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SOME APPLICATIONS OF THE THEORY OF ASSEMBLAGES.

BY ARNOLD EMCH.

(Presented to the Am. Math. Soc., 28th December, 1901.)

1. In order to be able to apply the theory of real numbers to measurable quantities,* in particular to linear magnitudes, it is convenient to make use of the theory of groups (displacements). Thus, if $B, C, D, \dots, Y$ are successive points on a continuous line between the extremities A and Z ,

$$AB + BC + CD + \dots + YZ = AZ,$$

where $AB, BC, \dots$ are arcs or consecutive displacements on the given line. Assuming $AB = u$ as a unit-displacement, then u may be repeated a certain number of times, say a times, so that

$$au < AZ < (a+1)u.$$

Designating the remainder YZ by u_1 ,

$$AZ - au = u_1.$$

With u_1 as a second unit-displacement operate on the original unit u in exactly the same manner as before on AZ . If b is the corresponding positive integer,

$$bu_1 < u < (b+1)u_1.$$

Continuing this process until it closes, or else indefinitely, the original displacement AZ measured by the unit-displacement $u=1$ becomes

$$AZ = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots,$$

and this is called the length of the line AZ . If the process closes with a finite number of terms, AZ is a rational number, if it does not close,

$$AZ = \lim \left(a + \frac{1}{b + \dots} \right)$$

is an irrational number.

This so-called *Euclidian algorithm* may also be applied to measure the length of a closed curve. From the form of the algorithm it is apparent that the length of every finite continuous line is represented by some real positive number, and conversely, every positive real number may be represented by the length of some continuous line. The theory may, of course, be immediately extended to negative real numbers. For, from $AB + BA = 0$ it follows $BA = -AB$, and from this standpoint negative and positive real quantities do not present essential differences in their treatment.

2. I shall now consider a circle† whose perimeter may be any real positive number. To prove this, let S be any real number, then the perimeter of a circle will be S if its radius is $\frac{S}{2\pi}$. Choose now any unit-displacement $u < S$ on the circle and suppose that n_1 is a positive integer, so that

$$n_1 u < S < (n_1 + 1)u.$$

* See in this connection,

Lagrange: *Lectures on Elementary Mathematics*, p. 3; translation by Th. J. Carnack.

Dedekind: *Essays on the Theory of Numbers*, pp. 1-27; translation by Prof. Beeman.

E. Borel, *Leçons sur la Théorie des Fonctions*, vol. i., p. 111.

† Any other closed curve would do; the circle has been chosen on account of its simplicity.

Put the remainder $S - n_1 u = u_1$, then $u_1 < u$. With u_1 as a new unit repeat the previous process, so that if n_2 is another positive integer

$$n_2 u_1 < S < (n_2 + 1) u_1.$$

Clearly $n_2 > n_1$. Put $S - n_2 u_1 = u_2$, where $u_2 < u_1$, and so forth. In this manner we get

$$\begin{aligned} S &= n_1 u + u_1 \\ S &= n_2 u_1 + u_2 \\ S &= n_3 u_2 + u_3 \\ &\dots\dots\dots \\ S &= n_k u_{k-1} + u_k, \\ &\dots\dots\dots \end{aligned}$$

where

$$u > u_1 > u_2 > \dots > u_k > \dots,$$

and

$$n_1 < n_2 < n_3 < \dots < n_{k+1} < \dots$$

Eliminating $u_1, u_2, u_3, \dots, u_{k-1}$ between the first k equations we obtain

$$S = \frac{u_k + (-1)^{k+1} n_1 n_2 n_3 \dots n_k u}{1 - n_k + n_k n_{k-1} - \dots + (-1)^{k+1} n_2 n_3 \dots n_k},$$

or

$$S = \frac{u_k + (-1)^{k+1} \frac{u_k}{n_1 n_2 n_3 \dots n_k}}{\frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \dots + (-1)^{k+1} \frac{1}{n_1 n_2 n_3 \dots n_k}}.$$

If this algorithm is limited, then S is rational, if it is unlimited, $\lim (n_k) = \infty$, $\lim (u_k) = 0$, and

$$(I) \quad S = \frac{u}{\frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \dots}.$$

If $u = 1$ and $S > 1$ we have

$$S = \frac{1}{\frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \dots}.$$

If S is a real number, $\frac{1}{S}$ is also a real number, so that for $u = 1$ and $S < 1$ we have

$$(II) \quad S = \frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \dots.*$$

Every real number may thus be represented by I. or II.

