\mathfrak{P}$; K'_i on $\mathfrak{P}'$, and classes $\mathfrak{M}$ on $\mathfrak{P}$; $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ we define properties $K_i\mathfrak{M}$; $K'_i\mathfrak{M}'(\mathfrak{M}'')$ of functions φ on $\mathfrak{P}$; $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ as follows:

(1) $\varphi^{K_1\mathfrak{M}}$: φ is such that there exists a function μ (belonging to the class $\mathfrak{M}$) such that for every e there is an m_e such that for every p in the relation $K_{p m_e}$ we have

$$A\varphi_p \leq e A\mu_{p'}$$

(2) $\varphi^{K_2\mathfrak{M}}$: There exists a function μ and for every e an m_e such that

$$K_{p_1 p_2 m_e} \cdot \supset \cdot A(\varphi_{p_1} - \varphi_{p_2}) \leq e A\mu_{p_1}$$

(3) $\varphi^{K'_1\mathfrak{M}'(\mathfrak{M}'')}$: There exist functions μ' , μ'' on $\mathfrak{P}'$, $\mathfrak{P}''$, and for every e an m_e such that

$$K'_{p'_1 m_e} \cdot p'' \cdot \supset \cdot A\varphi_{p'_1 p''} \leq e A\mu'_{p'_1} \mu''_{p''}$$

(4) $\varphi^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$: There exist functions μ' , μ'' and for every e an m_e such that

$$K'_{p'_1 p'_2 m_e} \cdot p'' \cdot \supset \cdot A(\varphi_{p'_1 p''} - \varphi_{p'_2 p''}) \leq e A\mu'_{p'_1} \mu''_{p''}$$

(5) $\varphi^{K_{12}\mathfrak{M}}$: φ has the property $K_1\mathfrak{M}$ and the property $K_2\mathfrak{M}$.

(6) $\varphi^{K'_{12}\mathfrak{M}'(\mathfrak{M}'')}$: φ has the properties $K'_1\mathfrak{M}'(\mathfrak{M}'')$, $K'_2\mathfrak{M}'(\mathfrak{M}'')$.

In definitions 3, 4, 6, φ is said to have the property $K'_i\mathfrak{M}'$ ($i = 1, 2, 12$) on the range $\mathfrak{P}'$ relatively uniformly as to the scale class $\mathfrak{M}''$.

From a property $B'\mathfrak{M}'$ of functions on $\mathfrak{P}'$ we derive two properties

$$B'_*; \quad B'$$

of classes $\mathfrak{M}'$ of functions on $\mathfrak{P}'$.

(7) $\mathfrak{M}'^{B'}$: A class $\mathfrak{M}'$ has the property B' in case its every function has the property $B'\mathfrak{M}'$.

From (7) on the basis of 1, 2, 5, we obtain the definitions of $\mathfrak{M}'^{K'_i}$ ($i = 1, 2, 12$).

(8) $\mathfrak{M}'^{B'_*}$: A class $\mathfrak{M}'$ has the property B'_* in case for every class $\mathfrak{M}''^{LCD}$ on every class $\mathfrak{P}''$ the $*$ -composite class $(\mathfrak{M}'\mathfrak{M}'')_*$ on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ may be characterized as the class of all functions on $\mathfrak{P}$ having the properties

$$B'\mathfrak{M}'(\mathfrak{M}''); \quad B_0\mathfrak{M}''(p').$$

In this case it is convenient to write

$$(\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \varphi^{B'\mathfrak{M}'(\mathfrak{M}'') \cdot B_0\mathfrak{M}''(p')}].$$

Since $\mathfrak{M}''$ is by hypothesis linear and closed, so that $\mathfrak{M}'' = \mathfrak{M}''_*$, it is evident that every function of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has for every p' the property $B_0\mathfrak{M}''$: is contained in $\mathfrak{M}''$ ³⁷). The property $B'\mathfrak{M}'(\mathfrak{M}'')$ is subject to definition for the various properties B' to be considered.

³⁶) Cf. loc. cit. ¹), Arts. 60, 61, 72, 77.

³⁷) Cf. Art. 4.1.

From (8) on the basis of definitions 3, 4, 6 we obtain the definitions of $\mathfrak{M}'^{K'}$.

We call attention to the composite property $(K'_{12}B_1)_*$ with the understanding that the property $B'\mathfrak{M}'(\mathfrak{M}'')$ of (8) is defined as the composite of the properties $K'_{12}\mathfrak{M}'(\mathfrak{M}'')$, $B_1\mathfrak{M}'(\mathfrak{M}'')$, where $B_1\mathfrak{M}'(\mathfrak{M}'')$ is defined to be $B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$, meaning: *dominated by the class $\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$* ³⁸).

15 a. Propositions concerning relations and classes of functions on $\mathfrak{P}$ ³⁹).

- (1.) $\mathfrak{N}^{K_1\mathfrak{M}} \cdot \supset \cdot (\mathfrak{A}\mathfrak{N})^{K_1\mathfrak{M}} \cdot (\mathfrak{A}\mathfrak{N})^{K_1\mathfrak{M}}$.
- (2.) $\mathfrak{M}_1^{B_1\mathfrak{A}\mathfrak{M}_2} \cdot \mathfrak{N}^{K_1\mathfrak{M}_1} \cdot \supset \cdot \mathfrak{N}^{K_1\mathfrak{M}_2}$.
- (3.) $\mathfrak{N}_1^{K_1\mathfrak{M}} \cdot \mathfrak{N}_2^{B_1\mathfrak{A}\mathfrak{M}_1} \cdot \supset \cdot \mathfrak{N}_2^{K_1\mathfrak{M}}$.
- (4.) $\mathfrak{M}^{D_1} \cdot \mathfrak{N}^{K_1\mathfrak{M}} \cdot \supset \cdot \mathfrak{N}_L^{K_1\mathfrak{M}}$.
- (5.) $\mathfrak{M}^D \cdot \mathfrak{N}_1^{K_1\mathfrak{M}} \cdot \mathfrak{N}_2^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}} \cdot \supset \cdot \mathfrak{N}_{1\mathfrak{N}_2}^{K_1\mathfrak{M}}$.
- (6.) $\mathfrak{M}^{D_1} \cdot \mathfrak{N}_1^{K_1\mathfrak{M}} \cdot \mathfrak{N}_2^{K_1\mathfrak{M}} \cdot \supset \cdot \mathfrak{N}_{1\mathfrak{N}_2}^{K_1\mathfrak{M}}$.
- (7.) $\mathfrak{M}^D \cdot \mathfrak{N}^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}} \cdot \supset \cdot \mathfrak{N}_L^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}} \cdot \mathfrak{N}_*^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}}$.
- (8.) $\mathfrak{M}^{D_1} \cdot \mathfrak{N}^{K_1\mathfrak{M}} \cdot \supset \cdot \mathfrak{N}_L^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}} \cdot \mathfrak{N}_*^{K_1\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}}$.
- (9.) $\mathfrak{M}^{DK_1} \cdot \supset \cdot \mathfrak{M}_L^{DK_1} \cdot \mathfrak{M}_*^{DK_1} \cdot \mathfrak{M}_*^{DK_1}$.
- (10.) $\mathfrak{M}^{D_1K_1} \cdot \supset \cdot \mathfrak{M}_L^{D_1K_1} \cdot \mathfrak{M}_*^{D_1K_1} \cdot \mathfrak{M}_*^{D_1K_1}$.
- (11.) $\mathfrak{M}^{K_1*} \cdot \supset \cdot \mathfrak{M}_* = [\text{all } \phi^{K_1\mathfrak{M}}] \cdot \mathfrak{M}_*^{AK_1}$.

