THEOREMS ON CONFORMAL TRANSFORMATION

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1. A conformal curve-factor in a complex variable $\zeta (\equiv \xi + i\eta)$ has been defined* as a function $\mathcal{C}(\zeta)$, such that the transformation

$$dz = \mathcal{C}(\zeta) d\zeta$$

specifies a conformal correspondence between the half-plane $\eta > 0$ and some definite region in the plane of z . Particular classes of curve-factor are:—(i) pure curve-factors, for which the boundary of the relevant region in the z plane, whether wholly or only partially curved, is free from corners; (ii) corner-factors or Schwarzian factors, of the kind familiar in the Schwarz-Christoffel transformation.

Use has already been made of the fairly obvious fact that the limit of a product of corner-factors with infinitesimal exponents is itself a curve-factor. The present short piece of work was undertaken with a view to proving a converse theorem, namely that every curve-factor is expressible as a limit product of corner-factors, that is as the exponential of a certain line-integral taken round the boundary. This result is established, and is shewn to have general theoretical consequences, including the theorem that any curve-factor is expressible as a product of corner-factors and pure simple† curve-factors. Corresponding results are obtained for periodic curve-factors.

A second and quite different expression for a curve-factor as the exponential of a line-integral round the boundary of the relevant region results from the same analysis, and is of some interest.

2. A general conformal curve-factor $\mathcal{C}(\zeta)$, as defined above, has no zeros, infinities or branch-points in the principal half-plane of ζ , *i.e.*, the half-plane $\eta > 0$. It may have branch-points on the real axis, as these do not complicate operations confined to the principal half-plane. It may

* J. G. Leathem, "Some Applications of Conformal Transformation to Problems in Hydrodynamics," *R. S. Phil. Trans.*, Sec. A, Vol. 215, 1915.

† *L.c.*, § 4.

have zeros or infinities at points on the real axis, but these will be restricted to the type in which the limit form is $A(\xi - a)^p$, where $1 \geq p > -1$.

It will be assumed that $\mathcal{C}(\xi)$ has a definite angular range $n\pi$, and therefore a definite order at infinity n , so that the form of the function for $|\xi|$ great is

$$\mathcal{C}(\xi) = K\xi^n \{1 + f(\xi)\},$$

where n is real and $f(\xi) \rightarrow 0$ for $|\xi| \rightarrow \infty$. There is no loss of generality in assuming K real. The vanishing of $f(\xi)$ must be such that

$$\xi f'(\xi) \log \xi \rightarrow 0,$$

as is true, for example, when $f(\xi)$ is a sum of negative powers of ξ .

3. Let ξ_0 be a value of ξ corresponding to a point in the principal half-plane. It is proposed, in the first instance, to consider the integral

$$\int (1/\pi) \log(\xi_0 - \xi) d \log \mathcal{C}(\xi),$$

taken round a contour in the principal half-plane of ξ , consisting mainly of the real axis and a semicircle of great radius R with its centre at the origin. The formulæ will be pressed to limit forms corresponding to R infinite.

$\log \mathcal{C}(\xi)$ has neither infinity nor branch-point in the region enclosed by this contour. But ξ_0 is in the region and is a branch-point of $\log(\xi_0 - \xi)$; accordingly ξ_0 is surrounded by a circular cavity of infinitesimal radius ϵ , and a cut is made from this cavity to some point on the semicircle. The infinitesimal circumference and the two sides of the cut are added to the contour.

An infinity or zero of $\mathcal{C}(\xi)$ on the real axis, typified by $\xi = a$, is an infinity of $\log \mathcal{C}(\xi)$. Near such a point the contour must be modified, being kept clear of the point of infinity by a semicircular detour of infinitesimal radius ϵ' on the positive side of the axis.

With these modifications the contour assumes the form typified in the

diagram, and encloses a region in which the subject of integration has no infinities or branch points. The contour integral must therefore equal zero.

The logarithm may be so chosen that, as ξ traverses the real axis from left to right, the vector-angle of $\log(\xi_0 - \xi)$ increases from zero to π .

As ξ makes a circuit round ξ_0 in the positive sense, $\log(\xi_0 - \xi)$ increases by $2i\pi$. Hence, by taking together corresponding elements of integration on opposite sides of the cut, it is seen that

$$\begin{aligned} \left(\int_C^D + \int_E^F \right) \frac{1}{\pi} \log(\xi_0 - \xi) d \log \mathcal{C}(\xi) \\ = 2i \int_C^D d \log \mathcal{C}(\xi) \rightarrow 2i \{ \log \mathcal{C}(\xi_0) - \log \mathcal{C}(\xi_C) \}. \end{aligned}$$

On the infinitesimal circumference round ξ_0 , $d\xi$ is proportional to ϵ , and $\epsilon \log \epsilon \rightarrow 0$, so this circumference makes no contribution to the contour integral, in the limit for $\epsilon \rightarrow 0$.

