Proofs That the Arithmetic Mean Is Greater than the Geometric Mean

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PROOFS THAT THE ARITHMETIC MEAN IS GREATER THAN THE GEOMETRIC MEAN.

What follows is a condensed statement and classification of various proofs of the above well-known theorem.

For two positive quantities, a and b , the proof may be put as follows :

$$\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2}, \text{ which is } > 0.$$

$$(1) \quad \therefore \frac{a+b}{2} > \sqrt{ab}.$$

Otherwise, let the two quantities be denoted by $x+y$ and $x-y$, then their arithmetic mean x is $>$ their geometric mean $\sqrt{x^2 - y^2}$.

Here we assume for the sake of simplicity, as we shall do throughout, that the given quantities are not all equal ; otherwise the symbol $\leq$ would have to be used instead of $>$. We shall use A and G to denote the means.

To extend the theorem to n positive quantities $a, b, c, \dots$, several methods are available, which may be classified as follows :

A. By substituting for the greatest and least another equal pair, which (α) leaves the A.M. unaltered and increases the G.M., or (β) leaves the G.M. unaltered and diminishes the A.M., but in both cases makes the A.M. and the G.M. continually approach equality.

B. By substituting for a and b , the greatest and least of the set, a pair of quantities having (α) the same G.M., or (β) the same A.M. as a and b , one of the pair being the mean of all ; and so by a *finite* number of steps completing the proof.

C. By proving the theorem for $n=2^m$, and extending it to the general case by introducing $2^m - n$ equal quantities (α) each equal to G , or (β) each equal to A .

D. By mathematical induction, making the case for n quantities depend on that for $n-1$. Here also are two varieties, (α) and (β), in which A and G exchange rôles, and a third, (γ), depending on the inequality

$$\left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^{n-1}.$$

E. By showing that A and G are the extremes of a set of means of cognate type which are in descending order of magnitude.

A α . Substitute for a and b , the greatest and least of the n given quantities, a pair of quantities each $=(a+b)/2$. We have thus a new set of n quantities whose sum is the same as that of the given set, but whose continued product is greater, since by (1),

$$\left(\frac{a+b}{2}\right)^2 > ab \quad \text{and} \quad \therefore \left(\frac{a+b}{2}\right)^2 \cdot cde \dots > abcde \dots$$

By taking a sufficient number of such steps we ultimately get a set having the same sum as $a, b, c, \dots$ but a greater product, the members of which are as nearly equal to one another, and therefore to their G.M., as we please. At the same time this G.M. approaches as nearly as we please to the A.M. which is the same for all the sets. Since the G.M. of the last set is $>$ that of the given set by a finite amount, it follows that the G.M. of the latter is less than its A.M.

A β . Substitute for a and b their G.M. $\sqrt{ab}$ taken twice, and we get a new set of n quantities having the same continued product as that of $a, b, c, \dots$, but a smaller sum, since

$$2\sqrt{ab} - a - b = -(\sqrt{a} - \sqrt{b})^2, \text{ which is } < 0.$$

Taking a sufficient number of such steps, we ultimately get a set of numbers as nearly equal to one another, to their A.M., and to their G.M., as we please, and having the same product as $a, b, c, \dots$, but a smaller sum. Hence we deduce that the A.M. of the original set is greater than their G.M.

In order to make the proofs A α and A β complete, it would be necessary to show that by taking a sufficient number of steps in each case, we *can* make the numbers of the resulting set approach mutual equality as nearly as we please.

B α . Assuming for the present that both the arithmetic and the geometric means of n quantities lie between b and a , the least and the greatest of them, G , the geometric mean, will lie between a and b . Let then $k = \frac{ab}{G}$, so that

$$a+b-(G+k) = a+b-G - \frac{ab}{G} = \frac{1}{G}(a-G)(G-b), \text{ which is } > 0;$$

$$\therefore a+b > G+k.$$

Thus we have a new set of quantities having the same continued product as the given set but a smaller sum, and including at least one quantity equal to G . By a series of similar steps, not more than $n-1$ in number, we arrive at n quantities G , having a smaller sum than the given set, whose A.M. is therefore $> G$.

