A SYMMETRICAL RELATION BETWEEN LEGENDRE'S FUNCTIONS WITH PARAMETERS $\cosh \alpha$ AND $\coth \alpha$

By F. J. W. WHIPPLE.

[Received and read June 8th, 1916.]

1. The relation

$$e^{-m\pi} Q_n^m(\cosh \alpha) = \sqrt{\left(\frac{\pi}{2}\right)} \frac{\Pi(m+n)}{\sqrt{\sinh \alpha}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth \alpha), \quad (1)$$

which is discussed in the following notes, was arrived at in the examination of certain integrals which occur in the determination of Green's function for a wedge.* It is found to explain the symmetry of various parts of the theory of Legendre's functions. Its geometrical interpretation is found in the applications of inversion to potential problems connected with toroidal coordinates.

2. It is convenient to write $\cosh \alpha = \mu$, $\coth \alpha = \nu$, so that μ and ν satisfy

$$(\mu^2 - 1)(\nu^2 - 1) = 1. \quad (2)$$

We shall show in the first place that if $V_n^m(\mu)$ is any solution of Legendre's equation with μ as variable, then there is a corresponding function $W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu)$, which is a solution of the equation with ν as variable, and is connected with $V_n^m(\mu)$ by the equation

$$V_n^m(\mu) = (\nu^2 - 1)^{\frac{1}{2}} W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu).$$

Legendre's differential equation is

$$\left[-\frac{\partial}{\partial \mu} (\mu^2 - 1) \frac{\partial}{\partial \mu} + n(n+1) + \frac{m^2}{\mu^2 - 1} \right] V = 0. \quad (3)$$

* Vide "Diffraction by a Wedge and Kindred Problems," p. 94 of the present volume, and particularly pp. 107-109.

This may be transformed to

$$\left[-(\nu^2-1)^{\frac{3}{2}} \frac{\partial}{\partial \nu} (\nu^2-1)^{\frac{1}{2}} \frac{\partial}{\partial \nu} + m^2(\nu^2-1) + n(n+1) \right] V = 0. \quad (4)$$

If $(\nu^2-1)^{\frac{1}{2}} W$ is substituted for V , the last equation becomes, after some reduction,

$$\left[-\frac{\partial}{\partial \nu} (\nu^2-1) \frac{\partial}{\partial \nu} + (m-\frac{1}{2})(m+\frac{1}{2}) + \frac{(n+\frac{1}{2})^2}{\nu^2-1} \right] W = 0. \quad (5)$$

Accordingly W is a Legendre function of ν of order $m-\frac{1}{2}$ and rank $n+\frac{1}{2}$. Thus

$$V_n^m(\cosh \alpha) = \sqrt{(\operatorname{cosech} \alpha)} W_{m-\frac{1}{2}}^{n+\frac{1}{2}}(\coth \alpha). \quad (6)$$

Obviously the signs of m and of $n+\frac{1}{2}$ may be changed without altering either of the equations (3) and (5), so that instead of (6) we may write

$$V_n^m(\cosh \alpha) = \sqrt{(\operatorname{cosech} \alpha)} W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth \alpha).$$

3. The identity (6) could be derived from Olbricht's result that the differential equation (3) is satisfied by either of Riemann's P -functions

$$P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ \frac{1}{2}m & -n & \frac{1}{2}m \\ -\frac{1}{2}m & n+1 & -\frac{1}{2}m \end{array} \quad \frac{1-\mu}{2} \right\}$$

and

$$P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ -\frac{1}{2}n & m & -\frac{1}{2}n \\ \frac{1}{2}(n+1) & -m & \frac{1}{2}(n+1) \end{array} \quad \frac{\mu+(\mu^2-1)^{\frac{1}{2}}}{2(\mu^2-1)^{\frac{1}{2}}} \right\}.$$

4. The Legendre functions are defined by different authors in ways which are not consistent. We shall accept the definitions given by Prof. Hobson, to whose paper "On a Type of Spherical Harmonics of Unrestricted Degree, Order and Argument," *Phil. Trans.*, 1896, frequent reference will be made. These definitions are in terms of contour integrals round looped paths. The equation (1) will be derived from a comparison of two of these contour integrals.

