THE PRESSURE OF RADIATION ON A CYLINDRICAL OBSTACLE

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[Received November 1st, 1911.—Read December 14th, 1911.—Revised version Received January 10th, 1912.]

PART I.

IN an earlier paper * the pressure experienced by a reflecting spherule on which plane waves impinge was examined numerically, with special reference to the maximum value of this pressure corresponding to an obstacle whose radius is of the same order of magnitude as the wave length of the incident radiation. The determination of a corresponding maximum, when it exists, for obstacles which are not spherical is a matter of importance; but the mathematical difficulties are so formidable that no solution has yet been given. As a preliminary to a more complete knowledge of the pressure which can be experienced by non-spherical bodies, the present paper takes up the only other case in which an analysis on similar lines to that used for a sphere can be employed profitably. This is the two-dimensional problem of the circular cylinder whose length is sufficiently great in comparison with a radius of the cross-section and with the wave length of light.

The general basis of the investigation is the electromagnetic theory. It appears that all possible cases can be built up from two when the incident waves are travelling perpendicularly to the axis of the cylinder. The two cases are those in which (1) the electric force and (2) the magnetic force are respectively parallel to that axis. In Part I of the paper we propose to consider the first case in detail. The second case, which is somewhat more complicated, is investigated in Part II.

Let a set of cylindrical coordinates (r, θ, z) be chosen, where z is parallel to the axis, so that r and θ are polar coordinates in the plane of a normal section of the cylinder. The problem being one of symmetry, none of the quantities concerned will be functions of z . If the wave length be $2\pi/k$, and if the incident waves are travelling along a direction x , the initial direction from which θ is measured, the magnetic

* *Monthly Notices of the R.A.S.*, May, 1910.

force in the incident waves may be taken as e^{ikx} , ignoring a time factor, acting along the direction y perpendicular to x and z . Moreover,

$$x = r \cos \theta, \quad y = r \sin \theta.$$

The waves are incident on the cylinder on the side for which x is positive.

If the cylinder is a perfect reflector, the magnetic force at its surface must be tangential and the electric force normal. A disturbance will therefore be thrown off, in which the radial magnetic force at the surface $r = a$ (a being the radius of the cylinder) will be equal and opposite to that of the incident waves. The latter is, of course,

$$\frac{y}{r} e^{ikx} \quad \text{or} \quad \sin \theta \cdot e^{ikr \cos \theta},$$

which may be expressed by a well known theorem ascribed by Nielsen* to Hansen, but probably used earlier by Jacobi, in the form

$$\sin \theta \cdot J_0(kr) + 2 \sin \theta \sum_1^\infty i^n J_n(kr) \cos n\theta,$$

where $J_n(kr)$ is the usual Bessel function of argument kr and order n , finite at the origin, or on the axis of the cylinder. This series may be written in the form

$$\sin \theta (J_0 + J_2) + i \sin 2\theta (J_1 + J_3) + i^2 \sin 3\theta (J_2 + J_4) + \dots,$$

and, by the use of a recurrence formula for the Bessel functions, this becomes

$$a_1 = \frac{2}{kr} \sum_1^\infty i^{n-1} n \sin n\theta \cdot J_n(kr). \quad (1)$$

This might also have been obtained by differentiating the original series with respect to θ . The electric force along the axis is

$$R = V e^{ikx} = V J_0(kr) + 2V \sum_1^\infty i^n J_n(kr) \cos n\theta. \quad (2)$$

Let us now consider the general equations of the external electromagnetic field. If (P, Q, R) and (a, b, c) be the components of the electric and magnetic vectors along the directions at any point in which r, θ , and z respectively increase, then, by the cylindrical form of the circuital electromagnetic relations,

$$\left. \begin{aligned} -r\dot{a} &= \frac{\partial R}{\partial \theta} - r \frac{\partial Q}{\partial z}, & \frac{r\dot{P}}{V^2} &= \frac{\partial c}{\partial \theta} - \frac{\partial b}{\partial z}, \\ -\dot{b} &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial r}, & \frac{r\dot{Q}}{V^2} &= \frac{\partial a}{\partial z} - \frac{\partial c}{\partial r}, \\ -r\dot{c} &= \frac{\partial}{\partial r} (Qr) - \frac{\partial P}{\partial \theta}, & \frac{r\dot{R}}{V^2} &= \frac{\partial}{\partial r} (br) - \frac{\partial a}{\partial \theta} \end{aligned} \right\}, \quad (3)$$

* *Handbuch der Cylinderfunctionen.*

with the solenoidal conditions

$$\frac{\partial}{\partial r}(ar) + \frac{\partial b}{\partial \theta} + r \frac{\partial c}{\partial z} = \frac{\partial}{\partial r}(Pr) + \frac{\partial Q}{\partial \theta} + r \frac{\partial R}{\partial z} = 0, \quad (4)$$

where V is the velocity of disturbances in the outer medium. For the present problem, $\partial/\partial z \equiv 0$, and therefore, by the solenoidal conditions, functions ϕ and ψ exist such that

$$\left. \begin{aligned} P &= -\frac{\partial \phi}{r \partial \theta}, & Q &= \frac{\partial \phi}{\partial r} \\ a &= -\frac{\partial \psi}{r \partial \theta}, & b &= \frac{\partial \psi}{\partial r} \end{aligned} \right\}. \quad (5)$$

If the disturbance has a wave-length $2\pi/k$, it will involve a time factor e^{ikVt} , and we accordingly write $\partial/\partial t = ikV$, and derive, on reduction,

$$R = ikV\psi, \quad c = ik\phi/V, \quad (6)$$

where ϕ and ψ appear as solutions of the equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + k^2 \psi = 0. \quad (7)$$

For the most general vibration of a given period there is no essential relation between ϕ and ψ . Thus, the most general vibration of given period in or about a system of coaxial cylinders of circular section, independent of z , consists of a superposition of two others, in one of which the electric force, and in the other the magnetic force, is parallel to the axis at all points. This is a special case of a more general result true for orthogonal sets of surfaces of revolution.

The general solution of (7) finite at the origin is

$$\psi = \sum_0^\infty A_n J_n(kr) \cos n\theta. \quad (8)$$

Let us consider the case in which the incident waves have no magnetic component parallel to the axis. Then a complete solution may be obtained in terms of functions of type ψ only, the electric force in both incident and scattered waves having a component R only parallel to the axis, and the magnetic force having components (a, b) in the plane of the circular section at any point.

The function ψ for the incident waves is, by (2) and (6), of the form

$$\psi_1 = -\frac{i}{k} \mathcal{J}_0(kr) + \frac{2}{k} \sum_1^\infty i^{n-1} J_n(kr) \cos n\theta, \quad (9)$$

leading by (4) at once to the proper formula (1).

