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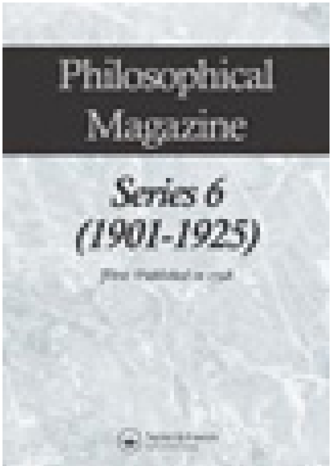

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XI. On the electron theory of matter and on radiation

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[SIXTH SERIES.]

FEBRUARY 1907.

XI. *On the Electron Theory of Matter and on Radiation.*

By G. A. Schott, *B.A.*, *B.Sc.*, *University College of Wales, Aberystwyth* *.

§ 1. ONE of the most important problems of the Electron Theory of Matter is to account for the spectra emitted by the several elements; the solution of this problem, rather than of any other, seems likely to lead to the construction of a working model of the atom. By Electron Theory of Matter I mean any theory which assumes matter to consist of electrical charges, acting upon each other with electromagnetic forces only. For stability it is necessary to have both positive and negative charges, and one or both must be in more or less rapid motion. Since the atom is permanent, or very nearly so, the orbits of all the charges must be closed and of atomic dimensions; since it is electrically neutral, except when ionized, positive and negative charges must be present in equal amount. There is no need at present to distinguish between the two possible alternatives: (1) the positive electricity constitutes a sphere of uniform electrification of atomic size, the negative electricity exists as corpuscles (negative electrons) moving inside the positive sphere (J. J. Thomson); (2) both positive and negative electricity exist as discrete charges—electrons—moving in closed orbits of atomic size. The following discussion applies to both types of theory, except where special exception is made.

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In the present paper we shall study the radiation from a ring of electrons in a controlling field, both when it is in steady motion and when it is disturbed. The general case of a system of rings is much more complicated; it can be studied by the same method, but for the sake of simplicity it will not be considered here.

It may be said at once that a single ring cannot be made to account for spectrum series or bands. It follows that the single ring cannot serve as a model of an atom; nevertheless its study is useful because it throws much light on the conditions which such a model must satisfy in order to account for spectra. They are three in number: (1) the electromagnetic waves emitted by the disturbed ring and received by a stationary observer must be of sufficient intensity to give observable lines; (2) their frequencies must lie within the limits corresponding to the spectrum; (3) they must be given by a formula, such as that of Deslandres for bands, or these of Balmer, Rydberg, or Kayser and Runge for series, and this formula must be satisfied within very narrow limits for every line.

Although the last condition is much the most difficult to study, since it requires us to write down and solve more or less complicated frequency equations, it is the only one of the three on which to my knowledge any work has been done. In the present paper we shall only consider the first two conditions; it will be found that they can only be satisfied for a number of the waves emitted by a single ring too small to account for the lines of even one series. Hence it is useless in this case to study the third condition at all. In this investigation we shall require a number of experimental data, which are known only very roughly: for instance, the intensities of strong and weak spectrum-lines, the ionization in a gas giving a line spectrum, the time for which a free ion exists on the average; but the margin of error is so large, that a knowledge of the order of magnitude of these quantities is quite sufficient for our purpose. For instance, any one of these quantities may be estimated ten times too large or small without affecting the conclusions of § 14. Nevertheless the estimates of the energy radiated per second per ionized ring given in §§ 15, 16 agree so nearly with those of the energy radiated in spectrum lines given in § 10, that the ideal ring of § 11 in this respect is a satisfactory model of radiating systems, such as we find in flames or vacuum-tubes.

§ 2. A first difficulty in a theory of this type is to explain how it is that a system of electrons in orbital motion has a definite structure at all. This difficulty I have examined in a previous communication*; it is there shown that, if the electron be supposed to expand, and n such electrons be moving in a circle at equal distances apart, any one of them is subject to two forces along the tangent to its orbit: (1) the reaction which the æther exerts on it in consequence of radiation; (2) a pull which the rest of the ring exerts on it as a result of the expansion. According as either force is in excess the electron is retarded or accelerated until its velocity reaches a perfectly definite equilibrium value; this value depends mainly on n; changes in the field of force in which the ring moves, in its radius, in the mass of the electron, in its rate of expansion, produce no appreciable effect, unless they involve a change in the *order* of magnitude of the quantity which is changed.