The definition of $P_n^m(\cosh \alpha)$ is [*l.c.*, equation (50)]

$$P_n^m(\cosh \alpha) = \frac{1}{2\pi i} 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \sinh^m \alpha \int_{e^{\alpha+}, e^{\alpha-}} \frac{h^{n+m}}{[(e^{\alpha}-h)(e^{-\alpha}-h)]^{m+\frac{1}{2}}} dh. \quad (7)$$

The integral is taken along a looped path which passes round e^a anti-clockwise and e^{-a} clockwise. The phases of the factors of the integrand are governed by the following conventions. The phase of h vanishes where the path cuts the real axis on the right of the origin. The phase of $e^a - h$ is that of e^a at the point where the path cuts the join of the origin to e^a , and the phase of $e^{-a} - h$ is determined in a similar way.

The definition of $Q_n^{m'}$ [*l.c.*, equation (9), p. 455] may be written

$$e^{-m'\pi} Q_n^{m'}(\coth a) = \frac{1}{4i \sin n'\pi} \frac{\Pi(n' + m')}{\Pi(n')} \frac{1}{\sinh^{m'} a} \int^{-1+, 1-} \frac{1}{2^{n'}} \frac{(t^2 - 1)^{n'}}{(\coth a - t)^{n' + m' + 1}} dt. \quad (8)$$

Here the phases of $t-1$ and $t+1$ vanish when t is real and greater than 1, and the phase of $\coth a - t$ is such that at the point E , where the line joining $\coth a$ and 1 cuts the path of integration, it is the angle (between $\pm \pi$) the line makes with the positive direction of the real axis of t .

Let us write $h = \cosh a - t \sinh a$ in (7), so that

$$e^{-a} - h = (t-1) \sinh a, \quad e^a - h = (t+1) \sinh a.$$

It follows that

$$P_n^m(\cosh a) = -\frac{1}{2\pi i} 2^m \frac{\Pi(m - \frac{1}{2})}{\Pi(-\frac{1}{2})} \sinh^n a \int^{-1+, 1-} \frac{(\coth a - t)^{n+m}}{(t^2 - 1)^{m + \frac{1}{2}}} dt. \quad (9)$$

In (8) write $m' = -n - \frac{1}{2}, \quad n' = -m - \frac{1}{2},$

$$e^{(n+\frac{1}{2})\pi} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth a) = -\frac{1}{4i \sin(m+\frac{1}{2})\pi} \frac{\Pi(-n-m-1)}{\Pi(-m-\frac{1}{2})} \sinh^{n+\frac{1}{2}} a \times \int^{-1+, 1-} \frac{2^{m+\frac{1}{2}} (\coth a - t)^{n+m}}{(t^2 - 1)^{m+\frac{1}{2}}} dt. \quad (10)$$

Now $\Pi(m - \frac{1}{2}) \Pi(-m - \frac{1}{2}) \sin(m + \frac{1}{2})\pi = \pi,$

and $\Pi(-\frac{1}{2}) = \sqrt{\pi}.$

Hence, from (9) and (10),

$$e^{(n+\frac{1}{2})\pi} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth a) = \sqrt{\left(\frac{\pi}{2}\right)} \Pi(-m-n-1) \sqrt{(\sinh a)} P_n^m(\cosh a), \quad (11)$$

provided that the rules determining the phases in the integrals in equations (9) and (10) are the same.

This can be shown to be the case as long as the real part of $\cosh a$ is

positive, so that the line joining e^α and $e^{-\alpha}$ passes to the right of the origin (Fig. 1).

FIG. 1.

