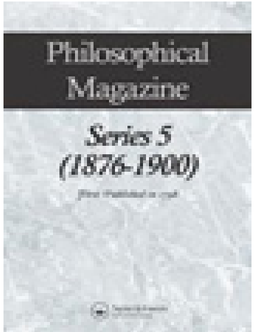

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XXXIX. On the highest wave of permanent type

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until the desired part of the spectral field is brought into the observing eyepiece. If the spectrum is now either too high or too low, it shows that the refracting edge of the prism is slightly inclined to the mirror-face, and the screw c is turned until the spectrum is centred. Then, if all the preliminary adjustments have been properly made, the angular deviation of the central ray in the field will be given by the relation

$$\theta=2(\beta-(90^\circ+\alpha)),$$

where β is the circle-reading for a deviation θ, and α is the zero-reading determined as already described.

In very accurate spectrometric work it is important to determine just what degree of accuracy is required in making the various adjustments of parts to each other in order to attain a given degree of accuracy in the final result. The theory of these adjustments is comparatively simple, but somewhat lengthy, and it will therefore be briefly indicated in a future paper.

Astro-Physical Observatory,
Washington, D.C, March 1893.

XXXIX. *On the Highest Wave of Permanent Type.*

By J. McCowan, *M.A., D.Sc., University College, Dundee**.

In a previous communication †, in which I discussed the general theory of the class of waves in water or other liquid which have no finite wave-length but which are of permanent type, that is to say, which are propagated with constant velocity without change of any kind, I gave a rough estimate of the maximum height to which such waves might attain without breaking. The paper dealt chiefly with an approximation which was specially suitable for waves of small or moderate elevation, and it is the object of the present paper, therefore, to supplement this by investigating an approximation better adapted to the discussion of the extreme case of the wave at the breaking height, and sufficiently exact for ordinary purposes. I trust, however, to be soon able to communicate a fuller discussion of the general theory of the solitary wave which I have almost completed.

1. *The General Equation of the Motion.*

The highest wave which can be propagated without change in water of any given depth is obviously the highest solitary

* Communicated by the Author, having been read before the Edinburgh Mathematical Society, June 8, 1894.

† "On the Solitary Wave," Phil. Mag. July 1891.

wave for such depth; for the height to which waves can attain without breaking must evidently increase with their length, and the solitary wave may be regarded as the limiting type to which each individual wave, reckoned from trough to trough, in a permanent train of finite waves approaches as the wave-length indefinitely increases. In fact this paper and the former, "On the Solitary Wave," may be regarded as giving a very close approximation to the form and motion of the individual waves in a train of finite waves if the wave-length is even so small a multiple of the depth as ten or twelve.

It will thus be convenient to follow to some extent the methods and notation of the paper "On the Solitary Wave," and references to it will be briefly indicated by an S prefixed.

Consider, then, a solitary wave propagated with uniform velocity U along the direction in which x increases in an endless straight channel of uniform rectangular cross section, the axis of x being taken along the bottom and that of z vertically upwards.

Let the motion be regarded as reduced to steady motion by having superposed on it a velocity equal and opposite to the velocity of propagation of the wave, and take $x=0$ and $z=c$ as the coordinates of the crest. Let u and w be the horizontal and vertical components respectively of the resultant velocity q in the steady motion at x, z, of which, further, ϕ is the velocity potential and ψ the current function.

We shall now, referring to the "General Theory of the Wave," (S. § 1), seek to determine a form of the relation between $\psi+\iota\phi$, or, as we shall here find more convenient, $u+\iota w$ and $z+\iota x$ corresponding to S. (1) and (2), but only containing so many disposable constants as will suffice for the degree of accuracy at present desired. Noting that for the limiting form the velocity at the crest must vanish, and remembering Sir George Stokes's expression * for the leading term in the velocity near the crest of a wave at the breaking-limit, we shall assume (compare S. (6)):—

$$u+\iota w=-\mathrm{U}\{1-fk^2\sec^2\tfrac{1}{2}m(z+\iota x)\}\sqrt{1-k^2\sec^2\tfrac{1}{2}m(z+\iota x)}, \quad (1)$$

where

$$k=\cos\tfrac{1}{2}mc\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as a form conveniently integrable with respect to $z+\iota x$ so as to give $\psi+\iota\phi$ in finite terms if so desired.

