174. The Criterion as to a Sequence Tending to a Limit
Author(s): E. B. Elliott
Source: *The Mathematical Gazette*, Vol. 3, No. 53 (Oct., 1905), pp. 236-237
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3602365
Accessed: 05-11-2015 18:08 UTC

The book is admirably written, and covers more ground than, from its moderate size and attractive print, one would at first imagine; nor has clearness been sacrificed for the sake of brevity, with perhaps one exception in Article 126, dealing with "centrifugal force." Here a diagram, in which the "centrifugal force" is shewn ranking with the real forces, may cause confusion, which the brief statement in the text may not fully remove.

The diagrams are remarkably neat and clear, and there are examples which, as mere algebraical and trigonometrical conundrums have been rigorously excluded, are really examples in mechanics.

On page 13 should not N. 53° E. be E. 53° N.? C. S. JACKSON.

Stereoscopic Views of Solid Geometry Figures. D. C. Heath & Co., Boston, U.S.A.

This set of slides, 93 in number, has been specially designed with reference to the 'Essentials of Solid Geometry' by Prof. W. Wells, a book which has been favourably noticed for the excellence of its text and the diagrams with which it is illustrated. The slides seem to be just what is wanted by a pupil temporarily puzzled as to the position and arrangement of the lines represented by those in the plane diagram. There is a future before the stereoscope in this department of education. Besides the direct use to a beginner of Bk. XI or an equivalent course, we believe that it is likely to act as a powerful incentive to a student to learn enough perspective to enable him to draw his own 'stereographs.'

E. M. LANGLEY.

MATHEMATICAL NOTES.

173. [**K. 13. a.**] The *Remarque Minuscule* (Note 167, *Gazette*, May 1905, p. 176) has also been made by Mr. Steggall (*Proc. Edin. Math. Soc.*, vol. **x**, 1892.

N. QUINT ('s-Gravenhage).

174. [**D. 2. a.**]. *The criterion as to a sequence tending to a limit.*

If $s_1, s_2, s_3, \ldots$ be the sequence of real or complex quantities, it does or does not tend to a limit according as it is or is not true that with every positive ϵ, assigned as small as we please, there can be associated a number n such that, for every p, $|s_n - s_{n+p}| < \epsilon$. The ordinary proof that the existence of a limit necessitates this inequality presents no difficulty, but proofs of the converse appear to me needlessly encumbered. The following, in two stages, seems conclusive.

(1) *If the sequence tends to no limit, a positive ϵ' can be taken so small that, whatever quantity s be, and whatever number n be, $|s - s_{n+p}| \nless \epsilon'$ for some p.*

For suppose this is not so. Then with any particular ϵ' goes at least one value of s such that, for some n and every p, $|s - s_{n+p}| < \epsilon'$. A value of s with this property for one ϵ' has it for every greater ϵ'. As ϵ' is diminished, it follows that no new s with the property can arise, but the range of values of s with the property may well diminish. We are supposing, however, that ϵ' cannot be so diminished, remaining assignable and positive, that all values of s cease to have the property. Let s' be one which retains it. Then we are told that, however small ϵ' be assigned, there is always some n such that, for every p, $|s' - s_{n+p}| < \epsilon'$; and this means that the sequence tends to the limit s', so that the supposition made is untenable.

(2) *If, however small ϵ be assigned, there is always some corresponding n such that, for every p, $|s_n - s_{n+p}| < \epsilon$, the sequence tends to a limit.*

For suppose the contrary. Then by (1) there is an assignable ϵ' such that with every value s and every m goes some q such that $|s - s_{m+q}| \nless \epsilon'$. Also our datum assures us that with the ϵ' in question goes some n such that, for every p, $|s_n - s_{n+p}| < \epsilon'$. In these two inequalities s and m may be chosen at will. Give them the values s_n and n, which are definite. Moreover, p may be chosen at will. Give it the value q, which is made definite by the definite choices of s and m. We thus get two inequalities,

$$|s_n - s_{n+q}| \nless \epsilon' \text{ and } |s_n - s_{n+q}| < \epsilon',$$

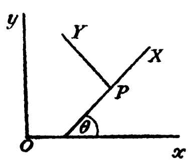
which are inconsistent. We have then supposed an impossibility.

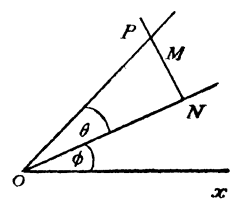
E. B. ELLIOTT.

175. [K. 20. a.]. *The Addition Formulae for Cosine and Sine.*

If PX makes an angle θ with Ox, then the projection of PX on Ox is $PX \cos\theta$. Hence if Oy, PY are obtained from Ox, PX by a turn through a right angle in the positive direction, it follows that the projection of PY on Ox is $PY \cos\left(\theta + \frac{\pi}{2}\right)$, and that the projection of PX on Oy is $PX \cos\left(\theta - \frac{\pi}{2}\right)$.

Now let ON make an angle ϕ with Ox and OP an angle θ with ON. Let NM be the direction obtained from ON by a turn through a right angle in





the positive direction. Let NM cut OP in P and take OP as the unit of length. Then the projection of OP on Ox is $\cos(\theta + \phi)$. But this projection is also the sum of the projections on Ox of ON, NP, that is of $\cos\theta$, $\cos\left(\theta - \frac{\pi}{2}\right)$. Now the projecting factors are $\cos\phi$, $\cos\left(\phi + \frac{\pi}{2}\right)$. Hence we have

$$\cos(\theta + \phi) = \cos\theta\cos\phi + \cos\left(\theta - \frac{\pi}{2}\right)\cos\left(\phi + \frac{\pi}{2}\right)$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The proof applies for angles of any size or sign.

Replacing ϕ by $\phi - \frac{\pi}{2}$, we get the addition formula for sine, and then, in the two addition formulae, replacing ϕ by $-\phi$, we get the formulae for the sine and cosine of $\theta - \phi$.

E. J. NANSON.

176. [D. 6. b.]. *The Fundamental Exponential Limit.*

Let the curve KJQ be the graph of $y = a^x$ and let $OI = OJ = 1$, so that $IA = a$. In IA or IA produced take any point B where $IB = b$ and through B draw DBC parallel to Ox cutting Oy in D and the graph in C. Then if $DC = c$, we have $b = a^c$, and therefore $b^x = a^{cx}$. Hence if through any point P on the graph JBP of $y = b^x$ MPQ is drawn parallel to Ox cutting Oy in M and the graph of $y = a^x$ in Q, then $MQ = c \, . \, MP$.

Thus from the graph of any one exponential a^x that of any other exponential b^x can be deduced by cutting all the ordinates to y in the proper ratio. Conversely, whatever ratio is used, the graph of some exponential is obtained. By properly choosing the ratio we can therefore make the derived graph have any slope we please at J. There must then be some value of b which gives unit slope at J. Denote this value of b by e, then from the definition of a tangent it follows that

$$\operatorname*{Lt}_{x=0} (e^x - 1)/x = 1.$$