II.—*Circular Magnetisations accompanying Axial and Sectional Currents along Iron Tubes.* By PROFESSOR CARGILL G. KNOTT, D.Sc., F.R.S.E. (With Plate.)

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The experiments now to be described have for their object the investigation of the magnetic induction in an iron conductor under the influence of a current passing through it. The method of experiment was briefly in this wise. An iron tube was magnetised circularly by a current passed from end to end along its entire length ; and the induction so produced in the iron was measured in terms of the current induced in a coil of wire wound longitudinally round the walls of the tube.

In the experiments four tubes were used, all of the same length and nearly the same external diameter. The internal diameters of one pair were approximately double those of the other. The various dimensions are given accurately in the following table, the tubes being distinguished as *A*, *B*, *a*, *b*. Each diameter measurement is the mean of eight measurements taken across different diameters.

Length of Tubes = 34·8 cm.				
Tube.	Diameters in cm.			
	Internal.		External.	
<i>A</i> ,	.	.	1·031 ± ·012	3·022 ± ·003
<i>B</i> ,	.	.	1·050 ± ·010	3·027 ± ·003
<i>a</i> ,	.	.	2·036 ± ·004	3·022 ± ·003
<i>b</i> ,	.	.	2·052 ± ·005	3·021 ± ·002

The tubes were all of wrought iron, turned on the lathe and bored. The *a* tube was made at a much later date than the other three. The results given below show that it was made out of quite a different specimen of iron.

To measure the circular magnetisation, four turns of insulated copper wire were coiled longitudinally round the walls of the tubes *A*, *B*, and *b* ; and sixteen round *a*.

Each tube could be magnetised circularly in two ways :—

- (1) By an *axial current* passing along a copper wire led through the axis.
- (2) By a *sectional current* passing longitudinally through the substance or wall of the tube.

It was by a direct comparison of the inductive effects of these two methods of applying the magnetising force that the action of the latter was studied. To this end two of the tubes were taken and arranged so that the same current could be passed axially or sectionally along them. The induced currents produced in the coils were then balanced on a galvanometer exactly as in the familiar method for comparing mutual inductances

of two pairs of coils. In this way the magnetic induction produced by the "sectional" current along, say, the A tube could be directly compared with the magnetic induction produced by the "axial" current along the B tube; and *vice versa*.

To complete the investigation it was necessary to know the laws governing the magnetic action of the axial current. At first it seemed sufficient to assume, in accordance with the usual view, that the magnetic force due to the axial current was inversely as the distance in the iron substance as well as in the air spaces, and that with the currents used the value of the force was small enough to warrant us taking the permeability as constant throughout the iron. There were hints, however, that these assumptions were not even approximately true. Accordingly, a series of experiments was undertaken in which the circular inductions due to various axial currents were carefully measured. In some experiments the induced currents were measured on a ballistic galvanometer; in others the induced currents were measured by balancing them against the induced currents produced in the secondary of a standard pair of coils in whose primary the axial current was made to flow. The results obtained by both methods were in good agreement; and I give here only the one series. It was the last in point of time, and was very carefully carried out by Mr SAWADA, a graduating student in physics of the Imperial University of Japan.

The quantity directly measured was the total induction across a radial section of the tube. It can obviously be written in the form

$$l \int_b^a \mu' R dr,$$

where l is the length of the tube, a and b the external and internal radii, R the magnetic force at a point distant r from the axis, and μ' the permeability at this point. In general μ' will be a function of R . If, however, μ' is unity, we know that R has the value $2i/r$ where i is the total current passing axially through the tube. In this case the integral gives us what might be called the total "normal induction" across the radial section of the tube. Its value is

$$li \log \frac{a^2}{b^2}.$$

Dividing by the area of the section, viz., $l(a^2 - b^2)$, we get the average magnetic force influencing the tube. Thus we may calculate what might be called the average permeability μ , where

$$\mu i \log \frac{a^2}{b^2} = \int_b^a \mu' R dr,$$

or

$$\mu h = \mathfrak{B}.$$

Here $\mathfrak{B}$ is measured experimentally, and h is calculated in terms of the current and the known dimensions of the tube. The values of these quantities are given in the following table.

TABLE I.