It should be noted that all the conditions required to be

* "On the Theory of Oscillatory Waves," Appendix B. 'Collected Papers,' vol. i.

satisfied for a solitary wave, with the exception of the condition of constant surface-pressure, are identically satisfied by (1): for w vanishes with z, and when $x=\pm\infty$, $u=-\mathrm{U}$ and $w=0$. We proceed, therefore, to determine the surface-pressure with a view to the determination of the available constants so as to satisfy as nearly as may be this condition for a *free* surface: as it is, (1) may be regarded as giving a particular *forced* wave.

2. *The Surface-Pressure near the Mean Level.*

Expanding (1), and writing for brevity

$$\alpha=1+2f, \quad \beta=1-4f, \text{ \&c.}, \quad . \quad . \quad . \quad . \quad (3)$$

we get

$$-(u+\iota w)/\mathrm{U}=1-\tfrac{1}{2}\alpha k^2\sec^2\tfrac{1}{2}m(z+\iota x)-\tfrac{1}{8}\beta k^4\sec^4\tfrac{1}{2}m(z+\iota x); \quad (4)$$

therefore, further, for x positive

$$-(u+\iota w)/\mathrm{U}=1-2\alpha k^2\epsilon^{-m(x-\iota z)}+2(2\alpha-\beta k^2)k^2\epsilon^{-2m(x-\iota z)} \text{ \&c.}; \quad (5)$$

therefore, integrating with respect to $z+\iota x$,

$$-m(\psi+\iota\phi)/\mathrm{U}=m(z+\iota x)+2\iota\alpha k^2\epsilon^{-m(x-\iota z)}-\iota(2\alpha-\beta k^2)k^2\epsilon^{-2m(x-\iota z)}. \quad (6)$$

Now if h be the mean depth, or the depth at an infinite distance from the crest, we must, taking $\psi=0$ at the bottom, have $\psi=-\mathrm{U}h$ at the surface. Also, if η denote the elevation of the surface at any point above the mean level, we must have $z=h+\eta$ at the surface. Substituting these values in the expression for ψ involved in (6), we obtain as the equation to the surface,

$$m\eta=2\alpha k^2\epsilon^{-mx}\sin m(h+\eta)-(2\alpha-\beta k^2)k^2\epsilon^{-2mx}\sin 2m(h+\eta)\ldots;$$

or, when η is small,

$$m\eta=2\alpha k^2\epsilon^{-mx}\sin mh-(2\alpha-\beta k^2-2\alpha^2k^2)k^2\epsilon^{-2mx}\sin 2mh\ldots; \quad (7)$$

which gives the equation to the surface in a convenient form for points not too near the crest.

Again, from (5) we get

$$q^2/\mathrm{U}^2=1-4\alpha k^2\epsilon^{-mx}\cos mz+4k^2\epsilon^{-2mx}\{(2\alpha-\beta k^2)\cos 2mz+4\alpha^2k^2\}; \quad (8)$$

therefore, by (7), we have at the surface where η is small,

$$q^2/\mathrm{U}^2=1-4\alpha k^2\epsilon^{-mx}\cos mh$$
$$+4k^2\epsilon^{-2mx}\{(2\alpha-\beta k^2)\cos 2mh+\alpha^2k^2(1+2\sin^2 mh)\}\ldots. \quad (9)$$

Now in a liquid of density ρ moving irrotationally acted on by no force but gravity, the pressure p at any point is given by

$$p=\text{constant}-\tfrac{1}{2}\rho q^2-g\rho z; \quad . \quad . \quad . \quad . \quad (10)$$

and therefore if δp denote the excess of pressure at any point on the surface over that at the mean level, we have

$$\delta p = \tfrac{1}{2}\rho(\mathrm{U}^2 - q^2) - g\rho\eta \; ; \quad . \quad . \quad . \quad . \quad (11)$$

whence, on substituting for η and q^2 from (7) and (9), we get

$$\delta p/\rho\mathrm{U}^2 = 2\alpha k^2\epsilon^{-mx}\{\cos mh - g/m\mathrm{U}^2 . \sin mh\} + \&\mathrm{c}. \quad . \quad (12)$$