If $K_n(kr)$ be the corresponding function of distance for the scattered waves, we may write, for these waves,

$$\psi_2 = -\frac{ia_0}{k} K_0(kr) + \frac{2}{k} \sum_1^\infty i^{n-1} a_n K_n(kr) \cos n\theta, \quad (10)$$

and a_n may be determined from the general surface condition of the continuity of tangential components of electric and magnetic force. For a perfect reflector it is sufficient to make a or $-\partial\psi/r\partial\theta$ and R or $ikV\psi$ vanish at the surface. Both conditions are secured if ψ itself vanishes, and therefore if

$$\left. \begin{aligned} a_0 &= -J_0(ka)/K_0(ka) \\ a_n &= -J_n(ka)/K_n(ka) \end{aligned} \right\}, \quad (11)$$

a now denoting the radius, and the form of $\psi = \psi_1 + \psi_2$ becomes

$$\begin{aligned} \psi &= -\frac{i}{k} \{J_0(kr) - J_0(ka) K_0(kr)/K_0(ka)\} \\ &\quad + \frac{2}{k} \sum_1^\infty i^{n-1} \{J_n(kr) - J_n(ka) K_n(kr)/K_n(ka)\} \cos n\theta. \end{aligned} \quad (12)$$

Let $Y_n(x)$ be the second solution of Bessel's equation of order n defined by *

$$Y_n(x) = \left\{ \frac{\partial J_n}{\partial n} - (-)^n \frac{\partial J_{-n}}{\partial n} \right\}_{n=\text{integer}}. \quad (13)$$

Then
$$\frac{K_n(kr)}{K_n(ka)} = \frac{Y_n(kr) + i\pi J_n(kr)}{Y_n(ka) + i\pi J_n(ka)}. \quad (14)$$

The electric force vanishes at the surface. The magnetic component is only b or $\partial\psi/\partial r$, and becomes on the surface

$$\begin{aligned} b &= -i \left\{ \frac{J'_0(ka) Y_0(ka) - Y'_0(ka) J_0(ka)}{Y_0(ka) + i\pi J_0(ka)} \right\} \\ &\quad + 2 \sum_1^\infty i^{n-1} \left\{ \frac{J'_n(ka) Y_n(ka) - Y'_n(ka) J_n(ka)}{Y_n(ka) + i\pi J_n(ka)} \right\} \cos n\theta. \end{aligned} \quad (15)$$

The two kinds of Bessel function are related by the identity

$$Y_n(x) J'_n(x) - J_n(x) Y'_n(x) = -2/x, \quad (16)$$

and the surface value of b becomes, inserting the time factor,

$$b = \frac{2i}{M_0 ka} e^{ikVt - i\eta_0} - \frac{4}{ka} \sum_1^\infty i^{n-1} \frac{\cos n\theta}{M_n} e^{ikVt - i\eta_n}, \quad (17)$$

* The referee points out that this definition may cause confusion. It was adopted mainly to keep the notation in line with that of earlier papers. The present function is more usually given a factor in English work, and is identical with $-2K_n$, where the latter symbol is used in the sense of Heine.

where

$$\left. \begin{aligned} M_n^2 &= Y_n^2(ka) + \pi^2 J_n^2(ka) \\ \tan \eta_n &= \pi J_n(ka) / Y_n(ka) \end{aligned} \right\}. \quad (18)$$

Determination of the Pressure.

Let E be the energy per unit volume of the incident radiation, half electric and half magnetic. Then the pressure on a unit length of the cylinder parallel to its axis has a resultant action along the direction of propagation of the incident waves, given by

$$p = E \iint b^2 \cos \theta dS, \quad (19)$$

the surface integral being taken over the unit length. This follows at once from the Maxwellian stress system, the electric force being zero. Expressed as a line integral round the circular section, the formula becomes

$$p = 2aE \int_0^\pi b^2 \cos \theta d\theta. \quad (20)$$

In this calculation the average value of b^2 is needed for the determination of the motive pressure; that is, the average value over a sufficiently large time, which neglects the periodic terms. It is a familiar relation that the average value of the square of a complex b is given by $\frac{1}{2}|b|^2$ or $\frac{1}{2}b\bar{b}$, where $\bar{b}$ is the conjugate complex. In the present case this average becomes B , where

$$\begin{aligned} 2B &= \frac{4}{k^2 a^2} \frac{1}{M_0^2} + \frac{16}{k^2 a^2} \sum_1^\infty \frac{\cos^2 n\theta}{M_n^2} + \frac{16}{k^2 a^2} \frac{1}{M_0} \sum_1^\infty \frac{\cos n\theta}{M_n} \cos\left(\eta_0 - \eta_n + \frac{n\pi}{2}\right) \\ &+ \frac{32}{k^2 a^2} \sum_1^\infty \sum_1^\infty \frac{\cos r\theta \cos s\theta}{M_r M_s} \cos\left\{(\eta_r - \eta_s + (s-r)\frac{\pi}{2})\right\}, \end{aligned} \quad (21)$$

where the double summation excludes equal values of r and s .

Now for all integral values of n, r, s ,

$$\int_0^\pi \cos^2 n\theta \cos \theta d\theta = 0, \quad (22)$$

$$\int_0^\pi \cos r\theta \cos s\theta \cos \theta d\theta = \frac{1}{2} \int_0^\pi \{\cos(r+s)\theta + \cos(r-s)\theta\} \cos \theta d\theta.$$

The first integral vanishes because $r+s$ is necessarily greater than unity. The second also vanishes unless $r-s=1$, in which case

$$\int_0^\pi \cos r\theta \cos s\theta \cos \theta d\theta = \frac{\pi}{4}. \quad (23)$$

Moreover, when $n = 1$,

$$\int_0^\pi \cos n\theta \cos \theta d\theta = \frac{\pi}{2}, \quad (24)$$

being zero for other integral values of n .

$$\begin{aligned} \text{Thus} \quad p &= 2aE \int_0^\pi B \cos \theta d\theta \\ &= -\frac{8\pi aE}{k^2 a^2} \sum_0^\infty (M_n M_{n+1})^{-1} \sin(\eta_n - \eta_{n+1}), \end{aligned} \quad (25)$$

all the terms falling into line with one another.

But $-\sin(\eta_n - \eta_{n+1}) = \pi(Y_n J_{n+1} - J_n Y_{n+1})/M_n M_{n+1}$,
by (18), and by an identical relation

$$Y_n J_{n+1} - J_n Y_{n+1} = 2/ka, \quad (26)$$

we deduce
$$\frac{p}{2aE} = \frac{8\pi^2}{k^3 a^3} \sum_0^\infty M_n^{-2} M_{n+1}^{-2};$$

or if z denotes ka , the argument of the Bessel functions,

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \sum_0^\infty (Y_n^2 + \pi^2 J_n^2)^{-1} (Y_{n+1}^2 + \pi^2 J_{n+1}^2)^{-1}, \quad (27)$$

gives the pressure per unit length of the cylinder. This result is of considerable elegance.