If, however, the real part of μ is negative, the definition of $P_n^m(\cosh \alpha)$ implies loops as in Fig. 2b passing to the right of the origin, whilst the definition of $Q_n^{m'}(\coth \alpha)$ implies loops, which do not surround the origin, as in Fig. 2a. As the two paths can not be reconciled, the equation (11) can not be proved in this case. Accordingly it must be restricted to values of α such that the real parts of $\cosh \alpha$ and $\coth \alpha$ are positive. With this proviso we can write (11) in the form

$$P_n^m(\cosh \alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\Pi(-m-n-1)} \frac{e^{(n+\frac{1}{2})i\pi}}{\sqrt{\sinh \alpha}} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth \alpha). \quad (12)$$

By substituting $\cosh \alpha$ for $\coth \alpha$, m for $-n-\frac{1}{2}$, and n for $-m-\frac{1}{2}$, we get

$$e^{-m\pi} Q_n^m(\cosh \alpha) = \sqrt{\left(\frac{\pi}{2}\right)} \Pi(m+n) \frac{1}{\sqrt{\sinh \alpha}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth \alpha), \quad (13)$$

which is the relation (1).

FIG. 2a.

FIG. 2b.

5. The definition adopted for Q in equation (8) suffers from the drawback that it makes Q complex for real values of α . It would be advantageous to take a new definition of Q , so that

$$Q_n^m(\cosh \alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Pi(m+n)}{\sqrt{\sinh \alpha}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth \alpha)$$

could be substituted for (1), when the real parts of $\cosh a$ and $\coth a$ were positive. Hobson's notation has been retained however in what follows.

6. The limitation of the scope of the relations (12) and (13) may be explained by a reference to the domains of $\mu (\equiv \cosh a)$ and of $\nu (\equiv \coth a)$ assumed in the definitions (7) and (8). In each case (Fig. 3) there is a

FIG. 3. $(\mu^2 - 1)(\nu^2 - 1) = 1$.

cross-cut from 1 to $-\infty$ along the real axis. Now the imaginary axis in the domain of μ corresponds to that portion of the real axis in the domain of ν which lies between 0 and 1. Accordingly the analytic functions which are identical in the first quadrant are "continued" across boundaries which do not correspond with each other, and there is no reason to expect identity in the second quadrant.

7. Formulae connecting P 's and Q 's of Associated Orders and Ranks.

Legendre's equation is satisfied by $P_n^m(\mu)$, $P_{-n-1}^m(\mu)$, $P_n^{-m}(\mu)$, $P_{-n-1}^{-m}(\mu)$, and by the four corresponding Q functions.

The eight solutions are not independent: they can all be expressed in terms of any two.

The first two solutions are identical (Hobson, *l.c.*, p. 452), *i.e.*

$$P_n^m(\mu) = P_{-n-1}^m(\mu). \quad (14)$$

If this equation is transformed by equation (12) it becomes

$$\frac{1}{\Pi(-m-n-1)} e^{+(n+\frac{1}{2})\pi} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu) = \frac{1}{\Pi(-m+n)} e^{-(n+\frac{1}{2})\pi} Q_{-m-\frac{1}{2}}^{+n+\frac{1}{2}}(\nu),$$

and by writing m for $-(n+\frac{1}{2})$, n for $-(m+\frac{1}{2})$, μ for ν , we find

$$\frac{1}{\Pi(m+n)} e^{-m\pi} Q_n^m(\mu) = \frac{1}{\Pi(n-m)} e^{+m\pi} Q_n^{-m}(\mu), \quad (15)$$

which is Hobson's equation (12).

Again, Hobson's equation (18),

$$P_n^m(\mu) = \frac{e^{-m\pi}}{\pi \cos n\pi} \{ Q_n^m(\mu) \sin(n+m)\pi - Q_{-n-1}^m(\mu) \sin(n-m)\pi \}, \quad (16)$$

transforms by a like process into

$$e^{-m\pi} Q_n^m(\mu) = \frac{\pi}{2 \sin m\pi} \left\{ P_n^m(\mu) - \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \right\}, \quad (17)$$

which is his equation (19).

