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LXXXIII. *A Method of Finding the Coefficients of Absorption of the Different Constituents of a Beam of Heterogeneous Röntgen Rays, or the Periods and Coefficients of Damping of a Vibrating Dynamical System.* By Sir J. J. THOMSON, O.M., F.R.S.*

I HAVE found the following method useful in finding the absorption coefficients of the various constituents of a beam of heterogeneous Röntgen rays. In this case the intensity of the rays after passing through a thickness d of aluminium can be represented by the expression $A_1\epsilon^{-\lambda_1 d} + A_2\epsilon^{-\lambda_2 d} + \dots A_n\epsilon^{-\lambda_n d}$; the problem is to find from the $2n$ observations which are necessary for this purpose the values of $A_1, A_2, \dots A_n; \lambda_1, \lambda_2, \dots \lambda_n$. Though I arrived at the method from the consideration of the absorption problem, it is of much more general application and applies to any quantity which can be represented as the sum of a series of exponentials of some variable, whether the coefficients of the exponent are real, imaginary, or complex. Since the amplitude of the vibrations, damped or undamped, of a dynamical system about a position of equilibrium can be represented by a series of the form $\sum A\epsilon^{(\alpha+i\beta)t}$, the method can be employed to find the periods and damping coefficients of a dynamical system, and is especially useful when we have a graph representing the variation of the displacement of the point in the system with the time. In the case of undamped vibrations it gives a simple method of finding the simple periodic terms into which the motion can be resolved.

The method depends upon the following theorem. Suppose $y_0, y_1, y_2, \dots y_{2n-1}$ are $2n$ observed values, the observations being made at equal intervals θ , of the variable: in the case of the X-ray problem this variable is the thickness of the aluminium leaf through which the radiation has passed; in the dynamical problem the variable is the time. We shall denote this variable by t . Then y_0 is the observation when $t=0$, y_1 when $t=\theta$, y_2 when $t=2\theta$, and so on.

Since

$$\left. \begin{aligned} y &= A_1\epsilon^{\lambda_1 t} + A_2\epsilon^{\lambda_2 t} + A_3\epsilon^{\lambda_3 t} + A_n\epsilon^{\lambda_n t}, \\ y_0 &= A_1 + A_2 + A_3 + A_n, \\ y_1 &= A_1\xi + A_2\eta + A_3\zeta + \dots, \\ y_2 &= A_1\xi^2 + A_2\eta^2 + A_3\zeta^2 + \dots, \\ &\dots\dots\dots \\ y_s &= A_1\xi^s + A_2\eta^s + A_3\zeta^s + \dots, \end{aligned} \right\}, \quad \dots \quad (\alpha)$$

* Communicated by the Author.

where

$$\xi = \epsilon^{\lambda_1 \theta}, \quad \eta = \epsilon^{\lambda_2 \theta}, \quad \zeta = \epsilon^{\lambda_3 \theta}, \dots$$

Eliminating $A_1 \dots A_n$ from the first $(n+1)$ equations, we have

$$\begin{vmatrix} y_0 & 1 & 1 & 1 \\ y_1 & \xi & \eta & \zeta \dots \\ y_2 & \xi^2 & \eta^2 & \zeta^2 \dots \\ & \dots & \dots & \dots \\ y_n & \xi^n & \eta^n & \zeta^n \dots \end{vmatrix} = 0.$$

Expanding the determinant, we get

$$y_0 D_0 + y_1 D_1 + y_2 D_2 + \dots = 0, \quad \dots \quad (1)$$

where D_s is the minor corresponding to y_s , and is equal to

$$\pm \begin{vmatrix} 1 & 1 & 1 \\ \xi & \eta & \zeta \\ & \dots & \dots \\ \xi^n & \eta^n & \zeta^n \end{vmatrix}$$

the row $\xi^s, \eta^s, \zeta^s \dots$ being left out.

We easily see that

$$D_s = P \cdot S_s,$$

where P is the product of all the differences of ξ, η, ζ , *i. e.*

$$\begin{aligned} & (\xi - \eta)(\xi - \zeta)(\xi - \epsilon) \dots \\ & (\eta - \zeta)(\eta - \epsilon) \dots \\ & (\zeta - \epsilon) \dots, \end{aligned}$$

and S the sum of the products ξ, η, ζ taken $n-s$ at a time.