Showing the relation between the average magnetic induction ($\mathfrak{B}$) in an iron tube under the influence of an axial current, and the average magnetic force (h) as it would be throughout the region occupied by the iron if the iron were replaced by a substance of unit permeability.

h	$\mathfrak{B}$	$\mu = \mathfrak{B}/h$	h	$\mathfrak{B}$	$\mu = \mathfrak{B}/h$
·0299	9·2	307·9	1·654	2251	1361
·0475	14·6	308·8	2·068	3612	1747
·0572	18·6	326·2	2·481	4900	1975
·0724	25·1	347·2	2·895	5688	1965
·1046	39·2	375·2	3·308	6391	1932
·1234	47·8	387·5	3·722	6967	1872
·1900	79·6	419·2	4·136	7377	1784
·2157	94·9	440·3	4·570	7823	1720
·3218	162·8	505·7	4·963	8276	1668
·4136	213·4	516·2	5·377	8692	1617
·4626	249·0	538·6	5·790	9054	1564
·8270	544·2	658·0	6·204	9452	1524
·8493	577·9	680·5	6·617	9825	1485
1·241	1266	1021			

It should be mentioned that in the tube used, A namely, one ampere of axial current corresponds to an average magnetic field of 0·2150. Thus the highest magnetic field corresponds to an axial current of fully 30 amperes. In the experiment this highest value was attained by using a multiple wire of many insulated strands, and passing a moderate current through all in series.

It is important to notice that with a current of one ampere the field at the internal surface of the tube is nearly 0·4 ; and similarly in any case the maximum field acting on the iron is about 1·8 times the estimated average field acting over the entire radial section.

The relations existing between the quantities tabulated above are shown graphically in the Plate. The induction curve and permeability curve are both shown complete, and on a large enough scale to bring out the deviations from smoothness. They are drawn through *all* the points.

The induction curve is very similar in form to the induction curves obtained when an iron rod is longitudinally magnetised. The same may be said in a general way of the permeability curve. One particular, however, in which the results differ appreciably from results obtained for iron rods and anchor rings is in the comparative values of the maximum permeability and the permeability for the smallest field. This smallest permeability is remarkably high as compared with the initial permeabilities given, for example, by ROWLAND, EWING, or Lord RAYLEIGH. On the other hand, the maximum permeability is comparatively small, falling short of 2000. How far this may be due to the *form* of the iron tube, or to the inaccuracy of the estimate of the average magnetic force,

it is impossible to say. It is more than probable, however, that it is largely referable to the mode in which the magnetisation is applied.

There is a second particular that appears to be an essential peculiarity; and that is, the way in which the permeability curve begins its ascent. It will be seen from the table that the permeability has nearly the same value for fields below 0.05; but that above this it begins to grow rapidly. It is evident, then, that for axial currents greater than half an ampere, we cannot assume a constant permeability throughout the *A* and *B* tubes. Even for the *a* and *b* tubes, for which one ampere gives an estimated average field of nearly 0.16, it would be unsafe to assume a constant permeability for stronger axial currents than half an ampere.

It will be seen from the curve that the permeability is almost accurately proportional to the magnetic field from a field of about unity up nearly to its maximum. The curious hump-like character of the earliest part seems difficult to explain as a result of heterogeneity in the metal. Whatever be its cause, its existence indicates a remarkably rapid increase of permeability when first it begins to increase. I am not aware that any like peculiarity has been noticed in other magnetic experiments. Permeability curves seem always to begin convex to the axis, and not to become concave until the maximum point is being approached.

In the experiments in which the inductions were measured by ballistic swings, the swings were taken at make and break of the magnetising current which was passed first in one direction then in the other. From the numbers so obtained it was an easy matter to calculate the residual magnetism in terms of the total induced magnetism. With an axial current varying from 0.25 of an ampere to 4 amperes, and a corresponding average field varying from 0.055 to 0.882, it was found that the ratio of the residual magnetism to the total induced magnetism varied from 0.05 to 0.20.

I now proceed to the discussion of the experiments which form the real subject-matter of this paper.

The method as already briefly described was a nul-method. Two tubes were taken and subjected to the inductive effect of the same current. This current could be purely *axial* along both tubes, or purely *sectional*; or it could be axial along the one and sectional along the other. To guard against any confusion, I shall here formally define these terms, although the definition is fully implied in what precedes. An Axial Current is a current which flows in a wire forming the axis of the tube, but quite insulated from it. A Sectional Current is a current flowing through the substance or walls of the tube parallel to the axis.