It should be noted that the transformation of equations (12) and (13) cannot be applied to the relations between Legendre functions of $+\mu$ and $-\mu$, as those equations only hold good when the real parts of μ and ν are positive. As might be anticipated from Fig. 2, the Q 's of the parameters $+\mu$ and $-\mu$ are more closely associated than the P 's. Actually

$$Q_n^m(-\mu) = -e^{\pm n\pi} Q_n^m(\mu),$$

$$\text{whilst} \quad P_n^m(-\mu) = e^{\mp n\pi} P_n^m(\mu) - \frac{2 \sin(n+m)\pi}{\pi} e^{-m\pi} Q_n^m(\mu).$$

8. Expansion in Series.

Many ways of expressing Legendre's functions as hypergeometric series are known. It is obvious that when our transformation formulæ are applied to any one of these expansions another will be found.

Two examples will suffice.

Hobson's equation (5),

$$P_n^m(\mu) = \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1} \right)^{\frac{1}{2}m} F \left(-n, n+1, 1-m, \frac{1-\mu}{2} \right), \quad (18)$$

is transformed into his equation (57), which may be written

$$\begin{aligned} & e^{-m\pi} Q_n^m(\cosh a) \\ &= \frac{\Pi(-\frac{1}{2}) \Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{e^{-(n+\frac{1}{2})a}}{\sqrt{(2 \sinh a)}} F \left(m+\frac{1}{2}, -m+\frac{1}{2}, n+\frac{3}{2}, \frac{-1}{e^{2a}-1} \right). \end{aligned} \quad (19)$$

Again, Hobson's equation (10),

$$e^{-m\pi i} Q_n^m(\mu) = \frac{\Pi(-\frac{1}{2}) \Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^{n+1} \mu^{n+m+1}} F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{\mu^2}\right), \quad (20)$$

transforms into

$$P_n^m(\mu) = \frac{2^m}{\Pi(-m)} \frac{\mu^{m+n}}{(\mu^2-1)^{\frac{1}{2}m}} F\left(-\frac{n+m-1}{2}, -\frac{n+m}{2}, 1-m, \frac{\mu^2-1}{\mu^2}\right). \quad (21)$$

By means of the known relations between hypergeometric functions, the last series can be shown to be equivalent to the two series in powers of μ^{-2} in Hobson's equation (22).

9. Definite Integrals along Real Paths.

We shall confine our attention to the cases in which $\mu \equiv \cosh \alpha$ is real and greater than unity. The formulæ are written down seriatim for convenience of comparison. Their sources will be mentioned later. There are four groups distinguished by different types of integrand.

Group I.—

$$P_n^{-m}(\cosh \alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\Pi(m-\frac{1}{2}) \sinh^m \alpha} \int_0^\alpha (\cosh \alpha - \cosh \phi)^{m-\frac{1}{2}} \cosh(n+\frac{1}{2}) \phi d\phi, \quad (22)$$

where $m+\frac{1}{2}$ is positive ;

$$P_n^{-m}(\cosh \alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Pi(m-\frac{1}{2}) \sinh^m \alpha}{\Pi(m+n) \Pi(m-n-1)} \int_0^\infty \frac{\cosh(n+\frac{1}{2}) \phi}{(\cosh \alpha + \cosh \phi)^{m+\frac{1}{2}}} d\phi, \quad (23)$$

where $m-n$ and $m+n+1$ are positive ;

$$e^{m\pi i} Q_n^{-m}(\cosh \alpha) = \sqrt{\left(\frac{\pi}{2}\right)} \frac{1}{\Pi(m-\frac{1}{2}) \sinh^m \alpha} \int_\alpha^\infty (\cosh \phi - \cosh \alpha)^{m-\frac{1}{2}} e^{-(n+\frac{1}{2}) \phi} d\phi, \quad (24)$$

where $n+\frac{3}{2} > m+\frac{1}{2} > 0$;

$$e^{-m\pi i} Q_n^m(\cosh \alpha) = \frac{\Pi(m-\frac{1}{2})}{\sqrt{(2\pi)}} \sinh^m \alpha \left\{ \int_0^\pi \frac{\cos(n+\frac{1}{2}) \psi d\psi}{(\cosh \alpha - \cos \psi)^{m+\frac{1}{2}}} - \cos n\pi \int_0^\infty \frac{e^{-(n+\frac{1}{2}) \phi}}{(\cosh \alpha + \cosh \phi)^{m+\frac{1}{2}}} d\phi \right\}, \quad (25)$$

where $1+n+m > 0$.