15 b. Propositions concerning relations and classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ ⁴⁰). — The following propositions are generalizations in the sense of relativity of propositions of Art. 15 a.

For a class $\mathfrak{P}'$ with a development Δ' , a class $\mathfrak{P}''$, classes $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, and a class $\mathfrak{N}$ on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, we have the following propositions:

- (1.) $\mathfrak{M}'^{D_1} \cdot \mathfrak{M}''^{D_1} \cdot \mathfrak{N}^{K'_1\mathfrak{M}'(\mathfrak{M}'')} \cdot \supset \cdot \mathfrak{N}_L^{K'_1\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''} \cdot \mathfrak{N}_*^{K'_1\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}$
- (2.) $\mathfrak{M}'^{D_1K'_1} \cdot \mathfrak{M}''^{D_1} \cdot \supset \cdot (\mathfrak{M}'\mathfrak{M}'')_L^{D_1K'_1\mathfrak{M}'(\mathfrak{M}'')} \cdot (\mathfrak{M}'\mathfrak{M}'')_*^{D_1K'_1\mathfrak{M}'(\mathfrak{M}'')}$
- (3.) $\mathfrak{M}'^{DK'_1} \cdot \mathfrak{M}''^D \cdot \supset \cdot (\mathfrak{M}'\mathfrak{M}'')_L^{DK'_1\mathfrak{M}'(\mathfrak{M}'')} \cdot (\mathfrak{M}'\mathfrak{M}'')_*^{DK'_1\mathfrak{M}'(\mathfrak{M}'')}$ ($i = 1, 2, 12$).

15 c. Invariance of properties K_1 , K_2 , K_{12} under composition of classes $\mathfrak{M}$ and ranges $\mathfrak{P}$ ⁴¹). — We have for $i = 1, 2, 12$:

(1.) Given $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$; $\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}''$ on $\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}''$, and develop-

³⁸) Cf. Art. 5.

³⁹) Propositions 1, 2, 3, 4, 5, 7, 9, 11 are from Art. 72 a, loc. cit. 1; 6, 8, 10 are to be compared with 1, 2, 3 respectively of loc. cit. 5), Art. 29. They were originally stated by MOORE for relations K_1 , K_2 more general than those which we are at present considering, under conditions satisfied by virtue of Art. 14.1.

⁴⁰) Cf. loc. cit. 1), Art. 72 b 4: loc. cit. 5), Art. 29.5, 6, 9. It is evident that propositions 1-10 of Art. 17 a may be generalized in this manner.

⁴¹) Cf. loc. cit. 1), Arts. 54 a, 74, 84.

ments Δ' , Δ'' of $\mathfrak{P}$, $\mathfrak{P}''$; $\Delta = \Delta' + \Delta''$ of $\mathfrak{P}$, then

$$\mathfrak{M}^{K'_i} \cdot \mathfrak{M}^{''K''_i} \cdot \odot \cdot \mathfrak{M}^{K_i}$$

where K'_i , K''_i , K_i are relative to the respective developments Δ' , Δ'' , Δ .

We say that the property K_i is *invariant* under adjunctional composition. Similarly:

(2_i) The property $D_i K_i$ is invariant under multiplicative composition.

From loc. cit. ¹), Art. 74.12 we have:

(3) The property $D K_{12*}$ is invariant under multiplicative composition.

From 1_i above, Art. 6.8 and Art. 7.1, we may readily show by taking into account the definition of K_{i*} :

(4_i) The property $LD K_{i*}$ is invariant under adjunctional composition.

16. The dominance property D_3 .—In terms of a development Δ of a class $\mathfrak{P}$ we define a dominance property D_3 of classes $\mathfrak{M}$ of functions on $\mathfrak{P}$, namely ⁴²):

(1) $\mathfrak{M}^{D_3}$: A class $\mathfrak{M}$ has the property D_3 relative to a development Δ of $\mathfrak{P}$ in case there exists a sequence $\{n_m\}$ ⁴³) of integers n_m such that for every sequence $\{\mu_n\}$ of functions μ_n of the class $\mathfrak{M}$ there exists a sequence $\{a_n\}$ of numbers a_n and a function μ of $\mathfrak{M}$ such that

$$(a) \quad K_{\mathfrak{P}\mathfrak{P}n_m} \cdot m \cdot \odot \cdot A\mu_{m\mathfrak{P}} = Aa_m\mu_{\mathfrak{P}}.$$

16a. Sufficient conditions that $(\mathfrak{M}'\mathfrak{M}'')^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$.—The following proposition is to be compared with Art. 15b.3₂. For $\mathfrak{P}'$ with a development Δ' , $\mathfrak{M}'$; $\mathfrak{P}''$, $\mathfrak{M}''$, in case

$$(1) \quad \mathfrak{M}'^{D_1 D_3 K'_2} \cdot \mathfrak{M}''^D$$

the class

$$\mathfrak{F} = [\text{all } \phi^{K'_2\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{M}'\mathfrak{M}''}]$$

is linear, closed and contains the class $(\mathfrak{M}'\mathfrak{M}'')^*$.

The method of proof is suggested by Art. 66f of the Introduction to a Form of General Analysis. We notice that $\mathfrak{M}^{K'_2}$ implies $(\mathfrak{M}'\mathfrak{M}'')^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$, and hence $\mathfrak{M}'\mathfrak{M}''$ belongs to $\mathfrak{F}$. We proceed to show that $\mathfrak{F}$ is linear and closed, and so contains $(\mathfrak{M}'\mathfrak{M}'')^*$.

The presence of the properties D_1 , D in the hypothesis insures the linearity of $\mathfrak{F}$. It remains to prove that $\mathfrak{F}$ is closed.

We consider a sequence $\{\varphi_n\}$ of functions of the class $\mathfrak{F}$ which converges on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ uniformly as to the scale class $\mathfrak{F}$ to a function $\approx$. It is easy to show that $\approx$ is dominated by the class $\mathfrak{M}'\mathfrak{M}''$. We now prove $\approx^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$ by proving its equivalent $\approx^{K'_2\mathfrak{M}'\mathfrak{M}''}$.

The convergence is by hypothesis uniform as to some scale φ :

$$(a) \quad \lim_{n \rightarrow \infty} \varphi_n = \approx \quad (\mathfrak{P}; \varphi).$$

⁴²) The properties D_3 and D_4 (Art. 19) are of the same general type, in that they are defined (implicitly) in terms of a sequence of subclasses of $\mathfrak{P}$. In their applications they replace the stronger property D . The property D_4 is used implicitly by Dr. PITCHER in a special case.

⁴³) If $\{n_m\}$ is effective for D_3 , then any sequence dominating $\{n_m\}$ is also effective.