General Remarks.

In the first place, before making a detailed investigation of the formula just obtained, we may shew that it approaches the correct limiting value when the diameter of the obstacle increases indefinitely.

On the usual theory of pressure, a plate of a reflecting body, on which radiation falls at an angle θ to the normal, experiences a normal pressure $2E \cos^2 \theta$ per unit area, where E is energy in a unit volume of the

incident beam. If the cylinder has a large diameter, and attention is

confined to a unit length along the axis, the pressure on the plate of unit length and breadth δs , an element of arc of the circular section, becomes $2E\delta s \cos^2 \theta$. This acts towards the axis of the cylinder and has a component $2E\delta s \cos^3 \theta$ parallel to the incident rays. The resultant pressure in this direction is obtained by integration over the unshadowed part, or just one half, of the surface and becomes

$$p = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2E \cos^3 \theta \cdot a \, d\theta = \frac{8}{3}aE \quad (28)$$

per unit length of cylinder.

Consider now the accurate formula

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \sum_0^\infty \frac{1}{(Y_n^2 + \pi^2 J_n^2)(Y_{n+1}^2 + \pi^2 J_{n+1}^2)}.$$

Writing

$$\left. \begin{aligned} J_n(z) &= \left(\frac{2R_n}{\pi z}\right)^{\frac{1}{2}} \sin \phi_n \\ Y_n(z) &= -\left(\frac{2\pi R_n}{z}\right)^{\frac{1}{2}} \cos \phi_n \end{aligned} \right\}, \quad (29)$$

where the functions R_n and ϕ_n have been examined in detail in earlier papers,* then

$$\frac{p}{2aE} = \frac{2}{z} \sum_0^\infty \frac{1}{R_n R_{n+1}}. \quad (30)$$

Now, when z is large, and n is not so large that $z-n$ is of so low an order as $z^{\frac{1}{2}}$,

$$R_n = z/(z^2 - n^2)^{\frac{1}{2}} \quad (31)$$

approximately, and the number of terms of the series for which this occurs is effectively z .

When $n-z$ or $z-n$ is of order $z^{\frac{1}{2}}$ or less, the number of terms thus concerned being also of this order, R_n and R_{n+1} are of order $z^{\frac{1}{2}}$. The sum of such terms is therefore of order $z^{-\frac{1}{2}}$ at most in the series $\Sigma(R_n R_{n+1})^{-1}$ and is negligible in comparison with the earlier terms, which have a sum of order z .

When n is greater than z , and $n-z$ is of higher order than $z^{\frac{1}{2}}$, the functions $R_n R_{n+1}$ become large in comparison with any power of z , and the sum of the terms involving them is not significant †.

* *Vide Phil. Mag.*, February, 1910.

† A similar problem is discussed in *Phil. Mag.*, April, 1910.

These points are all illustrated very clearly in the numerical tables which follow. Finally, then, when z is large, we may write

$$\frac{p}{2aE} = \frac{2}{z^3} \sum_{n=0}^{\infty} (z^2 - n^2)^{\frac{1}{2}} [z^2 - (n+1)^2]^{\frac{1}{2}}. \quad (32)$$

In determining the leading term in the sum of this series, expressed in inverse powers of z , we may replace it by an integral in the usual way, writing $n/z = x$ and $z^{-1} = dx$, so that

$$\frac{p}{2aE} = 2 \int_0^1 (1-x^2) dx = \frac{4}{3},$$

or

$$p = \frac{8}{3}aE.$$

This is in accordance with (28), and the formula (27) therefore tends to the proper limiting value for a large obstacle.

The function R_n used above is not identical with the R_n of the earlier paper on radiation pressure. The latter, in the present notation, would be $R_{n+\frac{1}{2}}$ or R_m , m denoting $n+\frac{1}{2}$, as throughout that paper. The functions R_n and ϕ_n possess the properties

$$\left. \begin{aligned} R_{n+1} &= R_{n-1} + \frac{4n^2}{z^2} R_n - \frac{2n}{z} \sqrt{(R_n R_{n-1} - 1)} \\ \sin(\phi_n - \phi_{n+1}) &= (R_n R_{n+1})^{-\frac{1}{2}} \end{aligned} \right\}, \quad (33)$$

the proofs of these relations in the earlier paper not being confined to the case in which n is half an odd integer.

The Small Obstacle.

It is also convenient, before proceeding to tabulation, to obtain a formula for the pressure on a very thin obstacle of cylindrical shape, due to its scattering action on the incident radiation. For this purpose attention may be confined to the first term of (27), and we may also write

$$\left. \begin{aligned} Y_0(z) &= 2 \left\{ \log \frac{z}{2} + \gamma \right\} \\ J_0(z) &= 1, \quad Y_1(z) = -2/z, \quad J_1(z) = 0 \end{aligned} \right\}, \quad (34)$$

where γ is Euler's constant. Thus

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \frac{z^2}{4} \frac{1}{\pi^2 + \frac{1}{4}(\gamma + \log \frac{1}{2}z)^2};$$

or, finally, since $z = 2\pi a/\lambda$,

$$p = 2\lambda\pi E / \{\pi^2 + 4(\gamma + \log \pi a/\lambda)^2\}, \quad (35)$$

and we note that the pressure tends to zero with the radius of the obstacle. The ratio $p/\frac{2}{3}aE$, on the contrary, tends to infinity, exhibiting a marked difference from the behaviour of the corresponding ratio in the case of the sphere.

Numerical Results.

A large amount of numerical work is necessary in the tabulation of the formula (27), and a fairly complete account of it is given, as many of the intermediate figures are useful for other purposes.

In the first two tables the values of the Bessel functions are given. The functions of the logarithmic type $Y_n(z)$ have only been tabulated hitherto for the cases $n = 0$ and $n = 1$. The values for these cases have been taken from B. A. Smith's tables,* and those for higher values of n have been deduced by the recurrence formula

$$Y_{n+1}(z) = \frac{2n}{z} Y_n(z) - Y_{n-1}(z). \quad (36)$$

The notation is different from that of Smith's tables. His function $(\log 2 - \gamma)J_n(z) - Y_n(z)$ is identical with the function $-\frac{1}{2}Y_n(z)$ of this paper. As Smith's tables are restricted to a few significant figures, the last figure in the present table for $Y_n(z)$ will sometimes be incorrect to the extent of a unit.

The functions of type $J_n(z)$ involved less calculation. $J_0(z)$ and $J_1(z)$ have been taken from Meissel's tables.† For integer values of z the functions of higher orders are also taken from these tables. But, for non-integral values such as $z = 1.5$, it was necessary to calculate them from J_0 and J_1 by the recurrence formula

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z). \quad (37)$$

* *Phil. Mag.*, 1898, Vol. 1, p. 122. They are reprinted by Jahnke and Emde, *Funktionentafeln*, Leipzig (B. G. Teubner).

