ON THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS, AND ON SOME RELATED THEOREMS

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[Received December 11th, 1911.—Read December 14th, 1911.]

THE fundamental lemma of the calculus of variations states that, if $f(x)$ be a function which is continuous in the interval (x_0, x_1), and be such that $\int_{x_0}^{x_1} \eta(x) f(x) dx$ vanishes for all functions $\eta(x)$ which satisfy certain prescribed conditions, then $f(x)$ must have the value zero throughout the interval (x_0, x_1). This lemma is required for the establishment of Euler's equation as the condition that the first variation of an integral

$$\int_{x_0}^{x_1} F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) dx$$

should vanish.

In the form in which this lemma was first established by Du-Bois-Reymond, the function $\eta(x)$ is prescribed to belong to the class of all those functions which vanish at x_0 and x_1 and which have a continuous derivative in the interval (x_0, x_1). The same writer further shewed that the theorem is still valid in case the function $\eta(x)$ is restricted to belong to the class of either (1) all functions which vanish at x_0, x_1 and which have continuous derivatives of the first p orders in the interval, or (2) all functions which have continuous derivatives of all orders. It has further been shewn by H. A. Schwarz that the theorem holds in case the functions $\eta(x)$ are restricted to the class of functions that are regular in the interval, *i.e.*, such that in the neighbourhood of any point a in the interval they are representable by a power-series in powers of $x-a$. It is clear that the more restricted the class of functions to which $\eta(x)$ is assumed to belong, that may be found sufficient for the establishment of the theorem, the greater is the degree of generality of that theorem. Moreover, if the function $f(x)$ need not be assumed to be continuous in the whole interval (x_0, x_1) the generality of the theorem will be still further increased. In the present communication it is shewn to be sufficient to assume that the function $\eta(x)$ is restricted to be any finite polynomial which, together with its first p derivatives, vanishes at the ends of the intervals, p being any fixed integer (including zero). Moreover, it is shewn that if no assumption be made as to $f(x)$, except that is summable in the interval, in the sense that, whether it be limited in the interval or not, it has a Lebesgue integral, the theorem still holds in the sense that $f(x)$ must be zero at every point of the interval, with the possible exception of points belonging to an exceptional set with the measure zero. Moreover, $f(x)$ is zero at every point at which it is continuous. When the theorem thus generalized is applied for the purposes of the calculus of variations it is seen that, in order to establish Euler's differential equation satisfied by the extremals, it is sufficient to take weak variations δy of the restricted type which consists only of finite polynomials, which, together with a prescribed number of their derivatives, vanish at the extremities of the interval.

A theorem* which is of use in the theory of isoperimetric problems, and which is due to Du-Bois-Reymond, states that if $f(x)$, $g(x)$ are continuous in the interval (x_0, x_1), and such that $\int_{x_0}^{x_1} f(x)\psi(x)\,dx$ vanishes for all functions $\psi(x)$ which vanish at x_0, x_1, have continuous differential coefficients in the interval (x_0, x_1), and are such that $\int_{x_0}^{x_1} g(x)\psi(x)\,dx$ vanishes, then a constant λ exists such that $f(x)+\lambda g(x)$ vanishes throughout the interval (x_0, x_1). This theorem is here generalized by assuming that $f(x)$, $g(x)$ are any summable functions, and that $\psi(x)$ is restricted to be a finite polynomial of the same type as in the former theorem. It is then shewn that a constant λ exists, such that $f(x)+\lambda g(x)$ vanishes at all points except those of an exceptional set with measure zero, and that it vanishes at every point at which $f(x)$, $g(x)$ are continuous.

Another known theorem which has an important application in the Calculus of Variations is that, if $\int_{x_0}^{x_1} \frac{d\eta(x)}{dx} f(x)\,dx$ vanishes for all functions $\eta(x)$ which have continuous differential coefficients in the interval (x_0, x_1), where $f(x)$ is a continuous function, then $f(x)$ must be constant in the interval. This was also established by Du-Bois-Reymond, and has been extended to the case in which $f(x)$, though limited, has a finite set of discontinuities in the interval.