Each function φ, φ_n ($n = 1, 2, 3, \dots$) has the property $K'_2 \mathfrak{M}'(\mathfrak{M}'')$ which involves the existence of a scale function of the class $\mathfrak{M}' \mathfrak{M}''$ for each instance. It may be shown directly from the definitions involved that $\mathfrak{M}'^{D_1}, \mathfrak{M}''^D$ imply the existence of a single scale function $\mu' \mu''$ effective for the property $K'_2 \mathfrak{M}'(\mathfrak{M}'')$ in each of the instances φ, φ_n ($n = 1, 2, 3, \dots$). Furthermore, by $\mathfrak{M}'^{D_1}, \mathfrak{M}''^D, \varphi^{B_1 \mathfrak{M}' \mathfrak{M}''}$ we may suppose this scale function $\mu' \mu''$ so chosen that there exists a positive number a such that

$$(b) \quad A\varphi_{p'} \leq Aa\mu'\mu''(p')^{44}.$$

For $\mu' \mu''$ satisfying the conditions stated we have by hypothesis, for every e an n_e such that:

$$(c) \quad n \geq n_e \cdot \sup A(\varphi_{p'} - \varphi_{np'}) \leq e A\varphi_{p'} \quad (p');$$

and for every e and n there exists an m_{en} such that:

$$(d) \quad K'_{p'_1 p'_2 m_{en}} \cdot \sup A(\varphi_{np'_1} - \varphi_{np'_2}) \leq e A\mu'_{p'_1} \mu'';$$

and for $e = 1$ there exists m_1 such that:

$$(e) \quad K'_{p'_1 p'_2 m_1} \cdot \sup A(\varphi_{p'_1} - \varphi_{p'_2}) \leq A\mu'_{p'_1} \mu''.$$

Now

$$A(\mathfrak{z}_{p'_1} - \mathfrak{z}_{p'_2}) \leq A(\mathfrak{z}_{p'_1} - \varphi_{np'_1}) + A(\varphi_{np'_1} - \varphi_{np'_2}) + A(\varphi_{np'_2} - \mathfrak{z}_{p'_2}).$$

Hence, in case: $n \geq n_e$; $K'_{p'_1 p'_2 m_{en}}$, we have from (c) and (d)

$$A(\mathfrak{z}_{p'_1} - \mathfrak{z}_{p'_2}) \leq e(A\varphi_{p'_1} + A\mu'_{p'_1} \mu'' + A\varphi_{p'_2}),$$

and in case further $K'_{p'_1 p'_2 m_1}$, from (e) and (b)

$$A(\mathfrak{z}_{p'_1} - \mathfrak{z}_{p'_2}) \leq e(2A\varphi_{p'_1} + 2A\mu'_{p'_1} \mu'') \leq e(2a + 2)A\mu'_{p'_1} \mu''.$$

Accordingly, taking m_e^* as the greater of m_1 and m_{en_e} , we have the relation, holding for every e :

$$(f) \quad K'_{p'_1 p'_2 m_e^*} \cdot \sup A(\mathfrak{z}_{p'_1} - \mathfrak{z}_{p'_2}) \leq e(2a + 2)A\mu'_{p'_1} \mu'',$$

and therefore

$$(g) \quad \mathfrak{z}^{K'_2 \mathfrak{M}'(\mathfrak{M}'')},$$

which was to be proved.

16b. Invariance of D_3 under composition and extension of classes $\mathfrak{P}$ and $\mathfrak{M}$. — From the proposition:

$$(1) \quad \mathfrak{M}_1^{D_3 \cdot B_1 \mathfrak{M}_2} \cdot \mathfrak{M}_2^{B_1 \mathfrak{M}_1} \cdot \sup \mathfrak{M}_2^{D_3},$$

which follows directly from the definition of D_3 we readily obtain

$$(2) \quad \mathfrak{M}^{D_1 D_3} \cdot \sup \mathfrak{M}_*^{D_1 D_3}.$$

Relative to classes $\mathfrak{P}', \mathfrak{P}''$ with developments Δ', Δ'' and classes $\mathfrak{M}', \mathfrak{M}''$ we

⁴⁴) In case the variable p'' is not in explicit evidence validity for every p'' is to be understood.

have for the cases of multiplicative and adjunctional composition:

$$(3) \quad \mathfrak{M}^{D_3'} \cdot \mathfrak{M}^{D_3''} \cdot \supset \cdot \mathfrak{M}^{D_3},$$

which is provable without much difficulty by direct consideration of the definitions involved. In the respective cases $\mathfrak{M} = \mathfrak{M}'\mathfrak{M}''$; $\mathfrak{M}' + \mathfrak{M}''$, and D_3 is relative to $\Delta = \Delta'\Delta''$; $\Delta' + \Delta''$.

From (2) and (3) we have, for $\mathfrak{M}$ and Δ as in (3):

$$(4) \quad \mathfrak{M}^{D_1 D_3'} \cdot \mathfrak{M}^{D_1 D_3''} \cdot \supset \cdot \mathfrak{M}^{D_1 D_3}.$$

17. The property D_4 . Definitions and notation ⁴⁵). — For convenience, following Dr. PITCHER, we introduce the relation $\bar{K}_{pm}$, the *negative* of the relation K_{pm} previously defined (Art. 14). We have, more explicitly ⁴⁶),

(1) K_{pm} : p is undeveloped at stage m or some later stage. The class $\mathfrak{P}_m$ is the class of all elements p such that K_{pm} .

(2) $\bar{K}_{pm}$: p is developed at stage m and every later stage. The class $\mathfrak{P}_{-m}$ is the class of all elements p such that $\bar{K}_{pm}$. The classes $\mathfrak{P}_m$ and $\mathfrak{P}_{-m}$ are complementary for every m .

In terms of the relation $\bar{K}_{pm}$ we proceed to define the dominance property D_4 , relative to a development Δ of $\mathfrak{P}$.

(4) $\mathfrak{M}^{D_4}$: A class $\mathfrak{M}$ has the property D_4 relative to a development Δ of $\mathfrak{P}$ in case there exists for every sequence $\{\mu_m\}$ of functions μ_m of the class $\mathfrak{M}$ a sequence $\{a_m\}$ and a function μ of the class $\mathfrak{M}$ such that

$$(a) \quad \bar{K}_{pm} \cdot \supset \cdot A\mu_{mp} \leq Aa_m\mu_p \quad (m).$$

We readily see that $\mathfrak{M}^D$ or $\mathfrak{M}^{D_3}$ implies $\mathfrak{M}^{D_4}$ ⁴⁷).

17 a. Sufficient conditions that $(\mathfrak{M}'\mathfrak{M}'')^{K'_{12}\mathfrak{M}'(\mathfrak{M}'')}$. For $\mathfrak{P}'$ with a development Δ' , $\mathfrak{M}'$; $\mathfrak{P}''$, $\mathfrak{M}''$, we have the following proposition:

$$(1) \quad \mathfrak{M}^{D_1 D_4' K'_{12}}, \quad \mathfrak{M}''^D \quad \text{imply that the class:}$$

$$\mathfrak{F} = [\text{all } \varphi^{K'_{12}\mathfrak{M}'(\mathfrak{M}'') \cdot B_1 \mathfrak{M}'\mathfrak{M}''}]$$

is linear, closed, and contains the class $(\mathfrak{M}'\mathfrak{M}'')^*$.

The method of proof is similar to that used in Art. 16 a ⁴⁸). It follows as in the preceding argument that $\mathfrak{F}$ contains $\mathfrak{M}'\mathfrak{M}''$ and is linear. It remains to prove that $\mathfrak{F}$ is closed.

⁴⁵) The present article is suggested by proposition (12) of loc. cit. ⁵), Art. 39 a, and exhibits in explicit form the principle which underlies Dr. PITCHER's proof.

⁴⁶) loc. cit. ⁵), Art. 39, 1, 2.

⁴⁷) Since $\mathfrak{P}_{-m_0}$ is contained in $\mathfrak{P}_{-m}(m \geq m_0)$ it was possible to omit the sequence n_m in defining D_4 without strengthening the property. This is not the case with D_3 .

⁴⁸) This proof is suggested by the proofs of MOORE, loc. cit. ¹), Art. 66 f, PITCHER, loc. cit. ⁵), Art. 39 a, 12.

Under the hypothesis we consider a sequence $\{\varphi_n\}$ of functions of the class $\mathfrak{F}$ converging relatively uniformly as to the scale class $\mathfrak{F}$ to a limit function $\mathfrak{z}$. It is easy to show that $\mathfrak{z}$ is dominated by the class $\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$. Hence by Art. 15 a.3 (generalized in the sense of relativity) $\mathfrak{z}$ has the property $K'_1\mathfrak{M}'(\mathfrak{M}'')$. We are now to prove $\mathfrak{z}^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$ by proving its equivalent $\mathfrak{z}^{K'_2\mathfrak{A}\mathfrak{M}'(\mathfrak{M}'')}$.

