ON AN ASSUMPTION CONTAINED IN EUCLID'S PROOFS OF
CERTAIN PROPOSITIONS IN HIS TWELFTH BOOK

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It is well known that Euclid in his proof of V, 18 made the assumption that, given any three quantities—the first two being of the same kind—there always exists a fourth proportional to them. Simson, who objected to the proof as it stands, devised another free from the supposed blemish. The assumption reappears in Euclid in five propositions of his Twelfth Book, viz., XII, 2, 5, 11, 12, 18, and Simson did not give any proof of these theorems removing the difficulty; nor has he put forward any grounds which entitle him to condone the want of rigour in the Twelfth Book while condemning it in the Fifth. As far as I know, none of the subsequent commentators has cleared up the matter. It seems interesting, therefore, to enquire to what extent the above assumption is really necessary, and to examine how Euclid could have managed to do without it, had he entertained any objection to it, while adhering in other respects to his own methods and his own definitions. In other words, we might wish to know whether it is possible to make for the five theorems of the Twelfth Book the same kind of amendment that Simson effected in V, 18.

After a careful examination as to the best way of removing the difficulty, I am led to believe that in the case of XII, 2 we can proceed most readily by making use of Stolz's theorem. This theorem tells us that in order to satisfy ourselves that four quantities, say A, B, C, D are in proportion, we need only show that $pA \gtrless qB$, according as $pC \gtrless qD$; where pA, pC are any equimultiples of A and C , and qB, qD are any independent equimultiples of B and D . The remaining test contemplated by Euclid as to $pA = qB$, when $pC = qD$, need not be applied. (See Stolz, *Vorlesungen über allgemeine Arithmetik*, Pt. 1, p. 87, published in 1885.)

The proof of XII, 2 on these lines will now be given.

To show that any two circles have the same ratio as the squares on their diameters.

Let C_1 and C_2 be the two circles ; take $n_1 C_1$, $n_2 C_2$ multiples of C_1 and C_2 respectively, and first suppose that $n_1 C_1 > n_2 C_2$.

Inscribe a square in each of the circles equal to C_1 ; then just as in Euclid we may show that each square is greater than the half of the circle in which it is inscribed ; and that on substituting for each square an inscribed regular polygon of double the number of sides we shall be removing from the segments left over from the squares more than their half. (See Euclid's proof ; Heath, Vol. III, pp. 371, 372.)

Thus, by starting with the inscribed squares and continually substituting inscribed regular polygons of double the number of sides, we shall at length leave some segments of the circles which will be together less than the excess by which $n_1 C_1$ exceeds $n_2 C_2$ (X, 1).

Let P_1 denote the inscribed polygon when this stage has been reached ; then

$$n_1 C_1 - n_1 P_1 < n_1 C_1 - n_2 C_2,$$

so that

$$n_1 P_1 > n_2 C_2.$$

Let P_2 denote a polygon similar to P_1 but inscribed in C_2 ; then

$$n_2 C_2 > n_2 P_2.$$

Hence

$$n_1 P_1 > n_2 P_2.$$

But similar polygons inscribed in circles are to one another as the squares on the diameters (XII, 1). Therefore

$$n_1 S_1 > n_2 S_2,$$

where S_1 , S_2 are the squares on the diameters of C_1 , C_2 respectively (V, Def. 5).

Secondly, if $n_1 C_1 < n_2 C_2$, we may show in a similar manner that

$$n_1 S_1 < n_2 S_2.$$

By Stolz's theorem these two cases are sufficient to prove that C_1 , C_2 , S_1 , S_2 are proportional.

An objection might be raised to the above proof, viz., that Stolz's theorem does not actually occur in Euclid, and that consequently we have no right to use the theorem in a proof conducted professedly on Euclidean lines. To this we might urge in reply that though Stolz's theorem is not actually included in Euclid's Fifth Book, it can easily be proved by Euclidean methods (see Hill's *Euclid*, 2nd ed., p. 29); and we may therefore imagine that it has been proved and added to the Fifth Book just as Simson, for example, added his propositions *A—K*.

The remaining propositions referred to above, viz., XII, 5, 11, 12, 18, can all be proved by Stolz's theorem, without making use of Euclid's assumption, on much the same lines as XII, 2 in the proof just given. It seems worth mentioning that for three of the propositions, XII, 5, 11, 12, we can easily devise a proof by ordinary geometrical manipulation, which shall avoid Euclid's assumption, and yet be independent of Stolz's theorem. For XII, 12, though not for XII, 5 and 11, the alternative proof is decidedly the simpler.