As ξ traverses the great semicircle there is a sudden drop of $2i\pi$ in the value of $\log(\xi_0 - \xi)$ in passing from C to F . Continuity is restored, for the purpose of this part of the integration, by continuing the value of $\log(\xi_0 - \xi)$, as from C , across the gap; this necessitates adding

$$-2i \int_F^G d \log \mathcal{C}(\xi) = -2i \{ \log \mathcal{C}(\xi_G) - \log \mathcal{C}(\xi_C) \}.$$

On the great semicircle, approximately,

$$d \log \mathcal{C}(\xi) = [n\xi^{-1} + f'(\xi) \{1 + f(\xi)\}^{-1}] d\xi,$$

and, if $\xi = R \exp i\theta$,

$$\log(\xi_0 - \xi) = \log R + i(\pi + \theta) + \text{negative powers of } R \exp i\theta.$$

So the integral on the semicircle has the same limit as

$$\frac{i}{\pi} \int_0^\pi \{n + \xi f'(\xi)\} \{ \log R + i(\pi + \theta) \} d\theta,$$

or

$$in \log R - \left(\frac{3}{2}\right)n\pi.$$

On the infinitesimal semicircle round a it suffices to put $\log(\xi_0 - a)$ instead of $\log(\xi_0 - \xi)$, and $p(\xi - a)^{-1} d\xi$ instead of $d \log \mathcal{C}(\xi)$. If

$$\xi - a = \epsilon' \exp i\theta',$$

the integration with respect to θ' is from π to zero, and ϵ' disappears. So the limit of the integral is

$$-ip \log(\xi_0 - a).$$

The line-integral along the real axis is generally semi-convergent as regards the infinity at a , and the choosing of an infinitesimal gap with a in the middle gives the Cauchy principal value. This is indicated by the letters pv .

On collection of the various parts of the contour-integral, and equation of the sum to zero, there appears the following formula, which must be understood as only approximate for any definite large value of R , but absolutely true in the limit for $R \rightarrow \infty$:—

$$2i \log \mathcal{C}(\xi_0) - 2i \log \mathcal{C}(\xi_G) + in \log R - \left(\frac{3}{2}\right)n\pi \\ - i\sum p \log(\xi_0 - a) + pv \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi) d \log \mathcal{C}(\xi) = 0. \quad (1)$$

It is convenient to alter the form of the last integral, wherein ξ is real, by putting

$$\mathcal{C}(\xi) = q \exp i\chi, \quad (2)$$

so that

$$d \log \mathcal{C}(\xi) = d \log q + i d\chi.$$

The infinity of $\mathcal{C}(\xi)$, for $\xi = a$, has to do with q and not with χ ; hence the part of the integral depending on χ is absolutely convergent, and does not require the symbol pv . The result now takes the form :—

$$2i \{ \log \mathcal{C}(\xi_0) - \log \mathcal{C}(\xi_G) \} + in \log R - \left(\frac{3}{2}\right)n\pi \\ - i\sum p \log(\xi_0 - a) + pv \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi) d \log q + i \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi) d\chi = 0. \quad (3)$$

4. Let ξ'_0 be the complex conjugate to ξ_0 , so that the two quantities are represented by points which are images of one another with respect to the real axis. And let the integral

$$\int (1/\pi) \log(\xi'_0 - \xi) d \log \mathcal{C}(\xi)$$

be taken round a contour identical with that of the previous article in all respects save one. As ξ'_0 is not situated in the region enclosed by the contour, the cut and the infinitesimal cavity which would be necessary for the treatment of a branch-point are not now needed. Thus the contour consists of the great and infinitesimal semicircles and the real axis, and the contour integral must be zero.

$\log(\xi'_0 - \xi)$ will be so defined that its vector-angle decreases from zero to $-\pi$ as ξ traverses the real axis from left to right.