By hypothesis $\mathfrak{z}$ is the limit of a sequence $\{\varphi_n\}$ converging relatively uniformly as to a scale function φ . Hence, for every ϵ there exists n_ϵ such that

$$(a) \quad n \geq n_\epsilon \cdot \odot \cdot A(\mathfrak{z}_p - \varphi_{n_p}) \leq \epsilon A\varphi_p \quad (p).$$

The function φ has the properties $K'_2\mathfrak{M}'(\mathfrak{M}'')$, $B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$; and $\mathfrak{M}'^{D_1}$, $\mathfrak{M}''^D$. Hence there exist a positive number a_0 and a function $\mu'_0\mu''_0$ such that

$$(b) \quad A\varphi_p \leq a_0 A\mu'_{0p'}\mu''_{0p''} \quad (p),$$

and for $\epsilon = 1$ there exists m_0 such that

$$(c) \quad K'_{p'_1p'_2m_0} \cdot \odot \cdot A(\varphi_{p'_1} - \varphi_{p'_2}) \leq A\mu'_{0p'_1}\mu''_0.$$

From (a) for $\epsilon = 1$, $\mathfrak{z}^{K'_1\mathfrak{M}'(\mathfrak{M}'')}$, $\varphi^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$, $(\mathfrak{M}'\mathfrak{M}'')^{K'_{12}\mathfrak{M}'(\mathfrak{M}'')}$ and (b) we may show without much difficulty that there exists n_1 , $\mu'_1\mu''_1$ such that for every ϵ we have m_ϵ such that

$$(d) \quad n \geq n_1 \cdot K'_{p'_1m_\epsilon} \cdot K'_{p'_1p'_2m_\epsilon} \cdot \odot \cdot A\varphi_{n_{p'_1}} + A\varphi_{n_{p'_2}} \leq \epsilon A\mu'_{1p'_1}\mu''_1.$$

From $\varphi_n^{K'_2\mathfrak{M}'(\mathfrak{M}'')}$ (n), $\mathfrak{M}'^{D_1D_4}$, $\mathfrak{M}''^D$, we obtain $\mu'_2\mu''_2$ such that for every ϵ and n there exists $m_{\epsilon n}$ such that

$$(e) \quad \bar{K}'_{p'_1n} \cdot K'_{p'_1p'_2m_{\epsilon n}} \cdot \odot \cdot A(\varphi_{n_{p'_1}} - \varphi_{n_{p'_2}}) \leq \epsilon A\mu'_{2p'_1}\mu''_2.$$

For $n \geq m_\epsilon$ the elements p' which are either $K'_{p'_1m_\epsilon}$ or $\bar{K}'_{p'_1n}$ are all the elements of $\mathfrak{P}'$ (cf. Art. 17.1, 2). Hence, if we take $\bar{n}_\epsilon$ equal to the greatest of n_ϵ , n , m_ϵ ; and take $\bar{m}_{\epsilon n}$ equal to the greater of m_ϵ , $m_{\epsilon n}$, we have from (d), (e),

$$(f) \quad K'_{p'_1p'_2\bar{m}_{\epsilon n}} \cdot \odot \cdot A(\varphi_{n_{p'_1}} - \varphi_{n_{p'_2}}) \leq \epsilon (A\mu'_{1p'_1}\mu''_1 + A\mu'_{2p'_1}\mu''_2)(n = \bar{n}_\epsilon).$$

If we now take (a new) m_ϵ equal to the greater of m_0 , $\bar{m}_{\epsilon n}$, then from (a), (b), (c), (f) it follows by the method used in Art. 16 a.1, that

$$(g) \quad K'_{p'_1p'_2m_\epsilon} \cdot \odot \cdot A(\mathfrak{z}_{p'_1} - \mathfrak{z}_{p'_2}) \leq \epsilon (2a + 1 A\mu'_{0p'_1}\mu''_0 + A\mu'_{1p'_1}\mu''_1 + A\mu'_{2p'_2}\mu''_2),$$

as may be easily verified. Then from (g) and $\mathfrak{M}'^{D_1}$, $\mathfrak{M}''^D$ it follows that

$$\mathfrak{z}^{K'_2\mathfrak{A}\mathfrak{M}'(\mathfrak{M}'')}.$$

We have shown that $\mathfrak{z}^{K'_{12}\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}$ and according to the hypothesis, therefore, $\mathfrak{z}$ belongs to $\mathfrak{F}$. Consequently $\mathfrak{F}$ is closed. This completes the proof.

As a corollary of (1) we have, for $\mathfrak{B}$ with development Δ :

$$(2) \quad \mathfrak{M}^{D_1D_4K_{12}} \text{ implies that the class}$$

$$\mathfrak{F} = [\text{all } \varphi^{K_{12}\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}}]$$

is linear and closed and contains $\mathfrak{M}_*$.

17 b. Invariance of D_4 under composition and extension of ranges $\mathfrak{P}$ and classes $\mathfrak{M}$.—
The following proposition follows directly from the definition of $\mathfrak{M}^{D_4}$.

$$(1) \quad \mathfrak{M}_1^{D_4 \cdot B_1 \mathfrak{M}_2} \cdot \mathfrak{M}_2^{B_1 \mathfrak{M}_1} \cdot \supset \cdot \mathfrak{M}_2^{D_4}.$$

A corollary of (1) is:

$$(2) \quad \mathfrak{M}^{D_1 D_4 \cdot \supset \cdot \mathfrak{M}_*^{D_1 D_4}}.$$

Relative to classes $\mathfrak{P}'$, $\mathfrak{P}''$ with respective developments Δ' , Δ'' , we have for the cases of multiplicative and adjunctional composition:

$$(3) \quad \mathfrak{M}^{D'_4} \cdot \mathfrak{M}^{D''_4} \cdot \supset \cdot \mathfrak{M}_*^{D_4},$$

where the class $\mathfrak{M} = \mathfrak{M}' \mathfrak{M}''$; $\mathfrak{M}' \neq \mathfrak{M}''$, and D_4 is relative to $\Delta' \Delta''$; $\Delta' \neq \Delta''$, respectively.

Remembering that in the case of adjunctional composition the classes $\mathfrak{P}'$, $\mathfrak{P}''$ have no common elements, the proposition is for this case obvious.

A reference to the definition of the composite development $\Delta' \Delta''$ (Art. 12 a) shows us that the classes $\mathfrak{P}_{-m}$ (Art. 17) of $\Delta = \Delta' \Delta''$ are in fact the corresponding product classes $\mathfrak{P}'_{-m} \mathfrak{P}''_{-m}$. The proposition follows readily.

From (2) and (3) we have for both cases of composition:

$$(4) \quad \mathfrak{M}^{D_1 D'_4} \cdot \mathfrak{M}^{D_1 D''_4} \cdot \supset \cdot \mathfrak{M}_*^{D_1 D_4}.$$

18. Relations between finite and infinite developments in regard to the property K_{12} .—
We have the proposition:

(1) There exists a range $\mathfrak{P}$ and a class of functions $\mathfrak{M}$ on $\mathfrak{P}$ which does not have the property K_{12} relative to any finite development of $\mathfrak{P}$, but which has the property K_{12} relative to an infinite development of $\mathfrak{P}$.