† Gray and Mathews's *Treatise on Bessel Functions*.

TABLE 1.

z	·1	·4	·5	·6	·8	·9	1·0
J_0	·9975	·9604	·9385	·9120	·8463	·8075	·7652
J_1	·0499	·1960	·2423	·2867	·3688	·4059	·4401
J_2	·0012	·0197	·0306	·0437	·0758	·0946	·1149
J_3	$< 10^{-3}$	·0013	·0026	·0044	·0102	·0144	·0196
$-Y_0$	4·820	1·904	1·397	·9692	·2726	— ·0176	— ·2772
$-Y_1$	20·29	5·595	4·623	3·960	3·073	2·743	2·454
$-Y_2$	401·0	26·07	17·095	12·23	7·410	6·113	5·186
$-Y_3$	$> 10^3$	255·1	132·1	77·57	33·98	24·43	18·29
$-Y_4$	$> 10^4$	$> 10^3$	$> 10^3$	763·5	247·4	156·7	104·5

In this and the following table, the values of the Y_n functions have been checked as far as three figures in the last function of any series by means of the relation

$$Y_n J_{n+1} - J_n Y_{n+1} = 2/z.$$

TABLE 2.

z	1·1	1·2	1·5	2	2·5	3	4	5	6
J_0	·7196	·6711	·5118	·2239	— ·0484	— ·26005	— ·3971	— ·1776	·1506
J_1	·4709	·4983	·5579	·5767	·4971	·3391	— ·0660	— ·3276	— ·2767
J_2	·1366	·15935	·2321	·3528	·4461	·4861	·3641	·0466	— ·2429
J_3	·0257	·0329	·06095	·1289	·2166	·3091	·4302	·3648	·1148
J_4	·0036	·005	·0117	·0340	·0738	·1320	·2811	·3912	·3576
J_5	$< 10^{-3}$	·0006	·0016	·0070	·0195	·0430	·1321	·2611	·3621
J_6	$< 10^{-4}$	$< 10^{-4}$	·0004	·0012	·0042	·0114	·0491	·13105	·2458
J_7	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-3}$	$< 10^{-3}$	10^{-3}	·0025	·0152	·0534	·1296
J_8	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-3}$	$< 10^{-3}$	·0040	·0184	·0565
$-Y_0$	— ·5096	— ·7166	— 1·2016	— 1·6034	— 1·5648	— 1·1840	+ ·0532	+ ·9692	+ ·9054
$-Y_1$	2·193	1·951	1·295	·3364	— ·4586	— 1·020	— 1·2502	— ·4646	·5498
$-Y_2$	4·497	3·969	2·929	1·940	1·198	·5040	— ·6783	— 1·155	— ·7221
$-Y_3$	14·16	11·28	6·515	3·544	2·375	1·692	·572	— ·460	— 1·031
$-Y_4$	72·74	52·42	23·13	8·69	4·50	2·88	1·536	·604	— ·3091
$-Y_5$	515	338	116·8	31·22	12·03	5·99	2·50	1·426	·619
$-Y_6$	$> 10^3$	$> 10^3$	755	147·4	43·62	17·08	4·715	2·247	1·341

The following additional values have also been employed :—

$Y_7(3) = -62.3, \quad Y_7(4) = -11.645, \quad Y_7(5) = -3.968,$
 $Y_7(6) = -2.063,$
 $Y_8(4) = -36.04, \quad Y_8(5) = -8.863, \quad Y_8(6) = -3.473,$
 $Y_9(5) = -24.395, \quad Y_9(6) = -7.198,$
 $Y_{10}(6) = -18.12.$

Let

$\rho_n = Y_n^2(z) + \pi^2 J_n^2(z),$

so that

$\rho_n = \frac{2\pi}{z} R_n. \tag{38}$

Then the values of ρ_n are exhibited in the next two tables :—

TABLE 3.

z	.1	.4	.5	.6	.8	.9	1.0
ρ_0	33.050	12.728	10.643	9.148	7.143	6.436	5.856
ρ_1	411.75	31.679	21.950	16.490	10.786	9.151	7.933
ρ_2	16.10 ¹	679.6	292.2	149.6	54.96	37.46	23.20
ρ_3	> 10 ¹¹	65.10 ³	17.10 ³	6017	1154	596.6	334.5
ρ_4	> 10 ⁷	> 10 ⁶	> 10 ⁶	> 10 ⁵	> 5.10 ⁴	25.10 ³	11.10 ³

TABLE 4.

z	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
1.1	5.3707	6.999	20.409	200.5	5291	26.10 ⁴	> 10 ⁷
1.2	4.9591	6.257	16.003	127.25	2746	13.10 ⁴	> 10 ⁶
1.5	4.0304	4.750	9.108	42.50	535.0	14.10 ³	> 10 ⁵
2	3.0660	3.396	4.991	12.72	75.52	974.4	16.10 ³
2.5	2.4715	2.649	3.399	6.103	20.32	144.8	1903
3	2.0693	2.175	2.570	3.806	8.466	35.87	291.7
4	1.5590	1.606	1.7685	2.154	3.140	6.425	22.255
5	1.2505	1.275	1.355	1.525	1.875	2.706	5.220
6	1.0436	1.058	1.104	1.1935	1.358	1.677	2.394

The following additional values were used :—

$$\rho_7(4) = 135.6, \quad \rho_7(5) = 15.77, \quad \rho_7(6) = 4.421,$$

$$\rho_8(5) = 78.56, \quad \rho_8(6) = 12.09,$$

$$\rho_9(6) = 51.82, \quad \rho_{10}(6) = 328.$$

The fifth and sixth tables exhibit the corresponding values of $1/\rho_n \rho_{n+1}$ for $n = 0, 1, 2, 3, \dots$

TABLE 5.

z	$(\rho_0 \rho_1)^{-1}$	$(\rho_1 \rho_2)^{-1}$	$(\rho_2 \rho_3)^{-1}$	z	$(\rho_0 \rho_1)^{-1}$	$(\rho_1 \rho_2)^{-1}$	$(\rho_2 \rho_3)^{-1}$
·1	736.10 ⁻⁷	2.10 ⁻⁸	< 10 ⁻¹⁰	·9	·01698	·00292	·00005
·4	·00248	·00005	< 10 ⁻⁷	1·0	·02153	·00447	·00011
·5	·004281	·000156	2.10 ⁻⁷	1·1	·02667	·00700	·00024
·6	·006629	·000405	10 ⁻⁶	1·2	·03223	·00999	·00051
·8	·01298	·00169	·00002	1·5	·05223	·02311	·00258

The value of $(\rho_3 \rho_4)^{-1}$ for $z = 1.5$ is only 5.10^{-5} , and it is therefore unnecessary in the above table.