The following combinations of tubes with currents were tried:—

<i>B</i> axial	with <i>A</i> axial.		<i>b</i> sectional with <i>a</i> sectional.
<i>B</i> sectional	„ <i>A</i> „		<i>b</i> axial „ <i>a</i> „
<i>B</i> „	„ <i>A</i> sectional.		<i>a</i> „ „ <i>A</i> axial.
<i>B</i> axial	„ <i>A</i> „		<i>a</i> sectional „ <i>A</i> „
<i>b</i> „	„ <i>a</i> axial.		<i>a</i> axial „ <i>A</i> sectional.
<i>b</i> sectional	„ <i>a</i> „		

The ratios of the inductions in these different pairs, calculated from the resistances needed to be inserted in the secondary circuits so as to obtain a balance on the galvanometer, are given in the succeeding table for three different strengths of current. The current strengths are given in amperes at the head of each column. The symbols A , B , a , b mean the inductions produced by the axial current; and A' , B' , a' , b' the inductions produced by the sectional currents.

Current =		·49	1·05	2·09
(1)	B/A	1·088	1·0548	1·0340
(2)	B'/A	·333	·2960	·2898
(3)	B/A'	3·60	3·857	3·893
(4)	B'/A'	1·122	1·077	1·0868
(5)	b/a	1·338	1·3395	1·3333
(6)	b'/a	1·556	·5437	·533
(7)	b/a'	3·105	3·303	3·404
(8)	b'/a'	1·325	1·340	1·360
(9)	a/A	·2735	·2536	·249
(10)	a'/A	·117	...	·0974
(11)	A'/a	...	1·080	...

From these the following pairs of same ratios are obtained, and final means taken of these pairs.

Current =		·49	1·05	2·09	
1/3	A'/A	·302	·2735	·2663	
2/4	"	·298	·2748	·2667	
(12)	"	·3007	·2739	·2664	
2/1	B'/B	·307	·2806	·2803	
4/3	"	·312	·2793	·2792	
	"	·3087	·2802	·2799	
5/7	a'/a	·431	·4054	·392	
6/8	"	·420	·4058	·392	
(13)	"	·427	·4056	·392	
6/5	b'/b	·423	·4059	·4000	
8/7	"	·426	·4057	·3995	
	"	·424	·4058	·3999	
9	a/A	·2735	·2536	·2490	
10/13	"	·274	...	·2485	
12/11	"	...	·2536	...	
	"	·2738	·2536	·2488	

It will be noticed that the means are not simple arithmetical means. They are obtained by weighting the individual values in accordance with the accuracy of the experiments on which these values depend.

Finally, we may calculate the ratio b/B for each current thus :—

$$\frac{b}{B} = \frac{b}{a} \cdot \frac{a}{A} \cdot \frac{A}{B}.$$

Collecting, then, all the results that are of importance, we obtain Table II., showing the ratios between the sectional and axial inductions in each tube, and also the ratios between the axial inductions in the thin-walled and thick-walled tubes.

TABLE II.

Current =	·49	1·05	2·09
A'/A	·3007	·2739	·2604
B'/B	·3087	·2802	·2799
a'/a	·427	·4056	·392
b'/b	·424	·4058	·3999
a/A	·2738	·2536	·2488
b/B	·3367	·3220	·3208

For a tube of radii a and β ($a > \beta$) the sectional current i produces (in accordance with the usual theory) at any point r a field

$$h = 2i \frac{r^2 - \beta^2}{a^2 - \beta^2} \frac{1}{r}.$$

If l is the length of the tube, the total sectional induction across a radial section is

$$\begin{aligned} \mathfrak{B}' &= \int_{\beta}^a l \mu' h dr \\ &= l \mu' i \left(1 - \frac{1}{p^2 - 1} \log p^2 \right) \end{aligned}$$

where, for simplicity, we assume μ' to be constant over the section, and where $p = a/\beta$.

With the same assumption, we find for the total axial induction the expression

$$\mathfrak{B} = l \mu i \log p^2.$$

It is advisable to distinguish μ from μ' , since clearly the average magnetic force due to a given axial current is greater than the average magnetic force due to a sectional current of the same value; and we have seen above how much the permeability varies with the force.

In the following table, the various dimensional quantities on which the inductions depend are shown for the four tubes :—

Tube	$2a$	2β	$\log p^2$	$p^2 - 1$	$\frac{1}{p^2 - 1} \log p^2$	$\mathfrak{B}'/l\mu'i$
<i>A</i>	3·022	1·031	2·1508	7·591	·2833	·7167
<i>B</i>	3·027	1·050	2·1211	7·315	·2900	·7100
<i>a</i>	3·022	2·036	·7898	1·203	·6565	·3435
<i>b</i>	3·021	2·052	·7735	1·168	·6625	·3375

The column headed $\log p^2$ gives the values of $\mathfrak{B}/l\mu i$.