To exhibit the example required by the proposition, take $\mathfrak{P} = \mathfrak{P}^{III}$; the class $\mathfrak{M}$ as the single function μ , where $\mu_p = p$. Since $\mathfrak{M}$ is to have the property K_1 relative to a development Δ there must be an m such that

$$K_{pm} \cdot \supset \cdot A\mu_p \leq \frac{1}{2} A\mu_p$$

which is impossible unless the development Δ in question is such that after some stage m , every stage is complete (Art. 13). But relative to a *finite* development of this type the class $\mathfrak{M}$ cannot have the property K_2 , since a finite stage m which is complete must then contain classes with more than a finite number of elements p , from which we readily see that there is no m such that

$$K_{p_1 p_2 m} \cdot \supset \cdot A(\mu_{p_1} - \mu_{p_2}) \leq A\mu_{p_1}.$$

On the other hand, if the development Δ is infinite complete and fine, the class $\mathfrak{M}$ has the property K_{12} relative to Δ .

The proposition just established shows that the present theory includes cases not covered by the theory limited to finite developments.

CHARACTERIZATIONS OF $(\mathfrak{M}'\mathfrak{M}'')$. SYSTEMS $(\mathfrak{A}; \mathfrak{B}; \Delta; \mathfrak{M})$.

Developmental systems. Composition of developmental systems.

The property Δ of classes $\mathfrak{M}$.

Articles 19-21.

19. *Developmental systems $\mathfrak{D}(\mathfrak{M})$ relative to a class $\mathfrak{M}$ of functions and a development Δ* ⁴⁹⁾. — In the remaining part of this paper we shall consider only those developments of $\mathfrak{B}$ for which the index classes $\mathfrak{L}^m$ are either finite or denumerably infinite⁵⁰⁾. In terms of a development of this type we define a developmental system of functions relative to a given class $\mathfrak{M}$ of functions.

A developmental system

$$(1) \quad \mathfrak{D}(\mathfrak{M}) = ((\delta^{mi}))$$

is a system of functions δ^{mi} on $\mathfrak{B}$ for which there exists a representative system $\mathfrak{R}^\Delta = ((r^{mi}))$ of the development Δ such that the following conditions are satisfied:

(1 a) For every ϵ there exists an m_ϵ such that the inequalities⁵¹⁾

$$A(\sum_g \delta_p^{mg} - 1) \leq \epsilon; \quad A(\sum_g A \delta_p^{mg} - 1) \leq \epsilon$$

hold for every $m \geq m_\epsilon$ and every p (uniformly on $\mathfrak{B}$).

(1 b) For every function μ of the class $\mathfrak{M}$ there exists a function μ_o of the class $\mathfrak{M}$, and for every ϵ an m_ϵ , such that⁵²⁾

$$\sum_b A \mu_{r^{mb}} \delta_p^{mb} \leq \epsilon A \mu_o$$

for every $m \geq m_\epsilon$ and uniformly on $\mathfrak{B}$.

(1 c) For every μ there exists a μ_i and an m_i such that the series⁵³⁾

$$\sum_i A \mu_{r^{mi}} \delta^{mi} \quad (\mathfrak{B}; \mu_i)$$

converges for all $m \geq m_i$.

Here the indices g, b are understood to be g^p, b^p (Art. 10); and the conditions

49) Cf. loc. cit. ¹⁾, Art. 78 which this follows, except for the introduction of the condition 1 c. This condition is introduced for the purpose of generalization and is always satisfied if the development Δ is finite.

50) The index classes $\mathfrak{L}^m$ may therefore be taken as classes of positive integers in the natural order.

51) It is understood that for every p the summation extends over the corresponding indices g only. Cf. Art. 10.

52) It is understood that for every p the summation extends over the corresponding indices b only.

53) Cf. Art. 3.

$1a$, $1b$ are understood to apply only in so far as such indices are available, that is, for the various stages m the condition $1a$ applies only for those elements p which belong to at least one class of stage m , and the condition $1b$ applies only for those elements p which do not belong to every existent class of stage m .

The condition $1a$ is an absolute condition. Conditions $1b$ and $1c$ are relative to the class $\mathfrak{M}$ ⁵⁴).

By Art. 8.4 we readily see that $\mathfrak{M}^D$ implies that the systems $\mathfrak{D}(\mathfrak{M})$ are the systems $\mathfrak{D}(\mathfrak{M}_L)$, $\mathfrak{D}(\mathfrak{M}_*)$.

20. Composition of developmental systems. — In terms of developments Δ' , Δ'' of classes $\mathfrak{P}'$, $\mathfrak{P}''$ we have defined (Art. 14) the composite development Δ of the composite class $\mathfrak{P}$ for the cases of multiplicative and adjunctive composition. We proceed to define the composite developmental system $\mathfrak{D}$ of developmental systems $\mathfrak{D}'(\mathfrak{M}')$, $\mathfrak{D}''(\mathfrak{M}'')$, relative to classes $\mathfrak{M}'$, $\mathfrak{M}''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ and developments Δ' , Δ'' .

20a. Multiplicative composition of developmental systems ⁵⁵). — We consider the case of multiplicative composition of two classes $\mathfrak{P}'$, $\mathfrak{P}''$ and their developments Δ' , Δ'' . The extension to any finite number of classes can be made by simple induction.

From two developmental systems:

$$\mathfrak{D}'(\mathfrak{M}') = ((\delta'^{mi'})); \quad \mathfrak{D}''(\mathfrak{M}'') = ((\delta''^{mi''})),$$

relative to systems $(\mathfrak{P}'; \Delta'; \mathfrak{M}')$, $(\mathfrak{P}''; \Delta''; \mathfrak{M}'')$ respectively, we obtain by multiplication a *composite system*:

$$\mathfrak{D} = ((\delta^{mi})) = ((\delta^{mi' i''}))$$

where

$$\delta^{mi' i''} = \delta^{mi'} \delta^{mi''}.$$

The developmental system so obtained will not in general satisfy the conditions $1a$, $1b$, $1c$, Art. 21, relative to the class $\mathfrak{M} = \mathfrak{M}'\mathfrak{M}''$. However, it happens that in the cases with which we are primarily concerned these conditions are satisfied ⁵⁶).

20b. Adjunctive composition of developmental systems. — As before we consider only the case of composition of two classes $\mathfrak{P}'$, $\mathfrak{P}''$ and their developments Δ' , Δ'' . The extension to the case of composition for any finite number of classes and developments is easily made.

The composite class of two classes of functions has been defined only for the case in which the classes $\mathfrak{P}'$, $\mathfrak{P}''$ have no common elements. We confine ourselves to this case.

From two developmental systems $\mathfrak{D}'(\mathfrak{M}')$, $\mathfrak{D}''(\mathfrak{M}'')$ we derive a developmental

⁵⁴) The system $\mathfrak{D}(\mathfrak{M})$ need not actually depend on the class $\mathfrak{M}$. For example, a system $\mathfrak{D}$ satisfying $1a$, and the further conditions: (b) there exists l_0 (independent of m) such that for every $l \geq l_0$ and m , $\delta_p^{ml} = 0$; (c) for every m , p , and h^p , $\delta_p^{mh} = 0$, is easily seen to be a system $\mathfrak{D}(\mathfrak{M})$ as to every class $\mathfrak{M}$.

⁵⁵) Cf. loc. cit. ¹), Art. 83.

⁵⁶) Cf. Art. 21.3, Art. 25.

system $\mathfrak{D}$ on $\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}''$, relative to the composite development $\Delta = \Delta' + \Delta''$ of $\mathfrak{P}$ and composite class $\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}''$ as follows:

The developmental system $\mathfrak{D} = ((\delta^{ml}))$ is composed of functions δ^{ml} (on the composite class $\mathfrak{P}$) defined in terms of the index ml . To each index ml there corresponds (from the method of composition, cf. Art. 12 *b*) a unique index ml' (or ml'') and therefore a unique function $\delta^{ml'}$ (or $\delta^{ml''}$). The function δ^{ml} is defined to be equal to the function ($\delta^{ml'}$ or $\delta^{ml''}$) corresponding to ml on the class for which that function is defined, and is defined to be equal to zero (identically) on the remaining class.

