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A CLASS OF ALGEBRAIC FUNCTIONS.

THE functions in question are those which involve only the differences of their arguments; and the object of this note is to suggest that an elementary discussion of these functions would be a valuable addition to the usual school course in algebra.

In his memoir on the Schwarzian derivative,* Cayley gives the name 'diaphoric' to these functions (Art. 13). Invariants, seminvariants, covariants of quantics are diaphoric functions of the roots. Norms of complex numbers and 'circulants' are diaphoric functions, a particular case of which already appears in books on algebra under the properties of cube roots of unity.

But quite apart from any applications, the diaphoric functions are remarkable for the facility with which they can be transformed, and variety of forms that can be obtained; and my suggestion is, that the theory is worthy of a place beside that of symmetrical functions.

It might be treated somewhat in this way :

I. A diaphoric function of $x, y, \dots z$ is unchanged when any quantity δ (the same for all the arguments) is subtracted from the arguments $x, y, \dots z$.

For every difference, $x - y$ is unchanged.

II. If a function of $x, y, \dots z$ is unchanged when any quantity δ is subtracted from each of the arguments, whatever value δ may have, it is diaphoric.

For we may give to δ the value of one of the arguments, and then the function is visibly diaphoric.

III. This gives a simple mode of deciding whether a given function is or is not diaphoric. As an example, take

$$U = x^2 + 2y^2 + 3z^2 - 4yz - 2zx.$$

If this is diaphoric in x, y, z , it is unchanged when z is subtracted from each argument, giving

$$U' = (x - z)^2 + 2(y - z)^2,$$

which on expansion is found to be the same as U .

Hence $U = (x - z)^2 + 2(y - z)^2$,
and is visibly diaphoric.

But if U' is different from U , the latter cannot be diaphoric, seeing that for a value of δ it does not satisfy the test.

It will be observed that one trial suffices to decide the point.

IV. When U is known to be diaphoric, expressions equal to U but of different forms can at once be written down by assuming different values of δ , and by linear combinations of these

expressions. For example, taking U as above, we have

$$\begin{aligned} U &= 3(y-z)^2 + (x-y)^2 - 2(z-y)(x-y) & (\delta=y) \\ &= 2(x-y)^2 + 3(z-x)^2 - 4(y-x)(z-x) & (\delta=x) \\ &= 4x^2 + 2(x+y)^2 + 3(z+x)^2 - 4(x+y)(x+z) - 4x(x+z) & (\delta=-x) \\ &= (x+y-z)(x-y-z) + 2x^2 + 3y^2 - 4yz, \text{ etc., etc.} \end{aligned}$$

V. We sometimes simplify the analysis of a given expression by assigning special values to one or more of the arguments. In the case of diaphoric functions there is the important advantage that, if we assign the value 0 to any argument, say x , the original function is immediately found again by replacing the remaining arguments $(y, \dots z)$ by $y-x, \dots z-x$.

It is obvious that when a diaphoric function is not homogeneous in the arguments it can always be expressed as a sum of diaphorics each of which is homogeneous, and there is no loss of generality in restricting ourselves to homogeneous diaphorics.

VI. Integral algebraical diaphoric functions have the property that every integral factor involving any of the arguments is also diaphoric.

Suppose W is a diaphoric function of $x, y, \dots z$, and is a product of two factors U, V . We shall show that U, V are diaphoric functions. When δ is subtracted from each argument, U, V become

$$U + U_1\delta + U_2\delta^2 + \dots + U_r\delta^r, \quad V + V_1\delta + \dots + V_s\delta^s$$

if r, s denote the degrees of U, V respectively. Also W becomes W' ,

$$W' = (U + U_1\delta + \dots + U_r\delta^r)(V + V_1\delta + V_2\delta^2 + \dots + V_s\delta^s).$$

But $W' = W = UV$, so that

$$\begin{aligned} 0 &= (UV_1 + U_1V)\delta + \dots + (UV_t + U_1V_{t-1} + \dots + U_tV)\delta^t \\ &\quad + \dots + U_rV_s\delta^{r+s} \end{aligned}$$

for all values of δ . Hence the coefficient of each power of δ vanishes; and this implies, as we shall show, that $U_1, U_2, \dots U_r, V_1, V_2, \dots V_s$ all vanish.

To fix the ideas, suppose $r=3, s=4$. We have then the following equations to satisfy:

$$U_3V_4 = 0, \dots \dots \dots (7)$$

$$U_3V_3 + U_2V_4 = 0, \dots \dots \dots (6)$$

$$U_3V_2 + U_2V_3 + U_1V_4 = 0, \dots \dots \dots (5)$$

$$U_3V_1 + U_2V_2 + U_1V_3 + UV_4 = 0, \dots \dots \dots (4)$$

$$U_3V + U_2V_1 + U_1V_2 + UV_3 = 0. \dots \dots \dots (3)$$

From (7) it appears that U_3 or V_4 is 0, and from the other equations it may be inferred that both vanish. For if $V_4 \neq 0$, $U_3 = 0$, and therefore (6) gives $U_2V_4 = 0$, i.e. $U_2 = 0$. In like manner (5) gives $U_1 = 0$, and (4) $U = 0$, which is absurd. Thus

$V_4=0$. In the same way, the supposition that $U_3 \neq 0$ leads to $V=0$, which is impossible, and therefore $U_3=0$.

Having $U_3=0$, $V_4=0$, we may omit all terms containing these letters; and we obtain with the equation

$$U_2V + U_1V_1 + UV_2 = 0, \dots\dots\dots (2)$$

a system of four equations similar to the original system which gives $U_2=V_3=0$.