Group II.—

$$e^{-m\pi i} Q_n^m(\cosh \alpha) = \frac{\Pi(m+n)}{\Pi(n)} \int_0^\beta (\cosh \alpha - \sinh \alpha \cosh \phi)^n \cosh m\phi d\phi,$$

where $n+1$ is positive and β is defined by $\cosh \beta = \coth \alpha$; (26)

$$e^{-m\pi i} Q_n^m(\cosh \alpha) = \frac{\Pi(n)}{\Pi(n-m)} \int_0^\infty \frac{\cosh m\phi}{(\cosh \alpha + \sinh \alpha \cosh \phi)^{n+1}} d\phi,$$

where $n+m+1$, $n-m+1$ are positive; (27)

$$P_n^{-m}(\cosh \alpha) = \frac{1}{\Pi(n) \Pi(m-n-1)} \int_\beta^\infty (\cosh \phi \sinh \alpha - \cosh \alpha)^n e^{-m\phi} d\phi,$$

where $m > n$, $n+1 > 0$; (28)

$$P_n^{-m}(\cosh \alpha) = \frac{1}{\pi} \frac{\Pi(-1-n)}{\Pi(m-1-n)} \left\{ \int_0^\pi (\cosh \alpha - \sinh \alpha \cos \psi)^n \cos m\psi d\psi \right. \\ \left. - \sin m\pi \int_0^\infty (\cosh \alpha + \sinh \alpha \cosh \psi)^n e^{-m\psi} d\psi \right\},$$

where $m > n$. (29)

Group III.—

$$P_n^{-m}(\cosh \alpha) = \frac{\sinh^m \alpha}{2^m \sqrt{\pi} \cdot \Pi(m-\frac{1}{2})} \int_0^\pi (\cosh \alpha + \sinh \alpha \cos u)^{n-m} \sin^{2m} u du,$$

where $m+\frac{1}{2}$ is positive; (30)

$$P_n^{-m}(\cosh \alpha) \\ = \frac{1}{\sqrt{\pi}} \frac{2^m \Pi(m-\frac{1}{2})}{\Pi(m+n) \Pi(m-n-1) \sinh^m \alpha} \int_\beta^\infty \frac{(\cosh u \sinh \alpha - \cosh \alpha)^{n+m}}{\sinh^{2m} u} du,$$

where $m-n$ and $m+n+1$ are positive; (31)

$$e^{m\pi i} Q_n^{-m}(\cosh \alpha) = \sqrt{\pi} \frac{\sinh^m \alpha}{2^m \Pi(m-\frac{1}{2})} \int_0^\beta (\cosh \alpha - \sinh \alpha \cosh u)^{n-m} \sinh^{2m} u du,$$

where $n+\frac{3}{2} > m+\frac{1}{2} > 0$. (32)

Group IV.—

$$e^{-m\pi i} Q_n^m(\cosh \alpha) = \frac{\sinh^m \alpha}{2^{n+1}} \frac{\Pi(m+n)}{\Pi(n)} \int_0^\pi \frac{\sin^{2n+1} u}{(\cosh \alpha + \cos u)^{n+m+1}} du,$$

where $n+1$ is positive ; (33)

$$e^{-m\pi i} Q_n^m(\cosh \alpha) = 2^n \frac{\Pi(n) \sinh^m \alpha}{\Pi(n-m)} \int_\alpha^\infty \frac{(\cosh u - \cosh \alpha)^{n-m}}{\sinh^{2n+1} u} du,$$

where $n+m+1$, $n-m+1$ are positive ; (34)

$$P_n^{-m}(\cosh \alpha) = \frac{1}{2^n} \frac{\sinh^{-m} \alpha}{\Pi(n) \Pi(m-n-1)} \int_0^\alpha \frac{\sinh^{2n+1} u}{(\cosh \alpha - \cosh u)^{n-m+1}} du,$$

where $m > n$, $n+1 > 0$. (35)