TABLE 6.

z	2	2.5	3	4	5	6
$(\rho_0 \rho_1)^{-1}$	·09605	·15274	·22219	·39938	·6272	·9058
$(\rho_1 \rho_2)^{-1}$	·05900	·11107	·17890	·35207	·5787	·8564
$(\rho_2 \rho_3)^{-1}$	·01575	·04821	·10225	·26255	·4839	·7591
$(\rho_3 \rho_4)^{-1}$	·00104	·00806	·03104	·1479	·3498	·6172
$(\rho_4 \rho_5)^{-1}$	·00001	·0003	·00329	·0496	·1971	·4391
$(\rho_5 \rho_6)^{-1}$	< 10 ⁻⁷	< 10 ⁻⁵	< 10 ⁻³	·0070	·0708	·2490
$(\rho_6 \rho_7)^{-1}$	< 10 ⁻⁹	< 10 ⁻⁷	< 10 ⁻⁴	·00033	·0122	·0945

The extra values used are

(1) for $z = 5$, $(\rho_7 \rho_8)^{-1} = .0008$,

(2) for $z = 6$, $(\rho_7 \rho_8)^{-1} = .0187$, $(\rho_8 \rho_9)^{-1} = .0016$, $(\rho_9 \rho_{10})^{-1} = .00006$.

If S denotes the series

$$S = \sum_{n=0}^{n=\infty} (\rho_n \rho_{n+1})^{-1}, \quad (39)$$

the values of S and of the pressure ratio $p/\frac{5}{3}aE$ are given in the final table, with the corresponding ratios of diameter to wave length. The pressure ratio is $6\pi^2 S/z^3$.

TABLE 7.

z	S	Diameter wave length	$p/\frac{5}{3}aE$	z	S	Diameter wave length	$p/\frac{5}{3}aE$
·1	736.10^{-7}	·032	4·358	1·2	·04273	·382	1·464
·4	253.10^{-6}	·127	2·342	1·5	·07797	·477	1·368
·5	444.10^{-5}	·159	2·102	2	·1713	·637	1·272
·6	$703.5.10^{-5}$	·190	1·929	2·5	·3204	·796	1·214
·8	·01469	·255	1·699	3	·5377	·955	1·179
·9	·01995	·286	1·621	4	1·219	1·27	1·128
1·0	·02611	·318	1·546	5	2·3205	1·59	1·099
1·1	·03392	·350	1·509	6	3·941	1·91	1·081

The character of this table is somewhat unexpected. The ratio of the pressure to its limiting value for a large obstacle approaches the value unity in a regular manner as the size of the obstacle increases. This is especially evident if the values are plotted on a curve. The difference in behaviour of the cylinder and sphere is remarkable; for, in the present case, the pressure may be made greater than a bodily action, such as gravitation, in any arbitrarily assigned ratio by taking the radius of the cylinder small enough. This is, of course, only true in practice if a cylinder composed of so small a number of molecules could continue to approximate to a perfect reflector; but it is evident that the ratio can be made very great. The importance of this result to astrophysics is contained in the fact that the cylinder is, perhaps, a better type of obstacle than the sphere for the representation of the state of things which may obtain, for example, in the tail of a comet or in the solar corona.

In order to illustrate further the manner in which the pressure tends towards its limiting value, the case $z = 8$ has been worked out in detail. The calculation by the previous method being very laborious in this case, another method was adopted. The functions R_0 , R_1 , and R_2 were calculated from the asymptotic series*

$$R_n(z) = 1 + \frac{n^2 - \frac{1}{4}}{z^2} \frac{1}{2} + \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{z^4} \frac{1.3}{2.4} + \dots, \quad (40)$$

* *Phil. Mag.*, February, 1910.

and the corresponding values of $R_3, R_4, \dots$, by the recurrence formula (33). The pressure is then given by

$$p = \frac{4aE}{z} \sum_1^\infty (R_n R_{n+1})^{-1}. \quad (41)$$

The results of the calculation are as follows:—

$$\begin{array}{lll} R_0 = .998062, & R_1 = 1.005831, & R_2 = 1.021631, \\ R_3 = 1.07016, & R_4 = 1.12982, & R_5 = 1.28546, \\ R_6 = 1.4568, & R_7 = 1.7607, & R_8 = 2.4709, \\ R_9 = 4.3223, & R_{10} = 17.3521, & (R_0 R_1)^{-1} = .99613, \\ (R_1 R_2)^{-1} = .97315, & (R_2 R_3)^{-1} = .91466, & (R_3 R_4)^{-1} = .82708, \\ (R_4 R_5)^{-1} = .68854, & (R_5 R_6)^{-1} = .53398, & (R_6 R_7)^{-1} = .38987, \\ (R_7 R_8)^{-1} = .22985, & (R_8 R_9)^{-1} = .09363, & (R_9 R_{10})^{-1} = .01333, \end{array}$$

and the next function is of order 10^{-3} .

The resulting value of the pressure is given by

$$p/\frac{4}{3}aE = 1.061, \quad (42)$$

and corresponds to an obstacle whose diameter is 2.54 wave lengths. The approach towards the limiting value is still regular.

PART II.

When the incident *magnetic* force is parallel to the axis of the cylinder the analysis proceeds on very similar lines, although the final tabulation is somewhat more troublesome. In the present section of the paper this case is worked out in detail. It is then possible to construct the general case of incidence, of waves travelling perpendicular to the axis, without difficulty and to submit it to examination.

The analysis is given briefly, being analogous throughout to that of the first section. It is now the function ϕ which exists instead of ψ , and the radial, transversal, and axial components of the vectors are

$$\left. \begin{array}{l} a = 0, \quad b = 0, \quad c = -ik\phi/V \\ P = -\partial\phi/r\partial\theta, \quad Q = \partial\phi/\partial r, \quad R = 0 \end{array} \right\} \quad (43)$$

The magnetic force in the incident waves is again e^{ikx} , but in a new direction. The time factor is e^{ikVt} , and the corresponding electric force is $-Ve^{ikx}$ perpendicular to the axis and to the incident ray. Its radial

component is $-V \sin \theta e^{ikr \cos \theta}$, which, as in (1), may be expressed in the form

$$P_1 = -\frac{2V}{kr} \sum_1^\infty i^{n-1} n \sin n\theta J_n(kr), \quad (44)$$

with the magnetic force

$$c = J_0(kr) + 2 \sum_1^\infty i^n J_n(kr) \cos n\theta, \quad (45)$$

so that the function ϕ_1 for the incident waves is given by

$$\phi_1 = \frac{iV}{k} J_0(kr) - \frac{2V}{k} \sum_1^\infty i^{n-1} \cos n\theta J_n(kr). \quad (46)$$