This theorem is here extended to the case in which $f(x)$ is assumed only to be summable in (x_0, x_1), and in which $\eta(x)$ is restricted to belong to the class of those finite polynomials which vanish at x_0, x_1. It is shewn that $f(x)$ has a constant value at all points which do not belong to

* See Bolza's *Vorlesungen über Variationsrechnung*, p. 462.

some set of measure zero, and that it has this constant value at all points of any interval contained in (x_0, x_1) in which it is continuous. The theorem is proved for the more general case of the integral

$$\int_{x_0}^{x_1} \frac{d^p \eta(x)}{dx^p} f(x)\, dx,$$

in which a polynomial of degree $p-1$ takes the place of the constant for the special case $p = 1$.

Lastly, the case of the vanishing of the more general expression,

$$\int_{x_0}^{x_1} \left\{ q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} \right\} f(x)\, dx,$$

is considered, where $q_0, q_1, \ldots, q_p$ are functions of x satisfying certain conditions.

It is shewn that, if y be restricted to belong to the class of those finite polynomials which, together with their first $p-1$ derivatives, vanish at x_0 and x_1, then, in any interval of continuity of $f(x)$, that function must be a solution of the differential equation which is adjoint to the equation

$$q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

1. Let $f(x)$ be a function which is summable (whether limited or not) in the interval (x_0, x_1). Let it be assumed that for every function $\eta(x)$ which consists of a finite polynomial such that the function and its first p differential coefficients vanish for the values x_0, x_1 of x, the integral $\int_{x_0}^{x_1} \eta(x) f(x)\, dx$ has the value zero.*

It is clear that $\eta(x)$ is of the form $(x-x_0)^{p+1}(x_1-x)^{p+1} P_r(x)$, where $P_r(x)$ is any finite polynomial. If we write

$$\phi(x) = (x-x_0)^{p+1}(x_1-x)^{p+1} f(x),$$

the above assumption is equivalent to the one that $\int_{x_0}^{x_1} P_r(x)\,\phi(x)\, dx$ vanishes for every polynomial $P_r(x)$.

Let ξ, ξ' denote any two numbers such that $x_0 < \xi < \xi' < x_1$, and let ϵ be an arbitrarily chosen positive number, less than both $\xi - x_0$ and $x_1 - \xi'$. Let $\chi(x)$ denote that function, continuous in the interval (x_0, x_1),

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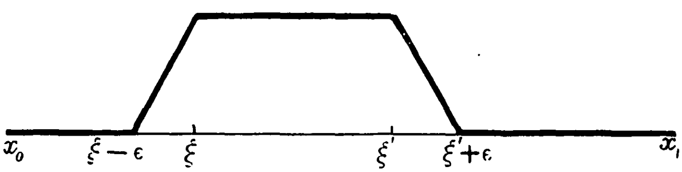
* It can be shewn that, if $f(x)$ is summable in the interval (x_0, x_1), then the product of $f(x)$ by any limited summable function is also summable in the same interval. The functions $\eta(x)$ fall into this class.

which has the values

$$\begin{array}{lll}\chi(x) = 0, & \text{for} & x_0 \leqslant x \leqslant \xi-\epsilon, \\ \chi(x) = \dfrac{1}{\epsilon}(x-\xi+\epsilon), & \text{for} & \xi-\epsilon \leqslant x < \xi, \\ \chi(x) = 1, & \text{for} & \xi \leqslant x \leqslant \xi', \\ \chi(x) = \dfrac{1}{\epsilon}(\xi'+\epsilon-x), & \text{for} & \xi' \leqslant x \leqslant \xi'+\epsilon, \\ \chi(x) = 0, & \text{for} & \xi'+\epsilon \leqslant x \leqslant x_1. \end{array}$$

Thus $\chi(x)$ is represented graphically by the thickened line in the figure.



In accordance with the well known theorem of Weierstrass, a finite polynomial $P_r(x)$ can be so determined that, if ζ be an arbitrarily chosen positive number, the condition

$$|\chi(x)-P_r(x)| < \zeta$$

holds for every value of x in the interval (x_0, x_1).

We have

$$\int_{x_0}^{x_1} \chi(x)\,\phi(x)\,dx = \int_{x_0}^{x_1} \phi(x)\,\{\chi(x)-P_r(x)\}\,dx + \int_{x_0}^{x_1} \phi(x)\,P_r(x)\,dx.$$

Now $$\int_{x_0}^{x_1} \chi(x)\,\phi(x)\,dx = \int_{\xi}^{\xi'} \phi(x)\,dx + \int_{\xi-\epsilon}^{\xi} \frac{1}{\epsilon}(x-\xi+\epsilon)\,\phi(x)\,dx + \int_{\xi'}^{\xi'+\epsilon} \frac{1}{\epsilon}(\xi'+\epsilon-x)\,\phi(x)\,dx,$$

and the second and third integrals on the right-hand side of this equation are less in absolute value than $\int_{\xi-\epsilon}^{\xi} |\phi(x)|\,dx$, $\int_{\xi'}^{\xi'+\epsilon} |\phi(x)|\,dx$; and thus, in accordance with a well known property of Lebesgue integrals, they converge to zero as ϵ converges to zero.

