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93. [Prof. Hill's Eucl. v and vi., Definition of Ratios, and Incommensurables]

Author(s): E. Budden

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and thus the point $(x_0y_0z_0)$ can be screwed along this line by an infinitesimal screw about the axis of z of pitch $\frac{\beta i}{2} \frac{a+1}{a-1} = \frac{\beta}{2} \cot \frac{\phi}{2}$ (using the expressions for the velocities just found). So that, in particular, if the original motion be infinitesimal, this value of the pitch is (β/ϕ) as it ought to be.

In concluding, it may be well to point out wherein the three-dimensional results differ from the two-dimensional. In the two-dimensional case, we have only to ignore z and β in the above; so that the line bisecting at right angles the join of (xy) , $(x'y')$, or

$$(x' - x)(\eta - y_0) + (y' - y)(\xi - x_0) = 0,$$

reduces to

$$\frac{1}{a} \xi y - x \eta = 0,$$

which passes through the origin whatever the point (x, y) may be; thus the origin is a marked point of the body which does not move in the displacement. This is of course a familiar property, and is only mentioned now for comparison with the other results.

If we assume that the usual conditions for orthogonality of lines, etc., can be extended to space of n dimensions, merely by adding on terms of the same type; I have proved that when n is *odd* there is no real point at a finite distance which does not move in a general displacement of a rigid body; but if n is *even*, there is always at least one such point (cf. R. F. Muirhead, *Proc. Edin. Math. Soc.*, Vol. xvi., p. 70). Of course by specializing the motion we can always keep a certain number of points fixed; but if n be odd, and a point is kept fixed, then a line is thereby also fixed.

Before leaving this subject, I should like to remark that the use of coordinates such as (x, y) above (*circular coordinates*) is often advantageous in geometrical problems that involve metrical relations; thus, properties of the epicycloids, etc., have been very simply obtained from these coordinates by Prof. F. Morley (*Am. Journal of Maths.*, Vols. xiii., xvi.): see also Vol. i., No. 2, of the *Transactions of the American Mathematical Society*.

T. J. F.A. BROMWICH.

93. [K.] Prof. Hill's Eucl. v. and vi., Definition of Ratios, and Incommensurables.

I am glad to have the opportunity of showing that my proposal of *measure* as the definition of ratio is free from the difficulties suggested by Prof. Hill* (Review of Prof. Hill's Eucl. v. and vi. Vol. I. No. 24, p. 411.

ART. 1. If A and B are commensurable so that $A = sC$, $B = rC$, then the measure, $A|B$ say, is the rational fraction s/r , since $A = s/r \cdot B$.

When A and B are incommensurable we find, as in the decimal system of weighing, a terminating decimal $\mu_n = \omega \cdot a_1 a_2 \dots a_n$, such that, the decimal μ'_n being the same as μ_n with $a_n + 1$ for a_n , and μ the true measure $A|B$,

$$A = \mu B > \mu_n B < \mu'_n B.$$

The process is supposed to be carried far enough to show any required difference between μ and any different decimal ν .

ART. 2. *Theorem I.* If the measures $A|B$, $C|D$ are μ , ν , and if, r and s being integers, whenever $rA > sB$, $rC > sD$, and when $<$, $<$, then $A|B = C|D$.

*The less important points raised by Prof. Hill are passed over for lack of space, as well as a detailed account of measurement in Art. 1 justifying (a) in Art. 2.

Suppose $\mu > \nu$. Find (Prof. Hill's Prop vii.) integers r, s such that

$$\begin{aligned} r\mu &> s.1 > r\nu \dots\dots\dots (a) \\ \therefore r\mu B &> sB, \text{ but } r\nu D < sD, \dots\dots\dots (b) \\ \therefore rA &> sB, \text{ but } rC < sD, \text{ contrary to hypothesis;} \\ \therefore \mu &\nless \nu, \text{ similarly } \mu \lessdot \nu; \therefore \mu = \nu. \end{aligned}$$

Note i. Professor Hill uses the same argument (a) in his *Art. Phil. Trans.* Prop. 17; and $r\mu$ is intelligible as an operator in (b) just so far as it is intelligible as a number $> s$ in (a). Prop. vii is a direct result of Archimedes' Principle.

Note ii. Professor Hill's 3 sets of conditions $>$, $=$, and $<$, of Prop. viii. are reducible by Prop. vii. to the 2, $>$ and $<$, of this article.

ART. 3. *Theorem II.* If $A|B = \mu$, the scale $[A, B] = [\mu, 1]$.

For, r and s being integers,
whenever $rA > sB$, $r\mu B > sB$
 $\therefore r\mu > s.1$.

Similarly, whenever $rA < sB$, $r\mu < s.1$;
 $\therefore [A, B] = [\mu, 1]$

Now when A, B are commensurable $= aN, bN$ say, Prof. Hill proves, Prop. xi,

$$[A, B] = [a, b] = [a/b, 1];$$

and says (Art. 44), " a/b is taken as the measure of $a:b$ " i.e. $A:B$.

Similarly, we say, " μ is taken as the measure of $\mu:1$ " (or $A:B$).

ART. 4. *Theorem III.* Hence, the measure $A|B$ is also the measure of the ratio $A:B$.

Among the advantages of defining ratio by this measure $A|B$ are:

(i.) We can prove the test (Art. 2), and it is therefore scientifically wrong to assume it; it is much easier to apply, and greater and less ratios become intelligible.

ii. The definition is that assumed in Trigonometry, etc.

iii. A fourth Proportional can be at once constructed.

ART. 5. As to the whole question of incommensurables:—I must point out that (i.) approximations being barred, Euclid's Book V. applies only to those kinds of magnitude for which a 4th Proportional can be found without approximation; these are, rational numbers, and certain figures in Euclidean space; and (ii.) it is significant that in general* when the magnitudes A, B , only are given, the scale $[A, B]$ and Euclid's ratio $A:B$ are indeterminate; since, however, many terms of the multiple order $1.B, 1.A, 2B$, etc. have been found, the number of possible continuations, and therefore of possible scales or ratios, is still infinite.†

Arts. 1, 2 above, however, show that the difficulties are largely imaginary, and are in any case contained in Archimedes' Principle, assumed in Euclid's postulates, in V., Def. 5, 7, and in Prof. Hill's Prop. vii. Euclid's and Prof. Hill's systems are therefore equally guilty with others. The merit of Euclid's test lies in applying the method of limits once for all.

I ought to say that I am deeply indebted to the suggestiveness of Prof. Hill's Book, and in particular to Prop. vii., for much of the above note and indeed for the suggestion of the 'measure' definition.

E. BUDDEN.

* The exception rational numbers.

† For instance, to the 3rd term given above, A is only known to be some magnitude between $1.B$ and $2.B$; $\therefore$ the number of possible continuations is infinite, and so on.