The contributions to the contour integral arising from the cut and the

discontinuity at its outer extremity are not represented in the present evaluation, but the other parts of the integral correspond closely to those of the previous article. So the equality which now takes the place of formula (1) is readily shewn to be

$$in \log R + \frac{1}{2}n\pi - i\Sigma p \log (\xi'_0 - a) + \mathbf{p}\mathbf{v} \int_G^B \frac{1}{\pi} \log (\xi'_0 - \xi) d \log \mathcal{C}(\xi) = 0, \quad (4)$$

which is the same as

$$\begin{aligned} in \log R + \frac{1}{2}n\pi - i\Sigma p \log (\xi'_0 - a) + \mathbf{p}\mathbf{v} \int_G^B \frac{1}{\pi} \log (\xi'_0 - \xi) d \log q \\ + i \int_G^B \frac{1}{\pi} \log (\xi'_0 - \xi) d\chi = 0. \end{aligned} \quad (5)$$

In this formula let the sign of i be changed wherever it occurs, which involves also changing ξ'_0 to ξ_0 . The result is

$$\begin{aligned} -in \log R + \frac{1}{2}n\pi + i\Sigma p \log (\xi_0 - a) + \mathbf{p}\mathbf{v} \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi) d \log q \\ - i \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi) d\chi = 0. \end{aligned} \quad (6)$$

5. From formulæ (3) and (6) either of the two line-integrals can be eliminated by addition or subtraction, as the case may be. After slight rearrangement the results take the forms

$$\log \mathcal{C}(\xi_0) = \log \mathcal{C}(\xi_a) - \frac{1}{2}in\pi + i\mathbf{p}\mathbf{v} \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi) d \log q, \quad (7)$$

$$\begin{aligned} \log \mathcal{C}(\xi_0) = \log \mathcal{C}(\xi_a) - in\pi - n \log R \\ + \Sigma p \log (\xi_0 - a) - \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi) d\chi. \end{aligned} \quad (8)$$

When R becomes great without limit

$$\log \mathcal{C}(\xi_a) - n \log R \rightarrow \log K + ni\pi,$$

and so the limit forms of formulæ (7) and (8) are

$$\log \mathcal{C}(\xi_0) = \lim_{R \rightarrow \infty} \left\{ \log K + n \log R + \frac{1}{2}ni\pi + i\mathbf{p}\mathbf{v} \int_{\xi=-R}^{\xi=R} \frac{1}{\pi} \log (\xi_0 - \xi) d \log q \right\}, \quad (9)$$

$$\log \mathcal{C}(\xi_0) = \log K + \Sigma p \log (\xi_0 - a) - \int_{\xi=-\infty}^{\xi=\infty} \frac{1}{\pi} \log (\xi_0 - \xi) d\chi. \quad (10)$$

These are two distinct expressions for the logarithm of a curve-factor as line-integrals along the axis of ξ real; they imply corresponding expressions for the curve-factor as exponentials of such line-integrals.

6. Of formulæ (9) and (10) the latter is probably the more useful and interesting. Its significance depends upon the geometrical interpretation of χ in connexion with the relation $dz = \mathcal{C}(\xi)d\xi$. When ξ is real, and so replaceable by ξ , formula (2) shews that

$$dz = q \exp(i\chi) d\xi,$$

wherein dz represents vectorially the element of arc (say ds) of the boundary of that region in the z plane which is conformally represented on the principal half-plane of ξ . Hence χ is the angle which the tangent to the boundary makes with the real axis in the z plane.

Formula (10) is equivalent to

$$\mathcal{C}(\xi_0) = K \Pi(\xi_0 - a)^p \text{Lim } \Pi(\xi_0 - \xi)^{-d\chi/\pi}, \quad (11)$$

and exhibits the general curve-factor as the limit of a product of corner-factors, these having definite exponents for all corners in the boundary, and infinitesimal exponents for all points where the boundary is smoothly curved. The terms of the product correspond to values of ξ ranging over the whole boundary.

This confirms a guess hazarded in the previous paper already cited, to the effect that the most general conformal transformation for the half-plane of ξ is of the type of formula (11), or, in other words, that every curve-factor is equal to the limit of a product of corner-factors.

7. A curve-factor of restricted linear range is one for which the boundary in the z plane consists partly of one or more definite curved portions, and, for the rest, of straight lines with which the curves join smoothly. For such a curve-factor $d\chi$ differs from zero only along one or more definite stretches of the real axis of ξ , which constitute the linear range of curvilinearity, and it is only values of ξ included in these stretches that contribute to the integral of formula (10) or the product of formula (11).