Now for a free surface δp ought to vanish; therefore for a first approximation we must take

$$\cos mh - g/m\mathrm{U}^2 . \sin mh = 0 \; ;$$

that is,

$$\mathrm{U}^2 = g/m \tan mh \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and (12) becomes, writing it out to the next term,

$$\delta p/\rho\mathrm{U}^2 = 2k^2\epsilon^{-2mx}\{(2\alpha - \beta k^2)\sin^2 mh - 3\alpha^2 k^2\}. \quad . \quad . \quad . \quad (14)$$

The coefficient of ϵ^{-2mx} in (14) ought of course to be made to vanish; but it will be preferable for our present purpose to retain it as a small pressure-error, and so leave another of our constants available to satisfy the conditions in the neighbourhood of the crest to which we proceed.

3. *The Surface-Pressure near the Crest.*

Put $z = c - \zeta$, so that ζ vanishes at the crest; then, writing for brevity

$$\left.\begin{aligned} p &= \tan \tfrac{1}{2} mc, \\ \mathrm{A} &= 1 - f, \\ \mathrm{B} &= \{(1 + 11p^2)f - (1 + 3p^2)\}/8p, \end{aligned}\right\} \quad . \quad . \quad (15)$$

we get, on expanding (1) in powers of $\zeta - \iota x$,

$$-(u + \iota w)/\mathrm{U} = \sqrt{pm(\zeta - \iota x)}\{\mathrm{A} + \mathrm{B}m(\zeta - \iota x) + \&\mathrm{c}.\} \; ; \quad (16)$$

whence, integrating with respect to $\zeta - \iota x$, we get

$$m(\psi + \iota\phi)/\mathrm{U} - m\psi_0/\mathrm{U} = 2\sqrt{p}\,\{m(\zeta - \iota x)\}^{\frac{3}{2}}\{\tfrac{1}{3}\mathrm{A} + \tfrac{1}{5}\mathrm{B}m(\zeta - \iota x)\}. \quad (17)$$

where ψ_0 is the value of ψ at the crest.

Put

$$\zeta = r\cos\vartheta, \quad x = r\sin\vartheta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

then (18) gives for the surface, determined by $\psi = \psi_0$,

$$\tfrac{1}{3}\mathrm{A}\cos\tfrac{3}{2}\vartheta + \tfrac{1}{5}\mathrm{B}\,mr\cos\tfrac{5}{2}\vartheta + \&\mathrm{c}. = 0. \quad . \quad . \quad . \quad (19)$$

Thus when $r = 0$, $\vartheta = \pm\dfrac{\pi}{3}$, showing that the crest is formed by two branches equally inclined to the bottom cutting at an angle of 120°.

Put then $\qquad \vartheta = \dfrac{\pi}{3} + \sigma \; ;$

$\therefore$ (19) gives when σ is small,

$$\sigma \doteq -\mathrm{B}mr/5\mathrm{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

as a convenient approximation for the form of the surface in the neighbourhood of the crest.

Again, (11) may be written

$$\delta p = \tfrac{1}{2}\rho(\mathrm{U}^2 - q^2) - g\rho(c - h - \zeta), \quad . \quad . \quad . \quad . \quad (21)$$

whence, since q and ζ vanish together, to make δp vanish we must have

$$\mathrm{U}^2 = 2g(c - h), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and this reduces (21) to

$$\delta p = g\rho\zeta - \tfrac{1}{2}\rho q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now from (16) we obtain

$$q^2/\mathrm{U}^2 pmr\{\mathrm{A}^2 + 2\mathrm{AB}m\zeta + \&c.\}, \quad . \quad . \quad . \quad (24)$$

and thus

$$\delta p = \rho r\{g \cos \vartheta - \tfrac{1}{2}m\mathrm{A}^2 p\mathrm{U}^2 + \mathrm{ABU}^2 m\zeta + \&c.\}. \quad . \quad (25)$$