Also $$\left|\int_{x_0}^{x_1} \phi(x)\,\{\chi(x)-P_r(x)\}\,dx\right| < \zeta \int_{x_0}^{x_1} |\phi(x)|\,dx.$$

By hypothesis $$\int_{x_0}^{x_1} \phi(x) P_r(x)\, dx = 0\,;$$

it then follows that $\left|\int_{\xi}^{\xi'} \phi(x)\,dx\right|$ has a value which can be made as small as we please by taking ϵ and ζ small enough. Therefore

$$\int_{\xi}^{\xi'} \phi(x)\, dx = 0,$$

where ξ, ξ' are any numbers such that $x_0 < \xi < \xi' < x_1$. Since the integral is a continuous function of either the upper or the lower limit, it follows that the integral vanishes for all pairs of values of ξ and ξ' such that $x_0 \leqslant \xi < \xi' \leqslant x_1$. It is a known theorem * that, if the integral of $\phi(x)$ vanishes when taken through any sub-interval whatever of (x_0, x_1), then $\phi(x)$ vanishes for every value of x, with the possible exception of those values belonging to a set with measure zero.

Since $f(x)$ vanishes for any value of x for which $\phi(x)$ vanishes, except for $x = x_0$, $x = x_1$, we see that $f(x)$ must vanish for all values of x except at most those belonging to some set with measure zero.

The fundamental lemma of the Calculus of Variations has now been established in the following generalized form :—

If $f(x)$ be a function which is summable in the interval (x_0, x_1), and is such that $\int_{x_0}^{x_1} f(x) P(x) dx$ vanishes, provided $P(x)$ is any finite polynomial which, together with its first p differential coefficients, vanishes at the end-points of the interval, then $f(x)$ must vanish for every value of x except those belonging to some set of which the measure is zero. Also $f(x)$ must vanish at any point at which it is continuous. In particular, if

* This theorem was first given by Vitali ; see *Rend. di Palermo*, Vol. XX, p. 136.

It may easily be proved as follows :—Let E be that set of points in the interval (x_0, x_1), at each of which $\phi(x) > 0$; then all the points of E can be enclosed in intervals of a set Δ, such that $m(\Delta) - m(E)$ is arbitrarily small; where $m(E)$, $m(\Delta)$ denote the measures of E and Δ respectively. The integral of $\phi(x)$ through Δ vanishes, by hypothesis, and thus

$$\int_{(E)} \phi(x)\, dx + \int_{(\Delta - E)} \phi(x)\, dx = 0.$$

The latter integral may be made arbitrarily small by choosing $m(\Delta) - m(E)$ small enough ; therefore

$$\int_{(E)} \phi(x)\, dx = 0.$$

It follows that the measure of that set of pcints of E at which $\phi(x) > 0$ must be zero. Similarly it is seen that the measure of the set of points in (x_0, x_1) at each of which $\phi(x) < 0$ must be zero. Thus the theorem is established.

$f(x)$ is continuous in an interval (α, β) which may be the whole or a part of (x_0, x_1), it must vanish everywhere in (α, β).

It is clear that, instead of $P(x)$, we may take the functions

$$(x-x_0)^{p+1}(x_1-x)^{p+1}x^k,$$

where k has the values 0, 1, 2, 3, ...

It will be observed that the proof here given establishes the theorem that, if $\phi(x)$ be any summable function such that $\int_{x_0}^{x_1}\phi(x)x^k dx$ vanishes for all positive integral values of k, including zero, then $\phi(x)$ vanishes in the interval (x_0, x_1), except possibly at points belonging to some set of measure zero; and that in particular $\phi(x)$ vanishes at any point at which it is continuous. The particular case of this theorem which arises when $\phi(x)$ is continuous throughout the interval has been proved by Landau,* who gives references to an earlier proof by Lerch.

2. Let $f(x)$, $g(x)$ be functions which are summable in the interval (0, 1), and let us suppose that $\int_0^1 f(x)\,Q(x)\,dx$ vanishes for every finite polynomial such that $\int_0^1 g(x)\,Q(x)\,dx$ vanishes, and such also that $Q(x)$ and its first p differential coefficients all vanish for $x=0$ and $x=1$. The number p may have any fixed integral value, including zero.