B β . Next let A be the arithmetic mean of the given numbers, and let $k = a+b-A$, where, as before, a and b stand for the greatest and least of the set considered. Then A lies between a and b . Again,

$$Ak - ab = A(a+b-A) - ab = (a-A)(A-b), \text{ which is } > 0.$$

Thus taking A and k for a and b , we have a new set of quantities having the same sum, but a greater continued product, and including at least one quantity equal to A . By a number of similar steps, $n-1$ at most, we arrive at a set of n quantities, each equal to A , which have a greater continued product than $a, b, c, \dots$, and whose G.M. is therefore $> G$. Hence $A > G$.

To prove the assumptions that A and G lie between a and b , we have only to note that

$$(2) \quad na - nA = (a-a) + (a-b) + (a-c) + \dots, \text{ which is } > 0, \text{ so that } a > A.$$

$$(3) \quad nA - nb = (a-b) + (b-b) + (c-b) + \dots, \text{ which is } > 0, \text{ so that } A > b.$$

$$(4) \quad a^n \div G^n = a^n \div (abc \dots) = \frac{a}{a} \cdot \frac{a}{b} \cdot \frac{a}{c} \dots, \text{ which is } > 1, \text{ so that } a > G.$$

$$(5) \quad G^n \div b^n = (abc \dots) \div b^n = \frac{a}{b} \cdot \frac{b}{b} \cdot \frac{c}{b} \dots, \text{ which is } > 1, \text{ so that } G > b.$$

C. To prove the theorem when $n = 2^m$, observe that

$$\begin{aligned} & a+b+c+\dots - n(a \cdot b \cdot c \dots)^{\frac{1}{n}} \\ &= (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + (c^{\frac{1}{2}} - d^{\frac{1}{2}})^2 + \dots \\ &+ 2\{(a^{\frac{1}{4}}b^{\frac{1}{4}} - c^{\frac{1}{4}}d^{\frac{1}{4}})^2 + (e^{\frac{1}{4}}f^{\frac{1}{4}} - g^{\frac{1}{4}}h^{\frac{1}{4}})^2 + \dots\} \\ &+ 2^2\{(a^{\frac{1}{8}}b^{\frac{1}{8}}c^{\frac{1}{8}}d^{\frac{1}{8}} - e^{\frac{1}{8}}f^{\frac{1}{8}}g^{\frac{1}{8}}h^{\frac{1}{8}})^2 + \dots\} \\ &+ \dots + \frac{n}{2}\{(ab \dots)^{\frac{1}{n}} - (uv \dots)^{\frac{1}{n}}\}, \text{ which is } > 0. \end{aligned}$$

Hence

$$(a+b+c \dots)/n > (abc \dots)^{\frac{1}{n}}.$$

Q.E.D.

Ca. Next suppose n not a power of 2, and let $n+p=2^m$.

$$\begin{aligned} a+b+c\dots+pG &> (n+p)(abc\dots G^p)^{\frac{1}{n+p}} \\ &> (n+p)(G^n G^p)^{\frac{1}{n+p}} \\ &> (n+p)G. \end{aligned}$$

$$\therefore a+b+c\dots > nG,$$

whence $A > G$.

Cβ. Again, $a+b+c+\dots+pA > (n+p)(abc\dots A^p)^{\frac{1}{n+p}}$.

$$\therefore nA+pA > (n+p)(G^n A^p)^{\frac{1}{n+p}},$$

$$\therefore A^{n+p} > G^n A^p.$$

$$\therefore A > G.$$

Da. Arrange $a_1 a_2 \dots$ in ascending order, so that $a_n > G_{n-1}$ (see (4)), and let

$$G_{n-1} \equiv (a_1 a_2 \dots a_{n-1})^{\frac{1}{n-1}} \quad \text{and} \quad G_n \equiv (a_1 a_2 \dots a_n)^{\frac{1}{n}}.$$

Assuming

$$a_1 + a_2 + \dots + a_{n-1} > (n-1)G_{n-1}.$$

$$\therefore a_1 + a_2 + \dots + a_{n-1} + a_n > (n-1)G_{n-1} + a_n.$$

$$\therefore \frac{a_1 + a_2 + \dots + a_n}{n} > G_{n-1} + \frac{a_n - G_{n-1}}{n}.$$

$$\therefore A_n^n > \left(G_{n-1} + \frac{a_n - G_{n-1}}{n} \right)^n$$

$$> G_{n-1}^n + n \cdot G_{n-1}^{n-1} \times \frac{a_n - G_{n-1}}{n}$$

by the Binomial Theorem, since $a_n - G_{n-1}$ is positive.

$$\therefore A_n^n > G_{n-1}^{n-1} a_n$$

$$> a_1 a_2 \dots a_{n-1} a_n.$$

$$\therefore A_n > G_n.$$

Hence, starting from (1), by mathematical induction, $A > G$.