As examples I mention

$$\begin{aligned} e &= \frac{1}{\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \dots}, \\ \frac{e}{e-1} &= \frac{1}{\frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots}. \end{aligned}$$

Applying the algorithm to the Ludolphian number π , it is found

$$\pi = \frac{1}{\frac{1}{3} - \frac{1}{3 \cdot 22} + \frac{1}{3 \cdot 22 \cdot 118} - \frac{1}{3 \cdot 22 \cdot 118 \cdot 390} + \dots}.$$

In this infinite expression it would be very interesting if the law of its general term could be established.

* See an interesting note of E. Maillet, *Comptes Rendus*, vol. cxxxiii., No. 20, pp. 782-784. "Sur les équations différentielles rationnelles."

3. Geometrically, if the unit-displacement u is repeated indefinitely, two cases may occur. The polygon formed by the extremities of all displacements closes, or it does not close. It closes when $\frac{S}{u}$ is a rational number, for if $\frac{S}{u} = \frac{m}{n}$, where m and n are positive integers with no common factors, $nS = mu$, i.e. if u is repeated m times, the circle is described n times. Hence, after m displacements u , we arrive at A , the starting point. If $\frac{S}{u}$ is irrational we easily conclude from our algorithm that after

$$n_1 + n_1 n_2 + n_1 n_2 n_3 + \dots + n_1 n_2 n_3 \dots n_k$$

displacements u we arrive at a point A_k so that

$$AA_k = n_k u_{k-1}; S - n_k u_{k-1} = u_k < u_{k-1}.$$

As

$$u > u_1 > u_2 > \dots > u_k > \dots,$$

we can make u_k as small as we please by making k sufficiently great.

Hence we can make

$$A_k A < \delta,$$

δ being arbitrarily small.

The points of the infinite polygon come therefore arbitrarily close to the starting-point A . Now, any point of the polygon may be chosen as a starting-point and the same conclusion holds for this point. Hence the theorem: *The vertices of a polygon of an infinite number of equal sides inscribed into a circle form an enumerable system of points which are everywhere dense.*

UNIVERSITY OF COLORADO,
December, 1901.

NOTES ON CONICS IN AREALS.

IN a recent number of the *Proceedings of the Cambridge Philosophical Society* (vol. x., 1900, p. 358), I have considered at some length the problem of reducing conics and quadrics to principal axes. Since that note was published I have succeeded in effecting the reduction by means of more familiar methods; the recent publication of Mr. Muggeridge's notes on Areal seems to lead up to mine.

The basis of my investigation is the following theorem, which should be familiar to most readers: If a quadratic form S is transformed to T by means of any linear substitution (whose determinant is M), then the determinant of T is M^2 times that of S ; or, symbolically,

$$|T| = |S| \times M^2.$$

(Salmon, *Conics*, 1879, note, p. 335.)

Suppose now that we wish to pass from areal coordinates (x, y, z) to rectangular (ξ, η) ; and let (ξ_1, η_1) , (ξ_2, η_2) , (ξ_3, η_3) be the rectangular coordinates of the angular points of the triangle of reference. Then we have the substitution

$$\xi = x\xi_1 + y\xi_2 + z\xi_3, \quad \eta = x\eta_1 + y\eta_2 + z\eta_3, \quad 1 = x + y + z.$$

Thus the determinant of the substitution is

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ 1 & 1 & 1 \end{vmatrix} = 2\Delta,$$

if Δ is the area of the triangle of reference.