Again, omitting the vanishing terms, and using the equation

$$U_1V + UV_1 = 0, \dots\dots\dots (1)$$

we find $U_1=V_2=0$, and then (1) yields $V_1=0$.

Now, in this analysis U is *any* factor (involving some of the arguments of W), and V is the co-factor. Hence we have shown that any factor U of a diaphoric function (W) is diaphoric. It will be observed that the proof though applied to a particular case is really quite general.

[It is perhaps proper to add that the theorem cannot be proved by means of the differential equation satisfied by diaphoric functions in general, and, indeed, is not true for such functions. Suppose, for example, U and V are diaphoric functions, so that e^{U+V} is likewise diaphoric. Then we have

$$e^{U+V} = e^{U+W} \times e^{V-W},$$

where W is any quantity whatever, and need not be diaphoric.]

VII. A few examples follow :

EXAMPLE 1. Factorize $\Sigma(b-c)^3 = U$.

$$\begin{aligned} U' &= b^3 - a^3 + (a-b)^3 \quad (\text{when } c=0) \\ &= 3ab^2 - 3a^2b \quad \text{,,} \\ &= 3ab(b-a); \quad \text{,,} \\ \therefore U &= 3(a-c)(b-c)(b-a) = 3(b-c)(c-a)(a-b). \end{aligned}$$

EXAMPLE 2. $U = \Sigma(b-c)^5$;

$$\begin{aligned} \therefore U' &= b^5 - a^5 + (a-b)^5 \quad (c=0) \\ &= 5ab(-a^3 + 2a^2b - 2ab^2 + b^3) \quad \text{,,} \\ &= 5ab(b-a)(b^2 - ab + a^2); \quad \text{,,} \\ \therefore U &= 5(a-c)(b-c)(b-a)\{\Sigma a^2 - \Sigma bc\} \\ &= 5(b-c)(c-a)(a-b)\{(c-a)^2 + (b-c)^2 + (c-a)(b-c)\}. \end{aligned}$$

EXAMPLE 3. The expression $\Sigma a^2 - \Sigma bc$, obtained in Example 2, being diaphoric in three arguments, is really a quadratic function of two arguments; and is therefore factorizable. The factors must be of the form $(a+hb+kc)(a+kb+hc)$, since the expression is symmetrical in (b, c) . Also since each factor is diaphoric, $1+h+k=0$, and observing the coefficient of b^2 (or c^2) in the product $hk=1$;

$$\therefore 1+h+h^{-1}=0,$$

and the factors are $(a+hb+h^{-1}c)(a+h^{-1}b+hc)$.

EXAMPLE 4. Determine A, B, C so that

$$U = Axyz + B(x+y+z)(yz+zx+xy) + C(x+y+z)^3$$

may be diaphoric,

Here $U' = B(x+y)xy + C(x+y)^3 \quad (z=0) ;$

and, if diaphoric, $U = B(x+y-2z)(x-z)(y-z) + C(x+y-2z)^3.$

Hence U is a multiple of $x+y-2z$, and, by symmetry,

$$U = k\Pi(x+y-2z) ;$$

$$\therefore U' = B(x+y)xy + C(x+y)^3 = k(x+y)(y-2x)(x-2y),$$

$$Bxy + C(x+y)^2 = k(-2x^2 + 5xy - 2y^2) ;$$

$$\therefore C = -2k, \quad B + 2C = 5k, \quad B = 9k.$$

Again, putting $x=y=z=1$, so that $U=0$,

$$0 = A + 9B + 27C ;$$

$$\therefore A = -81k + 54k = 27k ;$$

$$\therefore U, = (-27xyz + 9\Sigma x \cdot \Sigma yz - 2(\Sigma x)^3)k, = k\Pi(x+y-2z),$$

if it is diaphoric.

Hence if x, y, z are roots of an equation $ax^3 + 3bx^2 + 3cx + d = 0$,

$$\text{then} \quad U = 27 \frac{k}{a^3} (a^2d - 3abc + 2b^3) = 27k \left(-\frac{b}{a} - x \right) \left(-\frac{b}{a} - y \right) \left(-\frac{b}{a} - z \right),$$

$$\frac{a^3}{27k} U = a^2d - 3abc + 2b^3 = -(ax+b)(ay+b)(az+b).$$

H. W. LLOYD TANNER.

CAYLEY'S THEORY OF THE ABSOLUTE.

THIS is the title of an extremely interesting paper by PROFESSOR CHARLOTTE ANGAS SCOTT, in the *Bulletin* of the American Mathematical Society, 2nd Series, Vol. III., No. 7. Its object is "to show, as a matter of purely pedagogic interest, how simply and naturally Cayley's theory of the Absolute follows from a small number of very elementary geometrical conceptions, without any appeal to analytical geometry." The "contention is not that every step in the rigorous proof can be presented under the guise of elementary mathematics, but that it is quite possible to develop the theory so as to be intelligible and interesting to average students at a much earlier stage than is customary." The paper begins by explaining the principles of projection. If a plane figure be projected in any manner into another plane figure, it is easily seen "that the properties in which any difference is to be observed in the two figures all depend on measurement of lines or angles; these properties are called metric; the properties that are unaffected by the change from one figure to the other, unaffected, that is, by projection, are called projective." The simplest projective properties are that points project into points and straight lines into straight lines, and consequently collinear points into collinear points and concurrent lines into concurrent lines. By *assuming* that these general laws admit of no exception, and examining by what conception or artifice *apparent* exceptions may be brought under the general rule, we are led to postulate the existence in any