With this we associate a formula involving the function $K_n(kr)$ instead of $J_n(kr)$, so that the functions together satisfy the surface condition. The magnetic force is necessarily tangential by the vanishing of ψ . The electric force will be normal if $\partial\phi/\partial r = 0$. This is sufficient to secure the satisfaction of the surface conditions, and, determining the coefficients of the portion of ϕ representing the effect of the obstacle, the final form of ϕ for the external space is

$$\begin{aligned} \phi = & \frac{iV}{k} \{J_0(kr) - J'_0(ka) K_0(kr)/K'_0(ka)\} \\ & - \frac{2V}{k} \sum_1^\infty i^{n-1} \cos n\theta \{J_n(kr) - J'_n(ka) K_n(kr)/K'_n(ka)\}, \end{aligned} \quad (47)$$

or, in the notation of Part I,

$$\begin{aligned} \phi = & \frac{iV}{k} \{J_0(kr) Y'_0(ka) - Y_0(kr) J'_0(ka)\} / \{Y_0(ka) + i\pi J_0(ka)\} \\ & - \frac{2V}{k} \sum_1^\infty i^{n-1} \cos n\theta \{J_n(kr) Y'_n(ka) - Y_n(kr) J'_n(ka)\} \\ & / \{Y_n(ka) + i\pi J_n(ka)\}. \end{aligned} \quad (48)$$

$$\text{If} \quad Y'_n(ka) + i\pi J'_n(ka) = M_n e^{i\eta_n}, \quad (49)$$

the radial electric force at the surface is

$$\begin{aligned} P &= -(\partial\phi/r\partial\theta)_{r=a} \\ \text{or} \quad \rho &= -\frac{4V}{k^2 a^2} \sum_1^\infty \frac{n i^{n-1} \sin \theta}{M_n} e^{ikVt - i\eta_n}, \end{aligned} \quad (50)$$

and the magnetic force is $c = -ik\phi/V$, or

$$c = \frac{2}{kaM_0} e^{ikVt - i\eta_0} + \frac{4}{ka} \sum_1^\infty \frac{i^n \cos n\theta}{M_n} e^{ikVt - i\eta_n}, \quad (51)$$

where the usual identical relation between the two types of Bessel function has been employed and the time factor has been inserted. Now the pressure p acting on a unit length of the cylinder, derived as before

from the resultant of the Maxwellian tractions on its outer surface, is given by

$$p = E \iint (c^2 - P^2/V^2) \cos \theta \, dS, \quad (52)$$

the integral being taken over the curved surface, E being the density of incident energy, and average values of the squares being taken. This is equivalent to a line integral round the circular section, and we obtain

$$\frac{p}{2aE} = \int_0^\pi (c^2 - P^2/V^2) \cos \theta \, d\theta. \quad (53)$$

The average values of the squares are

$$\begin{aligned} \overline{c^2} = & \frac{2}{k^2 a^2 M_0^2} + \frac{8}{k^2 a^2} \sum_1^\infty \frac{\cos^2 n\theta}{M_n^2} + \frac{8}{k^2 a^2 M_0} \sum_1^\infty \frac{\cos n\theta}{M_n} \cos \left(\eta_0 - \eta_n + n \frac{\pi}{2} \right) \\ & + \frac{16}{k^2 a^2} \sum_1^\infty \sum_1^\infty \frac{\cos r\theta \cos s\theta}{M_r M_s} \cos \left\{ \eta_r - \eta_s + (s-r) \frac{\pi}{2} \right\}, \end{aligned} \quad (54)$$

$$\begin{aligned} \overline{P^2/V^2} = & \frac{8}{k^4 a^4} \sum_1^\infty \frac{n^2 \sin^2 n\theta}{M_n^2} \\ & + \frac{16}{k^4 a^4} \sum_1^\infty \sum_1^\infty \frac{r \cdot s \sin r\theta \sin s\theta}{M_r M_s} \cos \left\{ \eta_r - \eta_s + (s-r) \frac{\pi}{2} \right\}, \end{aligned} \quad (55)$$

with equal values of r and s excluded from the double summations. With the relations

$$\begin{aligned} \int_0^\pi (\cos^2 n\theta, \sin^2 n\theta) \cos \theta \, d\theta &= 0 \quad (n = 0, 1, 2, \dots), \\ \int_0^\pi \cos n\theta \cos \theta \, d\theta &= 0 \quad (n \neq 1) \\ &= \frac{\pi}{2} \quad (n = 1), \\ \int_0^\pi \sin r\theta \sin s\theta \cos \theta \, d\theta &= 0 \quad (s \sim r \neq 1) \\ &= \frac{\pi}{4} \quad (s \sim r = 1), \end{aligned} \quad (56)$$

we find
$$\int_0^\pi (\overline{P^2/V^2}) \cos \theta \, d\theta = \frac{4\pi}{k^4 a^4} \sum_1^\infty \frac{n(n+1)}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n), \quad (57)$$

$$\int_0^\pi \overline{c^2} \cos \theta \, d\theta = \frac{4\pi}{k^2 a^2} \sum_0^\infty \frac{1}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n), \quad (58)$$

the summations starting at different points; but the first can also be

regarded as starting at $n = 0$. Thus, by (53), with $ka = z$ as usual,

$$\frac{p}{2aE} = \frac{4\pi}{z^2} \sum_0^\infty \left(1 - \frac{n \cdot n + 1}{z^2}\right) \frac{1}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n). \quad (59)$$

But, by (49), $\sin(\eta_{n+1} - \eta_n) = \pi(Y_n J'_{n+1} - J'_n Y_{n+1})/M_n M_{n+1}$,

and it is not difficult to establish the identity

$$Y'_n(z) J'_{n+1}(z) - J'_n(z) Y'_{n+1}(z) = \frac{2}{z} \left(1 - \frac{n \cdot n + 1}{z^2}\right), \quad (60)$$

although it has apparently not been explicitly noticed. Accordingly,

$$\sin(\eta_{n+1} - \eta_n) = \frac{2\pi}{zM_n M_{n+1}} \left(1 - \frac{n \cdot n + 1}{z^2}\right),$$

and the pressure is finally given by

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \sum_0^\infty \left(1 - \frac{n \cdot n + 1}{z^2}\right)^2 / \{Y_n'^2 + \pi^2 J_n'^2\} \{Y_{n+1}'^2 + \pi^2 J_{n+1}'^2\}, \quad (61)$$

and this is the formula which must be tabulated.

Case of a Large Obstacle.