We shall assume that the structure of the system is rendered definite by this cause.

§ 3. The following notation will be used:—

Charge of the electron $=e$ electrostatic units,

Mass $=m$, velocity $=v$, velocity of light $=\mathrm{C}$, $\frac{v}{\mathrm{C}}=\beta$, radius of curvature of the orbit $=\rho$.

Occasionally we shall use a moving system of axes of $(\xi\eta\zeta)$ the origin being at the electron, $\mathrm{O}\xi$ along the tangent to the orbit, $\mathrm{O}\eta$ towards the centre of curvature, $\mathrm{O}\zeta$ along the binormal. The components of the mechanical force on the electron in these directions will be denoted by T, P, S.

The energy radiated per second will be denoted by R, for a single electron or for a group of electrons.

The following numerical values will be useful:—

$e=3\cdot4\times10^{-10}$ (J. J. Thomson).

$\frac{e}{\mathrm{C}m}=1\cdot88\times10^{7}$ for a slowly moving negative electron (Kaufmann).

$\rho=10^{-8}$ cm., or thereabouts.

Kinetic energy of α particle of radium $=5\cdot9\times10^{-6}$ erg (Rutherford).

Kinetic energy of β particle of radium $=7\times10^{-8}$ erg (Rutherford),

both of course average estimates.

Limits of the spectrum between which series lines have been observed: $\lambda=10^{-5}$ cm. and $\lambda=8\times10^{-5}$ cm.

With a time of revolution equal to the period of these extreme rays in a circle of atomic radius, a body has velocities given by $\beta=\cdot0063$ and $\beta=\cdot0008$ respectively.

* Phil. Mag. [6] vol. xii. p. 21.

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§ 4. We shall first show that stray electrons cannot exist in a permanent atom; all electrons must be arranged in groups or rings.

An electron loses energy by radiation whenever it has acceleration; in fact

$$R=\frac{2Ce^2}{3(1-\beta^2)^2}\left\{\frac{\beta^4}{\rho^2}+\frac{\dot{\beta}^2}{C^2(1-\beta^2)}\right\}.$$

This expression was first given by Liénard; it has been confirmed by different methods by Abraham and by myself. Both terms in R are positive; its least value is therefore given by $\dot{\beta}=0$ or $\beta=$constant. We find in this case, with the numerical values of § 3,

$$R=2{\cdot}4\times 10^7\frac{\beta^4}{(1-\beta^2)^2}\text{ erg/sec.},$$

for an orbit of atomic size, and greater in proportion for a smaller orbit. Comparing this with the value of the kinetic energy of an α or β particle, we see that β is necessarily extremely small. Hence we may calculate its energy and the radius of the orbit as if it obeyed the ordinary laws of mechanics.

Its kinetic energy is $E=\frac{1}{2}C^2m\beta^2$.

Let it move in a circle of radius ρ under the action of an opposite charge νe. If the charge be central we have

$$\frac{C^2m\beta^2}{\rho}=\frac{\nu e^2}{\rho^2}.$$

If it be a sphere of uniform electrification of radius b,

$$\frac{C^2m\beta^2}{\rho}=\frac{\nu e^2\rho}{b^3}.$$

In any case, $$C^2m\beta^2\geq\frac{\nu e^2\rho^2}{b^3}.$$

To the present approximation $R=\dfrac{2Ce^2\beta^4}{3\rho^2}$; hence

$$\frac{R}{E}=\frac{4e^2\beta^2}{3Cm\rho^2}\geq\frac{4\nu e^4}{3C^3m^2b^3}\geq\nu.2.10^9.$$

In other words, a single electron cannot move in a circle of atomic radius for any appreciable time. It will either pass out of the atom, fall into one of the groups of electrons already existing in the atom, or, as its velocity diminishes

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owing to radiation, fall into the centre of attraction and correspondingly reduce its strength. In every case we may treat the system as one of which all the electrons are arranged in groups. It remains to examine the radiation from a group of electrons; it is of two types: (1) that due to the permanent motion; (2) that due to disturbances produced by causes external to the group.