Hence, since $\vartheta = \frac{\pi}{3} + \sigma$ we must have, to make δp vanish to a first approximation,

$$g \cos \frac{\pi}{3} - \tfrac{1}{2}m\mathrm{A}^2 p\mathrm{U}^2 = 0,$$

that is

$$\mathrm{U}^2 = g/m\mathrm{A}^2 p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and (25) becomes

$$\delta p/\rho\mathrm{U}^2 = -\tfrac{1}{2}\left(1 - \frac{\sqrt{3}}{5}\right)p\mathrm{AB}m^2r^2 + \&c. \quad . \quad . \quad (27)$$

This cannot vanish unless B vanishes, in which case we see by (20) that the curvature of the surface vanishes close to the crest. This result is obviously independent of our approximation, but we have not taken enough of constants to secure it here: it will be found, however (v. § 5), that the other equations determining the constants we have at our disposal will make B very approximately vanish.

4. *Numerical Determination of the Constants.*

There is still one important condition to be satisfied. To ensure the connexion between our separate treatment of the neighbourhoods of the crest and mean level, we must secure that the stream-line $\psi = -\mathrm{U}h$, bounding the distant surface, shall pass through the crest: or, in other words, the flow across any infinitely distant section must be equal to the flow

2 B 2

across the axis of z. This condition may be written, by (1)

$$h=\int_0^c (1-fk^2\sec^2\tfrac{1}{2}mz)\sqrt{1-k^2\sec^2\tfrac{1}{2}mz}\,.\,dz, \quad . \quad (28)$$

which gives, remembering that $k=\cos\frac{1}{2}mc$,

$$mh=\pi\{1-\cos\tfrac{1}{2}mc-\tfrac{1}{4}f\sin\tfrac{1}{2}mc\,.\,\sin mc\}. \quad . \quad (29)$$

We may now proceed to evaluate the constants in (1) in terms of h the mean depth of the liquid in the channel.

Eliminating U^2 between (22) and (13) we get

$$mc=mh+\tfrac{1}{2}\tan mh, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

while (13) and (26) give

$$(1-f)^2=\cot mh\cot\tfrac{1}{2}mc. \quad . \quad . \quad . \quad . \quad (31)$$

If now we solve equations (29), (30), and (31) for m, c, and f, we shall find $mh=1{\cdot}0025$ approximately: hence, remembering that at best the surface-pressure is only to be approximately constant, it will be sufficient to take for our present purpose

$$mh=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and it is just possible that this may be the exact value.

Substituting this value in (30) we get

$$c \doteq 1{\cdot}78\,h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

This gives for the maximum wave-height

$$c-h=\eta_0={\cdot}78\,h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which differs by less than the experimental error from the value ${\cdot}75\,h$ which I have already {S. § 10} given as a fair average of some experiments I made in connexion with my former paper.

Again, from (13) and (32), or (22) and (34), we obtain for the velocity

$$U^2 \doteq 1{\cdot}56\,gh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

which shows that the maximum wave travels about 25 per cent. faster than low waves in the same depth of channel.

Finally, substituting from (32) and (33) in (31) we get

$$f \doteq {\cdot}28. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We have now only to substitute the values of the constants just determined in our general equations.

5. *The Final Equations of the Motion.*

The fundamental equation (1) may now be written

$$u+\iota w=-1{\cdot}25\sqrt{gh}\{1-{\cdot}11\sec^2\tfrac{1}{2}(z+\iota x)/h\}\sqrt{1-{\cdot}40\sec^2\tfrac{1}{2}(z+\iota x)/h}, \quad (37)$$

which completely determines the motion of the fluid.

It is convenient, however, to consider the formulæ specially suitable to the regions near to, and fairly distant from, the crest. Thus for regions fairly distant from the crest (where $exp(-x/h)$ is small) (5) and (6) give

$$-(u+\iota w)/\mathrm{U}\doteq 1-1{\cdot}24\,\epsilon^{-(x-\iota z)/h}+\&c., \quad (38)$$

and

$$-(\psi+\iota\phi)/\mathrm{U}h=(z+\iota x)/h+\iota\,1{\cdot}24\,\epsilon^{-(x-\iota z)/h}-\&c., \quad (39)$$

in which U has the value given by (35).