Let A and B be so chosen that

$$\int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}(Ax^n+Bx^m)\,dx=0,$$

where n and m are positive integers. When the two integrals

$$\int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}x^n\,dx, \qquad \int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}x^m\,dx$$

have values which are both different from zero, which we suppose to be the case, the assumed condition determines uniquely the ratio A/B.

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* See the *Rend. del Circ. Mat. di Palermo*, Vol. xxv, p. 343. A referee has called attention to the fact that the theorem is not valid for the case of an indefinitely great interval. Thus Stieltjes has pointed out (see Borel's *Théorie des séries divergentes*, p. 68) that if

$$\phi(x)=e^{-x^{\frac14}}\sin(x^{\frac14}),$$

then $$\int_0^\infty \phi(x)\,x^k\,dx=4\int_0^\infty e^{-y}\sin y\,.\,y^{4k+3}\,dy=\frac{1}{2^{2k}}\Gamma(4k+4)\sin(k+1)\pi,$$

which vanishes when $k=0, 1, 2, 3, \ldots$.

We have then, from the original assumption,

$$A\int_0^1 f(x)\,x^{p+1}(1-x)^{p+1}x^n\,dx+B\int_0^1 f(x)\,x^{p+1}(1-x)^{p+1}x^m\,dx=0.$$

From the two equations, we see that

$$\int_0^1 x^{p+1}(1-x)^{p+1}\{f(x)+\lambda g(x)\}\,x^n\,dx=0,$$

where λ has the value

$$-\int_0^1 x^{m+p+1}(1-x)^{p+1}f(x)\,dx\Big/\int_0^1 x^{m+p+1}(1-x)^{p+1}\,g(x)\,dx,$$

and thus, if m has any fixed value, λ is a determinate constant.

The integer m may be so chosen that $\int_0^1 g(x)x^{p+m+1}(1-x)^{p+1}dx$ is not zero, for if this integral were zero for every value 0, 1, 2, 3, ... of m, it would follow from the theorem of § 1 that $g(x)$ vanishes everywhere except at points of some set of measure zero, and we may provisionally suppose $g(x)$ to be such that this possibility is excluded. Moreover, there must be an indefinitely great number of values of m for which the integral does not vanish ; for, if it vanished for all values of m which exceed some fixed value m_1, we should see that $x^{m_1}g(x)$, and therefore $g(x)$, would vanish for all values of x except those of some set of measure zero. Further, for any value of n for which

$$\int_0^1 x^{p+1}(1-x)^{p+1}\,g(x)\,x^n\,dx=0,$$

we should also have $\int_0^1 x^{p+1}(1-x)^{p+1}f(x)\,x^n\,dx=0,$

in virtue of the original assumption. It has now been shewn that a constant λ exists, such that

$$\int_0^1 x^{p+1}(1-x)^{p+1}\{f(x)+\lambda g(x)\}\,x^n\,dx=0,$$

for $$n=0,\ 1,\ 2,\ 3,\ \ldots.$$

It follows from the theorem of § 1, that $f(x)+\lambda g(x)$ is zero at all points of the interval (0, 1) with the exception of those belonging to some set with measure zero. Also in any interval (α, β) in which $f(x)$, $g(x)$ are continuous, $f(x)+\lambda g(x)$ vanishes in the interval.

In case $g(x)$ vanishes everywhere in (0, 1) with the exception of points of some set with measure zero, the integral

$$\int_0^1 x^{p+1}(1-x)^{p+1}\,g(x)\,x^n\,dx$$

would vanish for every value of n; and therefore

$$\int_0^1 x^{p+1}(1-x)^{p+1} f(x)\, x^n\, dx$$

would also vanish for every value of n. It would then follow that $f(x)$ vanishes everywhere except at points of some exceptional set of measure zero; in this case we shall have $\lambda = 0$. It is clear that the limits of the integral can be taken to be x_0, x_1, instead of 0, 1.