Dβ. Assume $A_{n-1} > G_{n-1}$. By (2) $a_n > A_{n-1}$.

$$\therefore A_n = \frac{n-1}{n} A_{n-1} + \frac{a_n}{n} = A_{n-1} + \frac{a_n - A_{n-1}}{n}.$$

$$\therefore A_n^n > A_{n-1}^{n-1} + n A_{n-1}^{n-1} \times \frac{a_n - A_{n-1}}{n}$$

$$> a_n A_{n-1}^{n-1} > a_n G_{n-1}^{n-1} > G_n^n.$$

$$\therefore A_n > G_n.$$

As before, by mathematical induction, $A > G$.

Dγ. Thacker's proof depends on this lemma: if x is positive, and n a positive integer,

$$\left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^{n-1};$$

which is proved by expanding both sides by the Binomial Theorem, and observing that the r th term of the former expansion is $>$ that of the latter, since

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \dots \frac{n-r+2}{n} > \frac{n-1}{n-1} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-1} \dots \frac{n-r+1}{n-1}.$$

The following references, though not exhaustive, may be found useful :

A α is given by Todhunter and by Chrystal ; A α and A β are both given by C. Smith.

B α is due to G. H. Bryan (see the *Tutorial Algebra*), and B β to G. E. Crawford (*Edin. Math. Soc. Proceedings*, XVIII., p. 2).

C β is given in Todhunter's *Algebra*, and is due to Cauchy (*Analyse Algèbrique*, p. 457) ; and Ca is obviously suggested by it.

D β is given in H. Weber's *Algebra*, and Da is suggested by it. D γ is due to Thacker (*Camb. and Dub. Math. Journ.*, VI., p. 81). Another proof, of type D, but depending on a different fundamental inequality, is given by Chrystal (*Algebra*, XXIV., § 8).

E is given by Schlömilch in his *Zeitschrift* (III., p. 301), with a proof depending on Theory of Equations, and attributed to Fort. A proof of (7) by mathematical induction, not involving Theory of Equations, will be found in the *Edin. Math. Soc. Proceedings*, XIX., p. 36. R. F. MUIRHEAD.

MATHEMATICAL NOTES.

124. [D. 2. b.] Addition Series.

An Addition series is one in which $u_n = u_{n-1} + u_{n-2}$, the principal addition series being

$$1, 1, 2, 3, 5, 8, 13, 21, \text{etc.}, \dots \text{A.}$$

which was originally invented to obtain numbers by which to divide, practically, a line in extreme and mean ratio. In fact, in A.,

$$u_n = u_{n-1} + u_{n-2}, \text{ and } u_n u_{n-2} - u_{n-1}^2 = \pm 1,$$

which is as near an approximation to E. and M. ratio as is possible in integers.

Here,
$$u_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\};$$

and, by induction, the sum of n terms $= u_{n+2} - 1$.

The following are properties of A. which can be proved by induction, or from the value of u_n .

If $M = u_n u_{n-3} - u_{n-1} u_{n-2}$, then $M = \pm 1$ (M in G.P. and A.P. respectively being 0 and $-2b^2$).

$$u_0 = 0; \text{ and } u_{-n} = \pm u_n.$$

The mn^{th} term is divisible by the n^{th} .

$$u_n = u_m u_{n-m+1} + u_{m-1} u_{n-m}.$$

$u_n^2 - u_{n-2}^2$ and $u_n^2 + u_{n+1}^2$ are each equal to a term of A.

One of the two $u_n^2 \pm u_m^2$ is always equal to the product of two terms of A.

The general Addition series can be expressed in terms of A., for it is :

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, \text{etc.} \dots \text{G.A.}$$

The sum of n terms of G.A. is $au_n + b(u_{n+1} - 1)$ where u_n is a term of A.

In G.A., $M = \pm(a^2 + ab - b^2)$; and $U_n U_{n-2} - U_{n-1}^2 = M$; where U_n is a term of G.A.

The sum of n terms is $U_{n+2} - U_2$; and the sum of n terms after the m^{th} is $U_{n+m+2} - U_{m+2}$.

Also
$$U_{2n+1} = (1 + 1) \underbrace{U_1 U_2 U_3 \dots U_{n+1}}_n.$$