We must first show that this formula approximates to the correct result for a large obstacle when z becomes large. This result is, of course, again $p = \frac{8}{3}aE$. The method employed in Part I is again of service. We introduce the functions R_n and ϕ_n of (29) and of the earlier paper. Then

$$\begin{aligned} M_n^2 &= Y_n'^2 + \pi^2 J_n'^2 = \frac{1}{4} \{ (Y_{n-1} - Y_{n+1})^2 + \pi^2 (J_{n-1} - J_{n+1})^2 \} \\ &= \frac{1}{4} \left\{ \frac{2\pi}{z} R_{n-1} + \frac{2\pi}{z} R_{n+1} - \frac{4\pi}{z} (R_{n-1} R_{n+1})^{\frac{1}{2}} \cos(\phi_{n-1} - \phi_{n+1}) \right\}. \quad (62) \end{aligned}$$

But for a large value of z successive functions of type R will differ from one another by practically infinitesimal amounts, and thus, when they occur in the *same term* of an infinite series like (61), they may be equated. Accordingly, it is lawful to write

$$M_n^2 = \frac{\pi}{z} R_n [1 - \cos(\phi_{n-1} - \phi_{n+1})].$$

But, under the same conditions,

$$\phi_{n-1} - \phi_{n+1} = (\phi_{n-1} - \phi_n) + (\phi_n - \phi_{n+1}) = 2(\phi_{n-1} - \phi_n),$$

so that, by the recurrence formula (33),

$$1 - \cos(\phi_{n-1} - \phi_{n+1}) = 2 \sin^2(\phi_{n-1} - \phi_n) = 2/R_n R_{n-1} = 2/R_n^2.$$

Finally, then, we may write in *any single term* of the series (61),

$$M_n^2 M_{n+1}^2 = 4\pi^2/z^2 R_n^2; \quad (63)$$

and therefore
$$\frac{p}{2aE} = \frac{2}{z} \sum_0^\infty R_n^2 \left(1 - \frac{n \cdot n+1}{z^2}\right)^2 \quad (64)$$

for a large obstacle. This process is, of course, only valid in the region of variation of n such that R_n does not become large. But there is no necessity to repeat the argument which shewed, in Part I, that the important part of the sum of series such as (61), when z is large, is derived from the terms in which n ranges from unity to a value differing from z by a quantity of lower order. For this range R_n is not large. We may therefore write, in order to obtain the first term of the pressure expressed as a series in descending order of z ,

$$\frac{p}{2aE} = \frac{2}{z} \sum_0^z R_n^2 \left(1 - \frac{n \cdot n+1}{z^2}\right)^2, \quad (65)$$

and under the same conditions, by (31),

$$R_n^2 = 1 / \left(1 - \frac{n^2}{z^2}\right).$$

Replacing this summation by the asymptotically equivalent integral with $n/z = x$ as variable in the ordinary way,

$$\frac{p}{2aE} = 2 \int_0^1 (1-x^2) dx = \frac{4}{3}, \quad (66)$$

or $p = \frac{8}{3}aE$. The formula is therefore verified for a large obstacle.

The Small Obstacle.

As in Part I, we proceed now to a determination of the pressure on a cylinder of small section due to its scattering action on the waves. Attention may now be confined to the first term of (61), in which we may write,

retaining only the main terms,

$$\left. \begin{aligned} Y_1(z) &= -\frac{2}{z}, & Y'_0(z) &= -Y_1 = \frac{2}{z} \\ Y'_1(z) &= \frac{2}{z^2}, & J'_0(z) &= J'_1(z) = 0 \end{aligned} \right\}; \quad (67)$$

and therefore
$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \bigg/ \frac{4}{z^3} \frac{4}{z^4} = \pi^2 z^3/2,$$

or if λ be the wave length
$$p = 8\pi^5 a^4 E/\lambda^3. \quad (68)$$

In this case both the pressure itself and the ratio $p/\frac{8}{3}aE$ tend to zero with the radius of the obstacle in contradistinction to the result in Part I.

Numerical Tables.

We proceed now to a tabulation of the formula (61), the numerical work being somewhat heavier than before, but much facilitated by the intermediate tables of Part I.

Tables 8 and 9 give the values of the derivates of the Bessel functions which are necessary later. These have been calculated from the tables of the functions themselves (1 and 2) by means of the relations, for any argument z ,

$$\left. \begin{aligned} 2J'_n &= J_{n-1} - J_{n+1} \\ 2Y'_n &= Y_{n-1} - Y_{n+1} \end{aligned} \right\}. \quad (69)$$

Much labour is saved later by retaining the factor 2 which appears in these relations. The last figure is approximate. J'_0 and Y'_0 are respectively $-J_1$, $-Y_1$. The tables have been checked by the identity (60).

TABLE 8.

z	$2J'_1$	$2J'_2$	$2J'_3$	$2Y'_1$	$2Y'_2$	$2Y'_3$
·5	·908	·240	·030	15·70	127·5	$> 10^3$
·6	·868	·282	·043	11·26	73·61	751·3
·8	·7705	·359	·075	7·137	30·91	240·0
·9	·713	·3915	·093	6·131	21·69	150·6
1·0	·650	·4205	·110	5·463	15·84	99·3

TABLE 9.

z	1.2	1.5	2	2.5	3	4	5
$2J'_1$	·5118	·2797	—·1289	—·4945	—·7461	—·7612	—·2242
$2J'_2$	·4654	·4970	·4478	·2805	·0300	—·4962	—·6924
$2J'_3$	·1543	·2204	·3188	·3723	·3541	·0830	—·3446
$2J'_4$	·0323	·0593	·1219	·1971	·2661	·2981	·1037
$2J'_5$	·005	·012	·0328	·0696	·1206	·2320	·2602
$2J'_6$	$<10^{-3}$	$<10^{-2}$	$<10^{-2}$	10^{-2}	·0465	·1169	·2077
$2Y'_1$	4·686	4·131	3·543	2·763	1·688	—·7315	—2·124
$2Y'_2$	9·330	5·220	3·208	2·834	2·712	1·822	·005
$2Y'_3$	48·45	20·20	6·75	3·302	2·376	2·214	1·759
$2Y'_4$	327	110·3	27·68	9·66	4·30	1·928	1·886
$2Y'_5$	$>10^3$	732	138·7	39·12	14·20	3·179	1·643
$2Y'_6$	$>10^4$	$>10^3$	10^3	$2 \cdot 10^2$	56·3	9·145	2·542

The following additional values were employed :—

$$(1) \quad z = 4, \quad 2J'_7 = \cdot 015, \quad 2Y'_7 = 31\cdot 33, \quad 2Y'_8 = 120\cdot 9;$$

$$(2) \quad z = 5, \quad 2J'_7 = \cdot 113, \quad 2J'_8 = \cdot 048, \quad 2J'_9 = \cdot 017,$$

$$2Y'_7 = 6\cdot 616, \quad 2Y'_8 = 20\cdot 43, \quad 2Y'_9 = 78\cdot 96, \quad 2Y'_{10} = 302\cdot 5.$$

$$\text{Let} \quad \sigma_n = 4 \{ (Y'_n)^2 + \pi^2 (J'_n)^2 \}. \quad (70)$$

Then the values of σ_n are exhibited in the next two tables :—

TABLE 10.