The composite system $\mathfrak{D}$ of the systems $\mathfrak{D}'(\mathfrak{M}')$, $\mathfrak{D}''(\mathfrak{M}'')$ is easily seen to be a system $\mathfrak{D}(\mathfrak{M}' + \mathfrak{M}'')$ for the development $\Delta = \Delta' + \Delta''$, since the values of the sums appearing in conditions 1 *a*, *b*, *c* of Art. 19 are unchanged by composition.

21. *The developmental property Δ of classes $\mathfrak{M}$ relative to a development Δ ⁵⁷⁾. A class $\mathfrak{M}$ has the developmental property Δ (in notation $\mathfrak{M}^\Delta$) in case there exists a developmental system $\mathfrak{D}(\mathfrak{M})$ belonging to the class $\mathfrak{M}$. — We have the following propositions:*

$$(1) \quad \mathfrak{M}^\Delta \cdot \mathfrak{D} \cdot (\mathfrak{M}\mathfrak{M})^\Delta.$$

$$(2) \quad \mathfrak{M}^{D_1\Delta} \cdot \mathfrak{D} \cdot \mathfrak{M}_L^{D_1\Delta} \cdot \mathfrak{M}_*^{D_1\Delta}. \quad [\text{loc. cit. } ^5), \text{ Art. 18 } a. 3].$$

In the case of multiplicative composition of classes $\mathfrak{P}'$, $\mathfrak{P}''$, we have, relative to Δ' , Δ'' ; $\Delta = \Delta' \Delta''$: $\mathfrak{M}'$, $\mathfrak{M}''$; $\mathfrak{M} = \mathfrak{M}'\mathfrak{M}''$, the proposition:

$$(3) \quad \mathfrak{M}^{D_1K_2'\Delta'} \cdot \mathfrak{M}''^{D_1K_2''\Delta''} \cdot \mathfrak{D} \cdot \mathfrak{M}^{D_1\Delta} \cdot \mathfrak{M}_*^{D_1\Delta}.$$

We prove the proposition for $\mathfrak{M} = \mathfrak{M}'\mathfrak{M}''$. Then from (2) it follows for $\mathfrak{M}_*$. It is necessary to show that the composite developmental system $\mathfrak{D}$ of systems $\mathfrak{D}'(\mathfrak{M}')$, $\mathfrak{D}''(\mathfrak{M}'')$ as defined in Art. 20 *a* is a system $\mathfrak{D}(\mathfrak{M}'\mathfrak{M}'')$.

For a given $p = p'p''$, the indices $l = l'l''$ of a given stage m of Δ separate into indices $g = g'g''$; $h = h'h''$, $h'g''$ or $g'h''$ (Cf. Art. 10).

From condition 1 *a* of the hypothesis on $\mathfrak{D}'$, $\mathfrak{D}''$ (Art. 19), we have for sufficiently large m , and uniformly on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$,

$$(a) \quad \sum_{g'} A\delta_{p'}^{mg'} \leq 2; \quad \sum_{g''} A\delta_{p''}^{mg''} \leq 2.$$

Hence we may write the identity:

$$\sum_g A\delta_p^{mg} - 1 = \left(\sum_{g'} A\delta_{p'}^{mg'} - 1\right) \left(\sum_{g''} A\delta_{p''}^{mg''} - 1\right) + \left(\sum_{g'} A\delta_{p'}^{mg'} - 1\right) + \left(\sum_{g''} A\delta_{p''}^{mg''} - 1\right),$$

and from this formula it is easy to see that $\mathfrak{D}$ satisfies condition 1 *a*.

The proof that the system $\mathfrak{D}$ satisfies the condition 1 *b* rests upon the lemma:

LEMMA: Given a range $\mathfrak{P}$ with a development Δ , and a class $\mathfrak{M}$ of functions on $\mathfrak{P}$, $\mathfrak{M}$ having the properties D_1K_2 (relative to Δ), and having a developmental system $\mathfrak{D}(\mathfrak{M})$, then for every function μ of the class $\mathfrak{M}$ there exists an integer m_0 , a positive number α_0 , and a function μ_0 such that for every m greater than or equal to m_0 ,

57) Cf. loc. cit. ¹⁾, Arts. 79, 83.

and every p we have

$$\sum_g A\mu_{r,mg} \delta_p^{mg} \leq a_o A\mu_{op}.$$

The proof of this lemma is perfectly straightforward, and is omitted.

From 1 b , c of the hypothesis on $\mathfrak{D}'$, $\mathfrak{D}''$ it follows that we may write the identity:

$$\begin{aligned} \sum_h A\mu_{r,mh} \delta_p^{mh} &= \left(\sum_{h'} A\mu'_{r,mh'} \delta_{p'}^{mh'} \right) \left(\sum_{h''} A\mu''_{r,mh''} \delta_{p''}^{mh''} \right) \\ &+ \left(\sum_{h'} A\mu'_{r,mh'} \delta_{p'}^{mh'} \right) \left(\sum_{g''} A\mu''_{r,mg''} \delta_{p''}^{mg''} \right) + \left(\sum_{g'} A\mu'_{r,mg'} \delta_{p'}^{mg'} \right) \left(\sum_{h''} A\mu''_{r,mh''} \delta_{p''}^{mh''} \right), \end{aligned}$$

which holds for m sufficiently large.

If we now make use of the lemma and the fact that $\mathfrak{D}'$, $\mathfrak{D}''$ satisfy condition 1 b , we may readily reduce (by means of $\mathfrak{M}^{D_1}$, $\mathfrak{M}''^{D_1}$) the preceding identity to the form of the inequality in condition 1 b , and thus establish the desired conclusion.

Under the hypothesis of the Lemma, making use of the property 1 b , we have for every function μ of the class $\mathfrak{M}$ an integer m_o , a positive number a_o , and a function μ_o such that

$$m \geq m_o \cdot p \cdot \sum A\mu_{r,ml} \delta_p^{ml} \leq a_o A\mu_{op}.$$

By the aid of this proposition, the fact that $\mathfrak{D}'$, $\mathfrak{D}''$ satisfy condition 1 c , and $\mathfrak{M}^{D_1}$, $\mathfrak{M}''^{D_1}$; it becomes simply a matter of writing down the conditions to show that $\mathfrak{D}$ satisfies the condition 1 c relative to $\mathfrak{M} = \mathfrak{M}' \mathfrak{M}''$. This completes the proof ⁵⁸⁾.

In the case of adjunctional composition the classes $\mathfrak{P}'$, $\mathfrak{P}''$ have no common elements. From the considerations of Art. 20 b we easily see that the following proposition is true:

$$(4) \quad \mathfrak{M}^{L\Delta'} \cdot \mathfrak{M}^{L\Delta''} \cdot \mathfrak{M}^{L\Delta} \quad (\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}'').$$

22. Classes contained by $(\mathfrak{M}' \mathfrak{M}'')$. — The following proposition is a generalization of Theorem I of Article 80, Introduction to a Form of General Analysis. The proof is not given because it is not essentially different from the proof of the corresponding theorem of MOORE ⁵⁹⁾.

$$(1) \quad \mathfrak{M}^{D_1\Delta} \cdot \mathfrak{F} = [\text{all } \phi^{K_{12}\mathfrak{M}.B_1\mathfrak{M}\mathfrak{M}'}] \cdot \mathfrak{F}^{B_o\mathfrak{M}*}.$$

⁵⁸⁾ Cf. loc. cit. ¹⁾, Art. 13.6, 24.7, and Art. 6.7 of this paper.