By dividing the numbers in the $\mathfrak{B}'$ column by the corresponding numbers in the $\mathfrak{B}$ column, we obtain the following ratios of the inductions :—

$$\left. \begin{aligned} A'/A &= \cdot 3332 \frac{\mu'}{\mu} \\ B'/B &= \cdot 3347 \frac{\mu'}{\mu} \\ a'/a &= \cdot 4349 \frac{\mu'}{\mu} \\ b'/b &= \cdot 4363 \frac{\mu''}{\mu} \end{aligned} \right\} \dots \dots \dots E,$$

where it must be remembered that the ratio μ'/μ is not necessarily the same in all.

We also get the following ratios :—

$$\begin{aligned} a/A &= \cdot 3669 \frac{\mu_1}{\mu_2} \\ b/B &= \cdot 3647 \frac{\mu_1}{\mu_2} \end{aligned}$$

where the suffixes are attached to μ to show that the permeabilities for the two kinds of tubes subject to the same axial current are not necessarily the same.

In the light of these results we shall now discuss Table II. In the first place, it is evident at once that the *a* tube must differ specifically from the other tubes, being indeed, as already noted, of a different kind of iron. A comparison of the *b* and *B* ratios leads to the immediate conclusion that

$$\begin{aligned} \frac{\mu \text{ for } b \text{ tube}}{\mu \text{ for } B \text{ tube}} &= \cdot 9233 \text{ for axial current } \cdot 49 \text{ amp.} \\ &= \cdot 8830 \quad \text{,,} \quad \text{,,} \quad 1\cdot 05 \quad \text{,,} \\ &= \cdot 8800 \quad \text{,,} \quad \text{,,} \quad 2\cdot 09 \quad \text{,,} \end{aligned}$$

Now the average magnetic field acting within the substance of the tube is, in terms of the usual theory, $0\cdot 212i$ for the *B* tube, and $0\cdot 159i$ for the *b* tube; *i* being measured in amperes. Thus the average fields for the two tubes under influence of the currents given above are

$$\begin{aligned} &\cdot 104, \cdot 223, \cdot 445 \text{ for the } B \text{ tube, and} \\ &\cdot 078, \cdot 167, \cdot 333 \text{ for the } b \text{ tube.} \end{aligned}$$

From Table I. we may readily calculate that these corresponding pairs of magnetising forces give for the ratios of the inductions

$$\cdot 939, \cdot 914, \cdot 957$$

respectively. These are all somewhat higher than the experimental values just given. This, however, is not surprising inasmuch as the kinds of iron composing the two tubes may differ enough to render the numbers in Table I. not strictly applicable to the *b* tube. And, again, the theory itself that is made use of in estimating the average magnetic force and the induction due thereto is admittedly only approximately true. The comparison just made seems to show that it is nevertheless a tolerably fair approximation to the truth.

The ratios of the sectional and axial inductions call next for consideration. At a glance we see that the ratios given in Table II. are all less than the corresponding multipliers in equations *E*. This means that μ' is less than μ in every instance. In a general way, it is easy to see that this ought to be so. The fields due to axial and sectional currents of the same strength, although they are the same in the region outside the tube, have quite different values in the substance of the tube. The field due to the axial current increases as we pass inwards to the interior surface; while that due to the sectional current diminishes, and has the value zero at the interior surface. In any case, we may estimate the average field by dividing the total normal induction by the radial section. The values of the average fields, so estimated for the different tubes, are shown in the following table:—

Tube.	Average Field due to Current		
	$\cdot 49$	$1\cdot 05$	$2\cdot 09$
<i>A</i>	$\cdot 1058$	$\cdot 2268$	$\cdot 4514$
<i>A'</i>	$\cdot 0353$	$\cdot 0756$	$\cdot 1503$
<i>B</i>	$\cdot 1049$	$\cdot 2247$	$\cdot 4473$
<i>B'</i>	$\cdot 0350$	$\cdot 0749$	$\cdot 1491$
<i>a</i>	$\cdot 0785$	$\cdot 1682$	$\cdot 3348$
<i>a'</i>	$\cdot 0342$	$\cdot 0732$	$\cdot 1457$
<i>b</i>	$\cdot 0782$	$\cdot 1675$	$\cdot 3333$
<i>b'</i>	$\cdot 0341$	$\cdot 0731$	$\cdot 1455$