z	·5	·6	·8	·9	1·0
σ_0	87·80	65·96	43·144	36·604	31·732
σ_1	254·6	134·2	56·80	42·61	34·014
σ_2	16256	5419	956·7	472·0	252·6
σ_3	$>10^6$	$564 \cdot 10^3$	57600	22680	9861

TABLE 11.

z	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
1.1	28.00	28.425	145.2	4657	25.10 ⁴	> 10 ⁸	> 10 ¹²
1.2	25.03	24.54	89.19	2348	10 ⁵	> 10 ⁷	> 10 ¹⁰
1.5	19.00	17.84	29.69	408.5	12170	> 10 ⁵	> 10 ⁸
2	13.58	12.72	12.27	46.57	766.3	19240	> 10 ⁶
2.5	10.596	10.048	8.808	12.27	93.70	1530	> 4.10 ⁴
3	8.700	8.343	7.364	6.883	19.19	201.8	3170
4	6.424	6.254	5.750	4.970	4.594	10.637	83.77
5	5.100	5.007	4.732	4.266	3.663	3.368	6.888

The following are additional values which were required :—

- (1) $z = 4, \quad \sigma_7 = 981.6, \quad \sigma_8 = 15.10^3$
- (2) $z = 5, \quad \sigma_7 = 43.90, \quad \sigma_8 = 417.3,$
 $\sigma_9 = 6234, \quad \sigma_{10} = 91500.$

With these values of the σ 's the pressure is given from (61) by

$$\frac{p}{2aE} = \frac{128\pi^2}{z^3} \sum_0^\infty \left(1 - \frac{n.n+1}{z^2}\right)^2 / \sigma_n \sigma_{n+1}.$$

(71)

We may denote $(\sigma_n \sigma_{n+1})^{-1}$ by τ_n . Then the requisite values of τ_n are given in the next tables.

TABLE 12.

z	.5	.6	.8	.9	1.0
τ_0	4.474.10 ⁻⁵	1.130.10 ⁻⁴	4.081.10 ⁻⁴	6.480.10 ⁻⁴	9.263.10 ⁻⁴
τ_1	2.416.10 ⁻⁷	1.375.10 ⁻⁶	1.840.10 ⁻⁵	4.972.10 ⁻⁵	1.164.10 ⁻⁴
τ_2	< 10 ⁻¹¹	3.269.10 ⁻¹⁰	1.815.10 ⁻⁸	9.342.10 ⁻⁸	4.016.10 ⁻⁷

TABLE 13.

z	1.1	1.2	1.5	2	2.5	3	4	5
τ_0	$1.257.10^{-3}$	$1.628.10^{-3}$	$2.950.10^{-3}$	$5.787.10^{-3}$	$9.396.10^{-3}$	.01378	.02489	.03916
τ_1	$2.422.10^{-4}$	$4.569.10^{-4}$	$1.888.10^{-3}$	$6.407.10^{-3}$	$1.130.10^{-2}$	$1.628.10^{-2}$	.02781	.04220
τ_2	$1.478.10^{-6}$	$4.775.10^{-6}$	$8.246.10^{-5}$	$1.750.10^{-3}$	$9.253.10^{-3}$	$1.973.10^{-2}$	.03499	.04954
τ_3	$8.6.10^{-10}$	4.10^{-9}	$2.012.10^{-7}$	$2.802.10^{-5}$	$8.698.10^{-4}$	$7.571.10^{-3}$	.04380	.06399
τ_4	$<10^{-13}$	$<10^{-12}$	$<10^{-10}$	7.10^{-8}	$6.975.10^{-6}$	$2.583.10^{-4}$	.02046	.08106
τ_5	$<10^{-10}$	$<10^{-18}$	$<10^{-15}$	$<10^{-10}$	$<10^{-9}$	$1.563.10^{-6}$	$1.122.10^{-3}$	.04311

Additional values subsequently used are :—

$$(1) \quad z = 4, \quad \tau_6 = 1.216.10^{-5}, \quad \tau_7 = 6.97.10^{-8};$$

$$(2) \quad z = 5, \quad \tau_6 = 3.31.10^{-3}, \quad \tau_7 = 5.46.10^{-5}, \quad \tau_8 = 3.84.10^{-7}, \\ \tau_9 = 1.75.10^{-9}.$$

The final values of $p/\frac{8}{3}aE$, calculated from these data, by the formula

$$p/\frac{8}{3}aE = \frac{96\pi^2}{z^3} \sum_1^\infty \left(1 - \frac{n \cdot n + 1}{z^2}\right)^2 \tau_n, \quad (72)$$

are shewn in the last table.

TABLE 14.

z	.5	.6	.8	.9	1.0	1.1	1.2	1.5	2.0	2.5	3.0	4.0	5.0
Diameter wave length	.159	.190	.255	.286	.318	.350	.382	.477	.637	.796	.955	1.27	1.59
$p/\frac{8}{3}aE$	.429	.621	.898	.987	.998	.985	.957	.900	.940	.934	.950	.946	.965

The table shews that the pressure ratio exhibits a series of maxima and minima while tending towards the value unity on the whole. The principal maximum occurs near $z = 1$, where the ratio is almost exactly unity. This series of maxima and minima is analogous to that which occurs in the case of the sphere which is not perfectly reflecting. It could have been predicted from the type of the formula (61), in which the variation of $z^3 R_n R_{n+1}$ with z is, for any value of n , of the nature of a continuous increase with z . The values become smaller rapidly as n increases when z is small, but if z be larger they tend to the same order of magnitude.

Meanwhile the factor $(1 - n \cdot n + 1/z^2)^2$ vanishes whenever

$$z^2 = n \cdot n + 1,$$

and obliterates the corresponding term, and a definite minimum of the pressure ratio follows. As z increases, the obliterated term is otherwise a smaller fraction of the total, and the minima become less pronounced. The curve of the pressure ratio resembles a wave form of decaying amplitude tending asymptotically to a straight line. The table indicates this character clearly.

A combination of the two solutions yields the solution for the general two-dimensional problem. For, if the incident magnetic force H makes any angle β with the axis, it has components (1) $H \cos \beta$ along the axis, and (2) $H \sin \beta$ perpendicular to the axis and to the direction of propagation. These may be considered separately as in Parts I and II. If for a given wave length and diameter the pressure ratio $p/\frac{8}{3}aE$ becomes A in the case of Part I and B in that of Part II, the pressure in the general case must be

$$p = (A \sin^2 \beta + B \cos^2 \beta) \frac{8}{3} a E.$$

This appears at once from the nature of the analysis. Maxima and minima of this quantity, as a function of β , can only occur when β is zero or 90° ; that is to say, they occur in the two cases investigated in detail.

It follows that the cylinder can give extended support to the astrophysical theory based on radiation pressure, but only by virtue of the case discussed in Part I, and cases not too far removed from it. Here it gives valuable support in accordance with the brief discussion given in Part I.