⁵⁹⁾ Professor MOORE has suggested that we insert here the following amendment to the proof of loc. cit. ¹⁾, Art. 80, as there printed. Page 143, lines 14-26 should read:

«Then, for this element p , separating the indices l of stage m appearing in (9) into the indices g , h , we have:

$$\Theta_{mp} - \varphi_p = \sum_h \varphi_{r,mh} \delta_p^{mh} + \sum_g (\varphi_{r,mg} - \varphi_p) \delta_p^{mg} + \varphi_p \left(\sum_g \delta_p^{mg} - 1 \right).$$

so that, since by (19)

$$A(\varphi_p - \varphi_{r,mg}) \leq A(\varphi_p - \varphi_{r_1}) + A(\varphi_p - \varphi_{r_2}) \quad (g)$$

we have, by (12),

$$\begin{aligned} (20) \quad A(\Theta_{mp} - \varphi_p) &\leq a \sum_h A\mu_{o,r,mh} \delta_p^{mh} + (A\overline{\varphi_p - \varphi_{r_1}} + A\overline{\varphi_p - \varphi_{r_2}}) \sum_g A\delta_p^{mg} \\ &\quad + a A\mu_{op} A \left(\sum_g \delta_p^{mg} - 1 \right). \end{aligned}$$

Generalizing (1) in the sense of relativity ⁶⁰) we have; for $\mathfrak{P}'$ with development Δ' , $\mathfrak{M}'$, $\mathfrak{P}''$, $\mathfrak{M}''$:

$$(2) \quad \mathfrak{M}'^{L_1\Delta'} \cdot \mathfrak{M}''^{D_1} \cdot \mathfrak{F} = [\text{all } \varphi^{K'_{12}\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{M}'\mathfrak{M}'' \cdot B_0\mathfrak{M}''(p')}]. \supset \cdot \mathfrak{F}^{B_0(\mathfrak{M}'\mathfrak{M}'')*}$$

23. *Classes $\mathfrak{M}$ with the property $(K_{12}B_1)_*$* ⁶¹).

THEOREM I. — *From the hypothesis*

$$(a) \quad \mathfrak{P}', \Delta', \mathfrak{M}'^{D_1D_4K'_{12}\Delta'}$$

it follows that

$$(b) \quad \mathfrak{P}'', \mathfrak{M}''^{LCD}, \mathfrak{F} = [\text{all } \varphi^{K'_{12}\mathfrak{M}'(\mathfrak{M}'') \cdot B_1\mathfrak{M}'\mathfrak{M}'' \cdot B_0\mathfrak{M}''(p')}]$$

implies

$$(c) \quad \mathfrak{F} = (\mathfrak{M}'\mathfrak{M}'')_*, \mathfrak{F}^{LCD_1};$$

and if further

$$\mathfrak{M}'^{LC}$$

it follows that

$$\mathfrak{F}^{B_0\mathfrak{M}'(p')*}.$$

The proof of this theorem follows readily from preceding propositions. The class $\mathfrak{F}$ contains $(\mathfrak{M}'\mathfrak{M}'')_*$ by virtue of Art. 4.1, Art. 15b.2, Art. 17a.1. The class $(\mathfrak{M}'\mathfrak{M}'')_*$

Hence we have from (15, 14, 16) the relation:

$$(21) \quad p^m \cdot (m \geq m_0, m_{3e}, m_{2e}, m_{4e}) \cdot \supset \cdot A(\Theta_{mp} - \varphi_p) \leq e(aA\mu_{3p} + 4A\mu_{2p} + aA\mu_{0p}).$$

Here the understanding is that, if the element p belongs to every class of stage m , that is, if there are no indices h , the various sums as to h have the value zero ».

The functions Θ_m which appear in MOORE's proof have the definition

$$(a) \quad \Theta_m = \sum_l' \varphi_{r,ml} \delta^{ml}.$$

In the case considered by MOORE the sums are finite, and the functions Θ belong to $\mathfrak{M}_L$. Hence the limit of the sequence $\{\Theta_m\}$ (which is shown to converge to the function φ uniformly on the range $\mathfrak{P}$ as to the scale class $\mathfrak{M}$) belongs to $\mathfrak{M}_*$. In the present case the functions Θ_m are (for m sufficiently large) easily shown to belong to $\mathfrak{M}_*$, and therefore their limit φ belongs to $(\mathfrak{M}_*)_{\mathfrak{M}}$. In order to show that it belongs to $\mathfrak{M}_*$ we proceed as follows:

From $\mathfrak{M}^\Delta$, and $\varphi^{B_1\mathfrak{M}}$ it is easy to show that there exists an m_0 and a function μ_0 such that

$$(b) \quad m \geq m_0 \cdot \supset \cdot \Theta_m = \sum_l' \varphi_{r,ml} \delta^{ml} \quad (\mathfrak{P}; \mu_0)$$

Set

$$(c) \quad \Theta_{m\bar{l}} = \sum_{l=\bar{l}}^{\bar{l}} \varphi_{r,ml} \delta^{ml}.$$

Since

$$(d) \quad \varphi = \lim_m \Theta_m \quad (\mathfrak{P}; \mathfrak{M}),$$

it follows from (b, c) and that we may select a sequence $\{\bar{l}_m\}$ such that $\mathfrak{M}^{D_1}$:

$$(e) \quad \varphi = \lim_m \Theta_{m\bar{l}_m} \quad (\mathfrak{P}; \mathfrak{M}).$$

But the $\Theta_{m\bar{l}_m}$ belong to $\mathfrak{M}_L$, and it follows from (e) that $\varphi^{B_0\mathfrak{M}*}$.

In (a) and (b) the mark ' on the sign of summation is to restrict the index l to the indices of existent classes of stage m .

⁶⁰) Cf. loc. cit. ¹), Art. 80, Theorem II.

⁶¹) Cf. loc. cit. ¹), Art. 82, Theorem I; loc. cit. ⁵), Art. 49, Theorem I.

contains $\mathfrak{F}$ by virtue of Art. 23.2. Therefore $\mathfrak{F} = (\mathfrak{M}'\mathfrak{M}'')_*$. $\mathfrak{F}$ is linear and closed from Art. 17a.1 and D_1 from Art. 8.6. The last statement of the theorem is from Art. 4.1.

From Theorem I, by reference to Art. 15.8, Art. 17b.2, Art. 23.2, we easily obtain the following theorems ⁶²⁾:

THEOREM II. — *From*

- (a) $\mathfrak{P}, \Delta, \mathfrak{M}^{D_1 D_4 K_{12} \Delta}$
 it follows that
 (b) $\mathfrak{M}^{(K_{12} B_1)*}, \mathfrak{M}_* = [\text{all } \varphi^{K_{12} \mathfrak{M}, B_1 \mathfrak{M} \mathfrak{M}}],$
 (c) $\mathfrak{M}_*^{LCD_1 D_4 A \Delta K_{12} (K_{12} B_1)*, B_1 \mathfrak{M} \mathfrak{M}}.$

THEOREM III. — *From*

- (a) $\mathfrak{P}, \Delta, \mathfrak{M}^{LCD_1 D_4 K_{12} \Delta}$
 it follows that
 (b) $\mathfrak{M}^{A(K_{12} B_1)*}, \mathfrak{M} = [\text{all } \varphi^{K_{12} \mathfrak{M}, B_1 \mathfrak{M} \mathfrak{M}}].$

24. *Closure under extension and composition of classes* ⁶³⁾. From Art. 15c.1, 2, Art. 17a.2, Art. 17b.2, 3, 4, Art. 21.3, 4, Art. 23 Theorem II, we obtain the following theorem of closure.