The following theorem has now been established :—

Let $f(x)$ be a function summable in the interval (x_0, x_1), and such that $\int_{x_0}^{x_1} f(x)\, Q(x)\, dx$ vanishes for every finite polynomial $Q(x)$ which is such that

$$\int_{x_0}^{x_1} g(x)\, Q(x)\, dx = 0,$$

where $g(x)$ is another function summable in (x_0, x_1), and where $Q(x)$ also satisfies the condition that it and its first p derivatives vanish for $x = x_0, x_1$. A determinate constant λ then exists, such that $f(x)+\lambda g(x)$ vanishes for all points in the interval (x_0, x_1) except at most those belonging to some exceptional set which has the measure zero. In any interval (α, β), which may be a part or the whole of (x_0, x_1), and in which $f(x)$, $g(x)$ are continuous, $f(x)+\lambda g(x)$ vanishes without exception.

3. Let us assume that $f(x)$ is a function defined for the interval (0, 1), summable in that interval, and such that

$$\int_0^1 \frac{d^p \eta(x)}{dx^p} f(x)\, dx$$

vanishes for every value of $\eta(x)$ which is a finite polynomial such that it and its first $p-1$ differential coefficients all vanish at the end-points of the interval. Let $\eta(x) = x^p(1-x)^p x^k$, where k is a positive integer; we have then

$$0 = \int_0^1 f(x) \frac{d^p}{dx^p}\left[x^{p+k}(1-x)^p\right] dx$$

$$= \int_0^1 f(x) \Big[(p+k)(p+k-1)\dots(k+1)x^k - p(p+k+1)(p+k)\dots(k+2)x^{k+1}$$
$$+ \frac{p(p-1)}{2!}(p+k+2)(p+k+1)\dots(k+3)\, x^{k+2} - \dots\Big] dx.$$

Let $$u_k = (p+k)(p+k-1)\dots(k+1)\int_0^1 x^k f(x)\, dx;$$

the above equation becomes then

$$u_k - p u_{k+1} + \frac{p(p-1)}{2!} u_{k+2} - \ldots = 0;$$

which is a linear difference equation that can be written in the form

$$(1-E)^p u_k = 0,$$

where E is the usual operator, such that

$$E u_k = u_{k+1}.$$

The general solution of this difference equation is

$$u_k = A_0 + A_1 k + A_2 k^2 + \ldots + A_{p-1} k^{p-1};$$

where A_0, A_1, ..., A_{p-1} are constants independent of the value of k. It follows that

$$\int_0^1 f(x) x^k \, dx = \frac{a_1}{k+1} + \frac{a_2}{k+2} + \ldots + \frac{a_p}{k+p};$$

where $a_1, a_2, \ldots, a_p$ are independent of k. Hence

$$\int_0^1 x^k \left[f(x) - a_1 - a_2 x - a_3 x^2 - \ldots - a_p x^{p-1} \right] dx = 0.$$

Since the function $f(x) - a_1 - a_2 x - a_3 x^2 - \ldots - a_p x^{p-1}$ when multiplied by any finite polynomial and integrated through (0, 1) is such that the integral vanishes, it follows from the theorem in § 1 that this function vanishes for all points of the interval (0, 1) except possibly those belonging to a set of which the measure is zero. It is clear that, by a linear transformation, the limits of the integral can be taken to have any values x_0, x_1 instead of 0 and 1; thus we have established the following theorem:—

If $f(x)$ be a function which is summable in the interval (x_0, x_1) and is such that $\int_{x_0}^{x_1} f(x) \frac{d^p}{dx^p} P(x) \, dx$ vanishes provided $P(x)$ is any finite polynomial which, together with its first $p-1$ differential coefficients, vanishes at the end-points of the interval (x_0, x_1), then $f(x)$ must have the same values as some polynomial of degree $p-1$ for every value of x except possibly for those belonging to a set of measure zero. If in an interval which is a part of, or the whole of, the interval (x_0, x_1) the function $f(x)$ is continuous, then through the whole of such interval $f(x)$ must have the same values as the polynomial.

In the case $p = 1$, the theorem states that if

$$\int_{x_0}^{x_1} f(x) \frac{d}{dx} P(x)\, dx = 0,$$

for all polynomials $P(x)$ which vanish at x_0, x_1, then $f(x)$ must have a constant value at all points of the interval, with the possible exception of those belonging to a set of measure zero. Moreover, the function has this constant value at every point where it is continuous.