The equation to the free surface, as given by (7), becomes

$$\eta/h=1{\cdot}04\,\epsilon^{-x/h}-{\cdot}44\,\epsilon^{-2x/h}\ \&c., \quad (40)$$

and for the pressure-error given by (14) we have

$$\delta p\doteq-{\cdot}89\,\epsilon^{-2x/h}g\rho h. \quad (41)$$

Again, for the neighbourhood of the crest (16) and (17) give

$$-(u+\iota w)/\mathrm{U}\doteq{\cdot}80\sqrt{(\zeta-\iota x)/h}\{1+{\cdot}084(\zeta-\iota x)/h\} \quad (42)$$

and

$$(\psi+\iota\phi)/\mathrm{U}h=1+{\cdot}53\{(\zeta-\iota x)/h\}^{3/2}\{1+{\cdot}50(\zeta-\iota x)h\}. \quad (43)$$

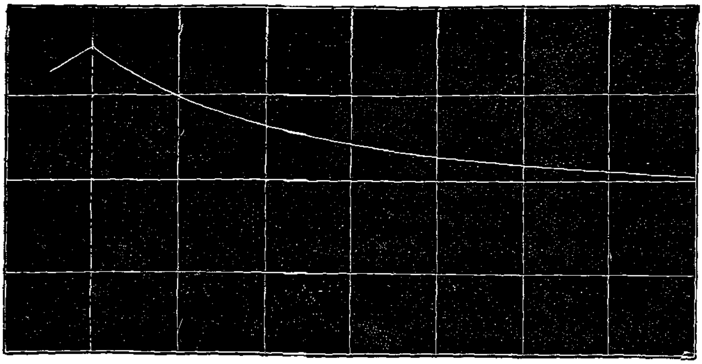
We have already seen that the crest is formed by two surfaces, equally inclined to the bottom, meeting at an angle of 120°, so that the summit of the wave has the form of a blunt wedge. We have also seen that in the free wave these surfaces must be plane or have an infinite radius of curvature at the crest; we have, however, made no effort to satisfy this condition, but on substituting the values of the constants in (20) we find that it gives the radius of curvature at the crest about equal to thirty times the depth of the water, a result sufficiently indicating the closeness of our approximation.

The pressure-error near the crest, as given by (27), is

$$\delta p\doteq{\cdot}03\,g\rho r^2/h. \quad (44)$$

By (41) and (44) we see that the deviation from constant surface-pressure is everywhere very small; there is a very slight excess near the crest but vanishing at the crest, and a slight defect near the mean level. The deviation has in fact only an appreciable value over a very limited region, say from $x={\cdot}5h$ to $x=1{\cdot}5h$; (41) and (44) are hardly applicable within this region, but I estimate that within it the maximum defect of pressure is less than that of a head of water, or whatever other liquid the channel may contain, of one tenth of the mean depth.

The accompanying figure shows the form of the wave, only half of it being drawn, however, as the wave is symmetrical



about the crest. The thicker straight line indicates the bottom of the channel, while the mean depth is shown by a finer line to which the surface approaches asymptotically.

XL. *On the Velocity of the Cathode-Rays.* *By* J. J. Thomson, *M.A., F.R.S., Cavendish Professor of Experimental Physics, Cambridge**.

THE phosphorescence shown by the glass of a discharge-tube in the neighbourhood of the cathode has been ascribed by Crookes to the impact against the sides of the tube of charged molecules driven off from the negative electrode. The remarkably interesting experiments of Hertz and Lenard show that thin films of metal when interposed between the cathode and the walls of the discharge-tube do not entirely stop the phosphorescence. This has led some physicists to doubt whether Crookes's explanation is the true one, and to support the view that the phosphorescence is due to ætherial waves of very small wave-length, these waves being so strongly absorbed by all substances that it is only when the film of the substance is extremely thin that any perceptible phosphorescence occurs behind it. Thus on this view the phosphorescence is due to the action of a kind of

* Communicated by the Author.