THEOREM. — *The respective genera of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}^P),$$

where P denotes any one of the composite properties:

- (a) $D_1 D_4; D_1 D_4 D_{12}; D_1 D_4 K_{12} \Delta [(K_{12} B_1)_*],$

are closed under the three operations:

A : Linear extension of classes $\mathfrak{M}$;

B : *-extension of classes $\mathfrak{M}$

C : Simultaneous multiplicative composition of classes $\mathfrak{P}$ and developments Δ and multiplication of classes $\mathfrak{M}$;

and the combinations of these operations.

The respective subgenera of all systems obtained by operation A are the respective genera of all systems:

$$(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}^{LP})$$

where P denotes any one of the properties (a). These genera are closed under the operations A, B, AC, BC ⁶⁴⁾, and their combinations, and also under the additional operation:

D : Simultaneous adjunctional composition of classes $\mathfrak{P}$, developments Δ and classes $\mathfrak{M}$.

The subgenus of all systems obtained from the third genus by operations B, BC and

⁶²⁾ Cf. loc. cit. ¹⁾, Art. 82, Theorems II, III. The property A (absolute, Art. 7) appears in II c, and III b because of the evident proposition: If φ has the properties $K_{12} \mathfrak{M}, B_1 \mathfrak{M} \mathfrak{M}$, then $A\varphi$ has the properties $K_{12} \mathfrak{M}, B_1 \mathfrak{M} \mathfrak{M}$.

⁶³⁾ Cf. loc. cit. ¹⁾, Art. 84, loc. cit. ⁵⁾, Art. 40, Theorem II.

⁶⁴⁾ AC , etc., denotes the operation C followed by the operation A . The genera are not closed under the single operation C .

their combinations is the genus of all systems:

$$(\mathfrak{A}; \mathfrak{B}; \Delta; \mathfrak{M}^{LCD_1 D_4 [A] K_{12} \Delta [(K_{12} B_1)_*]}).$$

This genus is closed under the operations A , B , D and their combinations; to this genus belong the systems I-IV ⁶⁵).

Here the properties $(K_{12}B_1)_*$; A , $(K_{12}B_1)_*$ appearing in brackets in the specification of various composite properties are implied in view of the propositions Art. 24. II, III, by the other constituent properties of the respective composite properties.

25. Theory for the property D . Finite developments. — MOORE ⁶⁶) has given for finite developments Δ theorems corresponding to those of articles 23, 24. Since for $\mathfrak{B}$ with development Δ :

$$(1) \quad \mathfrak{M}^D \cdot \mathfrak{D} \cdot \mathfrak{M}^{D_1 D_4},$$

we may replace $D_1 D_4$ in the theorems of Art. 23 by the property D , thus obtaining the direct generalization to infinite developments of the theorems of Art. 82, Introduction to a Form of General Analysis [loc. cit. ¹]).

From ⁶⁷):

$$(2) \quad \mathfrak{B}, \Delta^{\text{finite}}, \mathfrak{M}^D \text{ imply that every function } \varphi \text{ with the property } K_{12} \mathfrak{M} \\ \text{has the property } B_1 \mathfrak{A} \mathfrak{M},$$

and its generalization in the sense of relativity, which is easily seen to hold, we obtain:

$$(3) \quad \mathfrak{B}, \Delta^{\text{finite}}, \mathfrak{M}^D \text{ imply that } \mathfrak{M}^{(K_{12} B_1)_*} \text{ is equivalent to } \mathfrak{M}^{K_{12}*}.$$

It follows from (1, 3) that the theorems of Art. 23 specialize to the theorems of MOORE referred to.

It is easy to see in view of the preceding work that the theorem of Art. 84, loc. cit. ¹) becomes true for infinite developments if we replace the property K_{12*} by the property $(K_{12}B_1)_*$. The differences between this theorem and the theorem of Art. 24 above are due to the introduction of adjunctional composition and the use of the property $D_1 D_4$. We note that the genus $LCD_1 D_4$ is not necessarily closed under $*$ -composition, while the genus LCD is closed under $*$ -composition ⁶⁸).

26. Theory for the property D_1 . The property B_3 . Finite developments. — In his thesis Dr. PITCHER showed for the case of finite developments that characterizations of $(\mathfrak{M}'\mathfrak{M}'')_*$ could be obtained under various sets of hypotheses in which the class $\mathfrak{M}'$ is not required to have the property D .

In Art. 23 we have given a generalization of Theorem I, Art. 40 of Dr. PITCHER's

⁶⁵) Cf. Art. 11. The classes $\mathfrak{M}^{I-IV}$ have the properties L , C , D , and (relative to the ordinary developments Δ^{I-IV}) are shown in loc. cit. ¹), Arts. 65, 66, 67 to have the properties K_{12} , Δ , relative to Δ^{I-IV} , respectively.

⁶⁶) loc. cit. ¹), Arts. 82, 84.

⁶⁷) (3) This proposition is proved in loc. cit. ¹), Art. 80, p. 141, as a part of the proof of Theorem I.

⁶⁸) loc. cit. ¹), Art. 55, IV.

thesis, and have seen (Art. 25) that this generalization includes theorems of MOORE as special cases. We proceed to make explicit the relation to the work of Dr. PITCHER. Certain definitions are necessary.

Dr. PITCHER introduces properties B_3 , $B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)$ ⁶⁹:

(1) $\mathfrak{M}^{B_3}$: A class $\mathfrak{M}$ has the property B_3 in case there exists a function μ of the class $\mathfrak{M}$ such that for every m there exists an $e_m^{>0}$ such that

$$(a) \quad \bar{K}_{pm} \cdot \odot \cdot A\mu_p \geq e_m;$$

(2) $\mathfrak{M}^{B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)}$: A class $\mathfrak{M}$ has the property $B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)$ in case for every m there is a constant a_m such that

$$(a) \quad \bar{K}_{pm} \cdot \odot \cdot A\mu_p \leq a_m.$$

It follows readily from the definitions (1, 2) that

$$(3) \quad \mathfrak{M}^{D_1 B_3 B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)} \cdot \odot \cdot \mathfrak{M}^{D_1 D_4}.$$

From proposition (3) we see that we may replace $D_1 D_4$ in Arts. 24, 25 by $D_1 B_3 B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)$, and obtain theorems which specialize to the theorems of Dr. PITCHER, Art. 40, in the case of finite developments. The specialization results from the proposition ⁷⁰:

$$(4) \quad \mathfrak{P} \cdot \Delta^{\text{finite}} \cdot \mathfrak{M}^{K_{12}} \cdot \odot \cdot \mathfrak{M}^{B_1 \mathfrak{A} (\mathfrak{P}_{-m}/m)}.$$

We thus obtain the proposition ⁷¹:

$$(5) \quad \mathfrak{P} \cdot \Delta^{\text{finite}} \cdot \mathfrak{M}^{D_1 B_3 K_{12}} \cdot \odot \cdot \mathfrak{M}^{K_{12}*} \cdot \mathfrak{M}^{LCD_1 B_3 K_{12} K_{12}*}.$$

The present treatment shows that $\mathfrak{M}$ is linear and closed, otherwise (5) is the same as loc. cit. ⁵, Art. 40, Theorem I.

We refer to the thesis of Dr. PITCHER, [loc. cit. ⁵] Arts. 32-38 inclusive for a large body of interesting theorems which may readily be extended to the case of infinite developments. In general, as we have seen, the sole difference between the theorems for finite developments and the corresponding theorems for infinite developments lies in the way in which the property B_1 (dominated by) enters the theory.

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EDWARD WILSON CHITTENDEN.

⁶⁹) loc. cit. ⁵, Art. 39; also see Art. 17 of the present paper.

⁷⁰) Cf. loc. cit. ⁵, Art. 39 a, 4.

⁷¹) The proposition follows from Art. 23. Theorem II, by means of propositions (3, 4) of the present article.