Now since the product corresponding to a linear range made up of two separate portions is itself the product of two products, one corresponding to each of the sub-ranges, it follows that any curve-factor of multiple linear range can be factorised into curve-factors each having a simple linear range. This property of multiple curve-factors was surmised, but not proved, in the writer's previous paper (§ 29) referred to above.

8. If the functional relation between χ and ξ be $\chi = f(\xi)$, and if there be no corners, formula (10) is equivalent to

$$\mathcal{C}(\xi_0) = K \exp \left\{ -(1/\pi) \int f'(\xi) \log (\xi_0 - \xi) d\xi \right\}, \quad (12)$$

and in this form it constitutes the most general formula for a curve-factor of assigned linear and angular range. If the linear range is from $\xi = b$ to $\xi = c$, these must be taken as the limits of the line-integral; and, if the angular range is required to be $\kappa\pi$, f must be so chosen that

$$f(b) - f(c) = \kappa.$$

Any form may be assigned to f which does not introduce singularities already expressly excluded, and there is a form of f to correspond to every possible form of $\mathcal{C}$.

The formula may be used for the calculation of forms of curve-factor, and the only limitation upon this method of generating fresh types is the possible difficulty of effecting the integration. The indefinite integral is readily found if $f(\xi)$ is a rational algebraical function of ξ , or if it is the inverse of the square root of a linear or quadratic function. For example, the forms of $\mathcal{C}$ corresponding to

$$f(\xi) = 1/(\xi - k) \quad \text{or} \quad f(\xi) = 1/\sqrt{(\xi - k)}$$

can be calculated with little trouble. For some forms of f the definite integral could probably be calculated otherwise than through the indefinite integral. Thus the formula is substantially useful for extending the category of available curve-factors. Nevertheless its practical limitations leave plenty of scope for such other modes of generating fresh types as may result from different lines of investigation.

It is perhaps worth noting that

$$f'(\xi) = d\chi/d\xi = (d\chi/ds)(ds/d\xi) = q/\rho,$$

where ρ is the radius of curvature of the boundary in the z plane, and q the linear magnification in the z plane as compared with the ξ plane at the point considered.

9. Formula (9) is equally available, but probably less useful, for the calculation of forms of curve-factor; it has not the geometrically advantageous feature possessed by formula (10), namely that only the curved portions of the boundary contribute to the integral in the latter.

The explicit appearance of $\log R$ in formula (9) gives an appearance of divergence which is fallacious; the term actually serves to cancel a diver-

gence of the definite integral, and the limit on the right-hand side of the formula is definite. Peculiarities arising out of corners in the boundary, instead of appearing explicitly as in formula (10), manifest themselves in the semiconvergences which necessitate the restriction of the integral to the Cauchy principal value.

Formula (9) is susceptible of an interpretation of interest in hydrodynamics. If ξ be taken to be the velocity-potential and η the stream-function of two-dimensional steady continuous liquid flow, the relation $dz = C(\xi)d\xi$ specifies a configuration of flow in the z plane in which the boundary is the curve corresponding to $\eta = 0$. In this boundary the velocity is tangential, and its magnitude is $|d\xi/dz|$ or $1/q$. Now, if part of the boundary be a free stream-line, in the absence of a field of force the velocity is constant in magnitude along this line, and so $d \log q = 0$. Thus such parts of the boundary as are free stream-lines make no contribution to the integral in formula (9). The equivalent formula,

$$C(\xi_0) = Ke^{\frac{1}{2}ni\pi} \lim_{R \rightarrow \infty} \left\{ R^n \exp i p v \int \frac{1}{\pi} \log(\xi_0 - \xi) d \log q \right\}, \quad (13)$$

constitutes the most general specification of a motion with free stream-lines, and is in terms of an integral whose range of integration includes only those parts of the boundary which are fixed. By suitable choice of functional relations between q and ξ , with introduction of appropriate singularities for the corners, it is theoretically possible to make this formula represent continuous motion with free stream-lines past all types of fixed boundary.

If it were desired to use the formula for generating new specifications of motion with free stream-lines, it would seem best to take separate account of corners in the boundary. Thus, in the case of a free stream-line corresponding to the linear range $-\infty < \xi < 0$, it is known* that an angle $(1-p)\pi$ in the fixed boundary at $\xi = a$ is represented in the curve-factor

$$(a^{\frac{1}{2}} + \xi^{\frac{1}{2}})^{2p} / (\xi - a)^p, \quad (14)$$

whose modulus is constant for ξ real and negative. Hence, if P be the product of such functions representing all the corners, the transformation $dz = P C_s(\xi) d\xi$, where

$$C_s(\xi) = Ke^{\frac{1}{2}ni\pi} \lim_{R \rightarrow \infty} \left\{ R^n \exp i \int_0^R \frac{1}{\pi} \log(\xi - \xi) df(\xi) \right\}, \quad (15)$$

and $f(\xi)$ has no infinities, gives a motion of the type required.