The formulæ of Group I transform into those of Group II by applying formula (1), and then writing m for $-(n+\frac{1}{2})$, n for $-(m+\frac{1}{2})$, $\cosh \alpha$ for $\coth \alpha$.

The formulæ of Group IV can be derived in the same way from Group III.

Group III is derived from Group I by an algebraic process. For example, in (22), we write

$$e^\phi = \cosh \alpha + \sinh \alpha \cos u, \quad 2e^\phi (\cosh \alpha - \cosh \phi) = \sinh^2 \alpha \sin^2 u,$$

and obtain (30).

The formula (22) is (112) in Hobson's paper, (26) is his (92). (23) is not given by him except for the special case $m=0$ (111), but its transformation (27) is his (91). (24) is his (118), but he does not give the corresponding equation (28). The equation (29), which has been obtained in a previous paper,* is a generalization of Hobson's (83). The transformation (25) appears to be new.

Of the formulæ in Group III, (30), is Hobson's (67), (32) is his (87). The first formula in Group IV is also given by him (11). No equations corresponding to (25) and (29) occur in the third and fourth groups, as the substitution for e^ϕ in terms of α and u cannot be used with advantage when ϕ is imaginary.

10. Geometrical Interpretation.

The Legendre functions with argument greater than unity occur in mathematical physics as toroidal functions.

* *L.c.*, p. 109.

Toroidal coordinates may be defined as follows:—

The circular axis is a circle with centre O and radius a .

The straight axis ZZ' is a straight line through O perpendicular to the plane of the circular axis.

OK is a fixed radius of the circular axis.

P is any point.

The circular axis cuts the plane containing P and OZ in two points A and B , A being on the same side of OZ as P , B on the opposite side.

The toroidal coordinates of P are:—

α given by $PB/PA = e^\alpha$,

θ the angle BPA ,

ϕ the azimuth of POZ relative to KOZ .

The surfaces $\alpha = \text{const.}$ are anchor-rings,

$\theta = \text{const.}$ are spheres,

$\phi = \text{const.}$ are planes.

With this system of coordinates

$$V = \sqrt{(\cosh \alpha - \cos \theta)} P_{-\frac{1}{2}+n}^m(\cosh \alpha) \frac{\cos}{\sin} n\theta \frac{\cos}{\sin} m\phi, \quad (36)$$

is a solution of Laplace's equation.

Other solutions are obtained by changing the sign of m and by writing Q instead of P , but these solutions are not all independent. [See above, equations (14)–(17).]

Now invert the system with respect to the point K , and take $\sqrt{2}a$ as the radius of inversion. The circular axis inverts into a straight line OY through O ; the straight axis inverts into a circle equal to the original circular axis and in a plane at right angles thereto. The spheres $\theta = \text{const.}$ invert into planes, the planes $\phi = \text{const.}$ into spheres. The anchor rings $\alpha = \text{const.}$ will invert into anchor rings.

In the diagrams Figs. 4a, b, c, KK' is the common diameter of the two circular axes. Firm lines represent the original straight axis and circular axis and also sections of an anchor-ring by orthogonal planes. Broken lines represent the axes of the inverse system and sections of the corresponding anchor-ring. It will be seen that the anchor-ring and its inverse touch at points in the plane OYZ .