Thus

$$S_0 = \xi\eta\zeta\epsilon, \quad S_1 = \xi\eta\zeta + \xi\eta\epsilon + \dots$$

Since P is a common factor of the D 's, we may write equation (1) as

$$y_0 S_0 - y_1 S_1 + y_2 S_2 \dots = 0.$$

By eliminating $A_1, A_2, \dots A_n$ from the n equations beginning with y_1 instead of y_0 , we get

$$y_1 S_0 - y_2 S_1 + y_3 S_2 \dots = 0;$$

and similarly

$$y_2 S_0 - y_3 S_1 + y_4 S_2 \dots = 0,$$

down to

$$y_{n-1} S_0 - y_n S_1 + y_{n+1} S_2 \dots = 0.$$

But if x is equal to any of the quantities $\xi, \eta, \zeta, \&c.$,

$$S_0 - xS_1 + x^2S_2 \dots = 0.$$

Hence, eliminating $S_0, S_1, \dots S_n$ from the $n+1$ equations we get

$$\begin{vmatrix} y_0 & y_1 & y_2 & \dots & y_n \\ y_1 & y_2 & y_3 & \dots & y_{n+1} \\ y_2 & y_3 & y_4 & \dots & y_{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ y_{n-1} & y_n & \dots & \dots & y_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{vmatrix} = 0. \dots \dots (2)$$

Thus the n roots of this equation for x are the values of ξ, η, ζ —i. e., of $\epsilon^{\lambda_1\theta}, \epsilon^{\lambda_2\theta}$. Hence this equation determines the values of $\epsilon^{\lambda_1\theta}, \epsilon^{\lambda_2\theta}$; and since θ is known, the values of the λ 's can be immediately determined.

We can find the values of the coefficients $A_1, A_2, A_3, \&c.$ as follows:—

From the first n equations of the system a we get

$$A_1 = \begin{vmatrix} y_0 & 1 & 1 \\ y_1 & \eta & \zeta \\ \dots & \dots & \dots \\ y_{n-1} & \eta^{n-1} & \zeta^{n-1} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ \xi & \eta & \zeta \\ \xi^2 & \eta^2 & \zeta^2 \\ \dots & \dots & \dots \\ \xi^{n-1} & \eta^{n-1} & \zeta^{n-1} \end{vmatrix}$$

$$= -\frac{y_0S_{n-1} - y_1S_{n-2} + y_2S_{n-3} \dots}{(\xi - \eta)(\xi - \zeta)(\xi - \epsilon) \dots}, \dots \dots (3)$$

where S_r is the sum of the products of the $(n-1)$ quantities $\eta, \zeta, \epsilon, \dots$ taken r at a time. Thus

$$S_{n-1} = \eta\zeta\epsilon, \quad S_1 = \eta + \zeta + \epsilon + \dots$$

The value of a coefficient can be expressed in terms of the root corresponding to the coefficient, and does not require a knowledge of the other roots. For η, ζ, ϵ are the roots of the equation

$$x^{n-1} - S_1x^{n-2} + S_2x^{n-3} - \dots = 0;$$

so that the polynomial is equal to

$$(x - \eta)(x - \zeta)(x - \epsilon) \dots$$

Writing equation (1) in the form

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots = 0,$$

since the roots are $\xi, \eta, \zeta, \dots$, the polynomial is equal to

$$(x - \xi)(x - \eta)(x - \zeta) \dots$$

Hence

$$\begin{aligned} (x - \xi)(x^{n-1} - S_1 x^{n-2} + S_2 x^{n-3} \dots) \\ = x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3}, \end{aligned}$$

and equating coefficients we get

$$\begin{aligned} S_1 + \xi &= p_1, \\ S_2 + \xi S_1 &= p_2, \\ S_3 + \xi S_2 &= p_3, \\ &\dots \dots \dots \\ \xi S_{n-1} &= p_n. \end{aligned}$$

Thus all the S's can be found in terms of ξ .

Differentiating the identity

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} = (x - \xi)(x - \eta)(x - \zeta) \dots,$$

and putting $x = \xi$ after differentiation we find

$$\begin{aligned} n\xi^{n-1} - (n-1)p_1\xi^{n-2} + (n-2)p_2\xi^{n-3} - \dots \\ = (\xi - \eta)(\xi - \zeta)(\xi - \epsilon) \dots \end{aligned}$$

Thus both the numerator and denominator of (2) can be expressed in terms of ξ without knowing $\eta, \zeta, \dots$

If two roots are equal, say ξ and η , the expressions for both A_1 and A_2 become indefinite. $A_1 + A_2$, the quantity we require, has, however, the definite value

$$- \frac{y_0 q_{n-2} - y_1 q_{n-3} + y_2 q_{n-4} \dots}{(\xi - \zeta)(\xi - \epsilon) \dots},$$

where

$$\begin{aligned} q_1 + \xi &= S_1, \\ q_2 + \xi q_1 &= S_2, \\ q_3 + \xi q_2 &= S_3, \end{aligned}$$

where the S's are the same as in equation (3). The denominator is

$$(n-1)\xi^{n-2} - (n-2)S_1\xi^{n-3} + (n-3)S_2\xi^{n-4} \dots$$

Mr. R. A. Herman of Trinity College, Cambridge, has pointed out to me that the determinant in equation (2) is of the type discussed by Sylvester in a paper in the Philosophical Magazine, Nov. 1851, and called by him a canonizant.