* *L.c.*, § 36.

For example, the assumption

$$f(\xi) = \pi\xi/(c+\xi),$$

where $c > 0$, leads through formula (15) to the curve-factor

$$\xi^{ic/(c+\zeta)} (-c)^{ia/(c+\zeta)},$$

which has constant modulus for ξ real and negative. This could be combined with factors of the type of formula (14).

10. *Periodic Conformal Curve-Factors.* — A periodic curve-factor of linear period λ is a function $\mathcal{C}(\xi)$ such that the transformation $dz = \mathcal{C}(\xi)d\xi$ represents a doubly connected region in the z plane, having an internal but no external boundary, conformally and repeatedly upon successive semi-infinite strips of width λ in the principal half-plane of ξ . The writer has given an account of these functions elsewhere.* The typical periodic corner-factor is a power of $\sin \{\pi\lambda^{-1}(\xi-a)\}$.

The proof for periodic curve-factors of the theorems analogous to those of formulæ (9) and (10) depends on the application of two contour integrations to almost the same contour in the ξ plane. The integral is, in the one case

$$\int \frac{1}{\pi} \log \sin \frac{\pi}{\lambda} (\xi_0 - \xi) d \log \mathcal{C}(\xi),$$

and, in the other, the similar expression with the conjugate complex ξ_0 substituted for ξ_0 . The contour consists, in the main, of the rectangle determined by $\xi = a$, $\xi = a + \lambda$, $\eta = 0$, $\eta = t$, where $t > 0$; but, as ξ_0 is chosen to be within this rectangle, the first integration must include an infinitesimal cavity round ξ_0 and the two sides of a straight cut drawn from ξ_0 perpendicularly to the side $\eta = t$.

$\mathcal{C}(\xi)$ is supposed to have a definite angular period $2n\pi$, and therefore a definite exponential order at infinity n , so that, for η great and positive,

$$\mathcal{C}(\xi) = \exp(-2\pi ni\xi/\lambda) \sum c_s \exp(2\pi si\xi/\lambda), \quad (s = 0, 1, 2, \dots, \infty).$$

It is contemplated that the formulæ obtained will be pressed to limit forms corresponding to t infinite.

As details of a contour integration having considerable resemblance to those now required are included in the paper just referred to (§ 11), the corresponding work may be omitted here, and the results stated without

* "On Periodic Conformal Curve-Factors and Corner-Factors," *Proc. Royal Irish Academy*, Vol. xxxiii, Sec. A, 1916.

proof. The first integration gives

$$\int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d \log \mathcal{C}(\xi) + 2n \{ \pi - i \log 2 - \pi \xi_0 / \lambda \} \\ + i \{ 2 \log \mathcal{C}(\xi_0) - \log \mathcal{C}(a) - \log c_0 \} = 0, \quad (16)$$

and the second gives

$$\int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi'_0 - \xi) \right\} d \log \mathcal{C}(\xi) + 2n \{ \pi - i \log 2 - \pi \xi'_0 / \lambda \} \\ + i \{ \log \mathcal{C}(a) - \log c_0 \} = 0. \quad (17)$$

As before, let $\mathcal{C}(\xi) = q \exp(i\chi)$; let the sign of i be changed wherever it occurs in equation (17), which involves also changing ξ'_0 to ξ_0 ; and let the resulting equation be combined with (16) by addition and subtraction. There emerge the formulæ:—

$$\log \mathcal{C}(\xi_0) = i \int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d \log q + \log \mathcal{C}(a) + 2ni\pi \left(1 - \frac{\xi_0}{\lambda} \right), \quad (18)$$

$$\log \mathcal{C}(\xi_0) = - \int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d\chi + 2n \log 2 + \log c_0. \quad (19)$$

These are analogous to formulæ (9) and (10). Formula (19) is equivalent to

$$\mathcal{C}(\xi_0) = 2^{2n} c_0 \exp \left[- \int_0^{2n\pi} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d\chi \right], \quad (20)$$

and exhibits the general periodic curve-factor as the limit of a product of periodic corner factors. The inferences which can be drawn from this result correspond to those made in articles 6 to 8 above.