4. Let $q_0, q_1, q_2, \ldots q_p$ be functions of x which are continuous and of limited total fluctuation in the interval (0, 1). Moreover, let it be assumed that q_p has a continuous differential coefficient of order p, q_{p-1}, a continuous differential coefficient of order $p-1$, and so on. Let it be assumed that a summable function $f(x)$ is such that

$$\int_0^1 \left\{ q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2y}{dx^2} + \ldots + q_p \frac{d^py}{dx^p} \right\} f(x)\, dx$$

vanishes for every value of y that is represented by a finite polynomial which, together with its first $p-1$ derivatives, vanishes at the end-points 0, 1 of the interval. The method of integration by parts being applicable to Lebesgue integrals, we find, by successive employment of the process, that

$$\int_0^1 q_{p-r} f(x) \frac{d^{p-r}y}{dx^{p-r}}\, dx = (-1)^r \int_0^1 \left\{ \frac{d^py}{dx^p} \int_0^x dx \int_0^x dx \ldots \int_0^x q_{p-r} f(x)\, dx \right\} dx.$$

Hence, when y is any polynomial having the above property, we see that

$$\int_0^1 \left[q_p f(x) - \int_0^x q_{p-1} f(x)\, dx + \int_0^x dx \int_0^x q_{p-2} f(x)\, dx - \ldots \right] \frac{d^py}{dx^p}\, dx$$

vanishes. From the theorem established in § 3, we see that

$$q_p f(x) - \int_0^x q_{p-1} f(x)\, dx + \int_0^x dx \int_0^x q_{p-2} f(x)\, dx - \ldots$$

must, at every point x of the interval (0, 1), with the possible exception of the points of a set of measure zero, have the value of some polynomial $a_0 + a_1 x + \ldots + a_p x^{p-1}$, of degree $p-1$.

Remembering that an integral $\int_0^x U dx$ has a differential coefficient with respect to its upper limit x, equal to U, at any point at which U is continuous, we see that at any interior point of an interval (α, β) contained in (0, 1), such that $f(x)$ is continuous in (α, β), $f(x)$ has a continuous differential coefficient; since throughout this interval we have

$$q_p f(x) = a_0 + a_1 x + \ldots + a_p x^{p-1} + \int_0^x q_{p-1} f(x)\, dx - \int_0^x dx \int_0^x q_{p-2} f(x)\, dx + \ldots.$$

Thus, throughout this interval,

$$\frac{d}{dx}\{q_p f(x)\}$$
$$= a_1 + 2a_2 x + \ldots + (p-1) a_p x^{p-2} + q_{p-1} f(x) - \int_0^x q_{p-2} f(x)\, dx + \ldots;$$

it now follows that $q_p f(x)$ has a continuous second differential coefficient through the interior of (α, β), and thus that this holds also for $f(x)$. Proceeding in this way we see that, in the interior of the interval (α, β), the first p differential coefficients of $f(x)$ all exist and are continuous, and also that $f(x)$ there satisfies the condition

$$\frac{d^p}{dx^p}\{q_p f(x)\} - \frac{d^{p-1}}{dx^{p-1}}\{q_{p-1} f(x)\} + \frac{d^{p-2}}{dx^{p-2}}\{q_{p-2} f(x)\} - \ldots = 0.$$

This is the differential equation known as the one adjoint * to the differential equation

$$q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2y}{dx^2} + \ldots + q_p \frac{d^py}{dx^p} = 0.$$

In case $f(x)$ is continuous in the whole interval (0, 1) it must be a function which everywhere satisfies this adjoint equation.

The following theorem has now been established:—

Let $q_0, q_1, \ldots q_p$ be functions of x which are continuous and of limited total fluctuation in the interval (x_0, x_1), and let q_r be such that it has a continuous differential coefficient of order r throughout the interval. Let $f(x)$ be a function summable in the interval (x_0, x_1) and such that

$$\int_{x_0}^{x_1} \left\{ q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2y}{dx^2} + \ldots + q_p \frac{d^py}{dx^p} \right\} f(x)\, dx$$

vanishes for every value of y that consists of a finite polynomial which, together with its first $p-1$ differential coefficients, vanishes at x_0 and x_1. Then, if $f(x)$ be continuous in the interval (x_0, x_1), it must be a solution of

* See Forsyth's *Theory of Differential Equations*, Part III.

the equation which is adjoint to the equation

$$q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

Moreover, if $f(x)$ is continuous only in some interval contained in (x_0, x_1), its value must satisfy the adjoint equation at all points in the interior of the interval of continuity.

It will be observed that, as in the former cases, no assumption has been made *a priori* as to the existence of differential coefficients of $f(x)$, which has been assumed only to be summable, whether limited or not, in the sense that it is assumed to be such as to possess a Lebesgue integral.