Now each of these fields has its appropriate permeability; and by means of Table I. we may calculate the permeabilities in the case of Tube *A*. From Table II. we see that Tube *B* behaves very similarly to Tube *A*; and that Tube *a*, notwithstanding the fact that the permeability of its material is comparatively small, behaves almost exactly like Tube *b*. This is significant as showing that the relation between the sectional and axial inductions is largely independent of the numerical values of the permeabilities involved. Hence we may reasonably discuss the case of the Tubes *a* and *b* by applying to them the results of Table I. In this discussion we shall take the means of the various values

for the a and b tubes ; but shall leave Tube B out of consideration, since Table I. applies directly to Tube A .

Calculating, then, the permeabilities for the fields due to the several currents, we find as follows :—

Tube.	Permeabilities due to Current		
	·49	1·05	2·09
A	376	447	533
A'	308	350	400
ab	352	408	507
$a'b'$	308	348	398

From these we at once obtain the ratios (μ'/μ) for the different currents and tubes. These we shall call the *calculated ratios*; while the ratios obtained from equations E by use of the values of Table II., we shall call the *observed ratios*. It remains now to compare these sets of ratios, as in Table III.

TABLE III.

Current Strengths in Amperes.	Ratios of Permeabilities of Sectional and Axial Inductions.			
	A'/A		$a'b'/ab$	
	Obs.	Calc.	Obs.	Calc.
·49	·902	·819	·978	·874
1·05	·832	·783	·932	·853
2·09	·781	·750	·910	·785

It will be seen at once that a ready explanation is found for the diminution of the ratio of the two inductions as the field is taken stronger. The numbers in the “calculated” columns fall off according to much the same law as those in the “observed” columns. In all cases, however, the calculated ratios are smaller than the corresponding observed ratios. This may be explained in several ways, thus: either the average magnetic field acting on the iron is over-estimated for the axial induction or under-estimated for the sectional induction; or the permeability is really greater under the influence of the current passing through the iron than it is under the influence of an equal average magnetic field unaccompanied by such a sectional current.

If the last be the true explanation, then a direct experiment should show an appreciable increase in the axial induction when a current is in addition sent along the iron tube. Now, the effect of a sectional current sustained steadily while the axial is being

reversed will be to impart to the tube a mean circular magnetisation of definite amount. The induction will not, at reversal of the axial current, oscillate to equal amounts on the opposite sides of zero induction. On the contrary, it will oscillate about a mean value of definite amount, so that the total induction will be less on one side than on the other. In the ballistic method of measuring inductions, however, this one-sidedness or bias would not be apparent. But if the explanation suggested above be the true one, the induction due to reversal of the axial current should be greater when this current is also directed steadily without reversal along the tube. The experiment was accordingly tried in January of 1892, not with the tubes investigated above (which being in Japan were unfortunately not at my disposal), but with a short tube supplied me by Professor TAIT, to whom I would here express my thanks for liberty to work in the Physical Laboratory of the University of Edinburgh. The tube was 10 cm. long, had an external diameter of 4.1 cm., and an internal diameter of 2.7. The average field throughout the region occupied by the iron wall is consequently 0.122 for one ampere of current along the axis. The following are some of the results obtained, the first column giving the current in amperes, the second column containing numbers proportional to the induction produced by reversal of the axial current, and the third column similar induction numbers when the current was in addition sectional but steady :—

Current in Amperes.	Axial Induction.	
	Pure.	With Sectional Current.
3.26	29.8	30
4.44	43.3	43
8.43	87.3	85.6
16.85	194.6	194.5

There is no evidence here that the sectional current has any distinct effect upon the susceptibility. At any rate, what effect is produced is certainly not increase.

The discrepancy in Table III. is consequently not to be explained in terms of a direct effect upon the susceptibility. We must look for the explanation in the other directions indicated. That is to say, the ordinary theory of the circular magnetisation of iron under the influence of an axial, or a sectional, current does not strictly apply. It will be noticed, however, that the discrepancy is not large, amounting only to 7 per cent. To this approximation, therefore, we may regard the ordinary theory as holding for sectional currents whose current density does not exceed 0.2 ampere per square centimetre. For much greater current densities, as when, for example, a fairly strong current passes along a thinnish iron wire, nothing can be asserted; and it is difficult to see how an experimental investigation into the circular magnetisation of a solid wire could be undertaken.