To find the parameter β of one of the inverse rings with reference to the new circular axis, we need only consider the circles in which it is

cut by the plane $\phi = 0$. (Fig. 4a.) These circles are concentric with

FIG. 4a.

the new circular axis. If OK cut the ring $a = \text{const.}$ at P_1, P_2 , and its inverse at Q_1, Q_2 , we have

$$KP_1 = \frac{2a}{e^a - 1}, \quad KP_2 = \frac{2a}{e^a + 1},$$

$$KQ_1 = (e^a - 1)a, \quad KQ_2 = (e^a + 1)a,$$

so that

$$e^\beta \equiv \frac{K'Q_1}{KQ_1} = \frac{KQ_2}{KQ_1} = \frac{e^a + 1}{e^a - 1} = \coth \frac{a}{2}.$$

Thus

$$\cosh \beta = \coth a.$$

Accordingly it appears that, if a, θ, ϕ are the toroidal coordinates of a point P with respect to the original circular axis and the plane ZOK , then

β, ϕ, θ are the toroidal coordinates of the inverse point Q with respect to the new circular axis and the plane YOK .

FIG. 4b.

FIG. 4c.

Now the general theory of inversion shows that, if V_P is a solution of

Laplace's equation in the domain of P , then

$$V'_Q \equiv \frac{KP}{\sqrt{2}a} \times V_P$$

is a solution in the domain of Q .

It can be shown that

$$KP = \sqrt{2} a \sqrt{\left(\frac{\cosh \alpha - \sinh \alpha \cos \phi}{\cosh \alpha - \cos \theta} \right)};$$

and therefore, when V_P has the special value given in (36),

$$V'_Q = \sqrt{(\cosh \alpha - \sinh \alpha \cos \phi)} P_{-\frac{1}{2}+n}^m(\cosh \alpha) \frac{\cos n\theta}{\sin} \frac{\cos m\phi}{\sin}.$$

If the transformation of equation (12) is introduced, and $\cosh \beta$ is written for $\coth \alpha$, we get

$$V'_Q = \sqrt{(\cosh \beta - \cos \phi)} \sqrt{\frac{2}{\pi}} \frac{e^{ni\pi}}{\Pi(-m-n-\frac{1}{2})} Q_{-\frac{1}{2}-m}^{-n}(\cosh \beta) \frac{\cos n\theta}{\sin} \frac{\cos m\phi}{\sin}, \quad (37)$$

as a solution of Laplace's equation in the domain of Q . Thus the formulæ which have been under discussion fall into place as an essential part of the theory of toroidal functions.

11. An Important Example.

As an example the formula for the reciprocal of the distance between two points P_1, P_2 with coordinates $(\alpha_1, \theta_1, \phi_1), (\alpha_2, \theta_2, \phi_2)$, such that $\alpha_1 < \alpha_2$, may be considered. In the notation of this paper

$$\begin{aligned} \frac{1}{P_1 P_2} &= \frac{(\cosh \alpha_1 - \cos \theta_1)^{\frac{1}{2}} (\cosh \alpha_2 - \cos \theta_2)^{\frac{1}{2}}}{a\pi} \\ &\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-)^m \frac{\Pi(n-m-\frac{1}{2})}{\Pi(n+m-\frac{1}{2})} P_{n-\frac{1}{2}}^m(\cosh \alpha_1) Q_{n-\frac{1}{2}}^m(\cosh \alpha_2) 2 \cos m(\phi_1 - \phi_2) 2 \cos n(\theta_1 - \theta_2), \quad (38) \end{aligned}$$

where the corresponding 2 in front of a cosine is dropped when m or n is zero.*

When the system is inverted with respect to K , the condition $\alpha_1 < \alpha_2$, taken in conjunction with the relations

$$\sinh \alpha_1 \sinh \beta_1 = 1 = \sinh \alpha_2 \sinh \beta_2,$$

leads to $\beta_1 > \beta_2$. The application of formulæ (12) and (13) leads to the interchange of the P 's and the Q 's on the right of equation (38), which is therefore transformed into another of the same type.

* Heine, *Kugelfunctionen*, 1881, Vol. 2, p. 287.