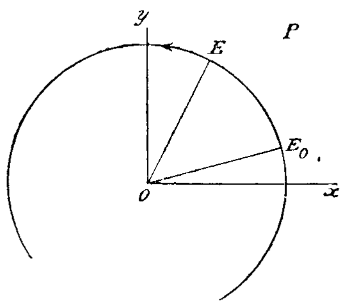
§ 5. J. J. Thomson* has shown that each of n equidistant electrons, moving uniformly in a circle, radiates energy at a very much slower rate than a single electron does for the same orbit and same velocity. He finds

$$\mathrm{R}=\frac{2\mathrm{C}e^2\beta^2}{\rho^2}\frac{n^3(n+1)}{2n+1}\frac{(n\beta)^{2n}}{\lfloor 2n}.$$

This expression is only true for small values of $n\beta$. We require one which holds for all values of β less than unity.

Strictly speaking, in all that follows β must be less than unity by a small amount depending on the ratio of the radius of the electron to that of the orbit. For the negative electron this ratio is of the order 10^{-5}; if $\beta=\cdot 99$ the results would be in error to about one part in one thousand, if $\beta=\cdot 999$ to one part in one hundred.

Let us consider the case of n equidistant electrons moving uniformly in a circle. Take the centre as origin, the axis



as Oz. The azimuth of the Oth electron E_0 may be taken to be $\omega t+\delta$, where ω is the angular velocity, so that $\beta=\omega\rho/\mathrm{C}$. The azimuth of the ith electron will be $\omega t+\delta+\dfrac{2\pi i}{n}$. Let the polar coordinates, referred to Oz as axis and xOz as initial plane, of a point P be (r, θ, ϕ); and let the components along the radius, meridian, and parallel at P be, of the electric

* Phil. Mag. [6] vol. vi. p. 681.

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force (P, Θ, Π), and of the magnetic force (Λ, M, N). At a point so distant that terms of order $\frac{\rho}{r^2}$ may be neglected in comparison with those of order $\frac{1}{r}$ I find

$$P=\Lambda=0,$$

$$\Theta=N=\frac{2e\beta}{r\rho}n^2\cot\theta\sum_{s=1}^{s=\infty}s\,J_{sn}(sn\beta\sin\theta)\sin sn\left[\omega\left(t-\frac{r}{C}\right)+\delta+\frac{\pi}{2}-\phi\right],$$

$$\Pi=-M=-\frac{2e\beta^2}{r\rho}n^2\sum_{s=1}^{s=\infty}s\,J'_{sn}(sn\beta\sin\theta)\cos sn\left[\omega\left(t-\frac{r}{C}\right)+\delta+\frac{\pi}{2}-\phi\right].$$

By Poynting's Theorem we easily find

$$R=\frac{2Ce^2\beta}{\rho^2}n^2\sum_{s=1}^{s=\infty}\left[sn\beta^2J'_{2sn}(2sn\beta)-s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)dx\right].$$

Using Duhamel's asymptotic value for the Bessel function

$$J_n(n\beta)=\frac{1}{\sqrt{2\pi n\gamma}}\exp.\,n\left(\gamma-\tfrac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right),\ \gamma=\sqrt{1-\beta^2},\ \beta<1,$$

we see that the series converges, and that for large values of n we may take the first term only.

When $n\beta$ is small we may content ourselves with the lowest power; in that case we get J. J. Thomson's result. Generally we cannot neglect the higher powers and must write

$$R=\frac{Ce^2\beta^2}{\rho^2}\sqrt{\frac{n^3\gamma}{2\pi}}\exp.\,n\left(\gamma-\tfrac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right)$$

approximately for n large.

For $n=1$ the series gives by direct summation

$$R=\frac{2Ce^2\beta^4}{3\rho^2(1-\beta^2)^2},$$

in agreement with the general result of § 4.

The reduction in the intensity of the radiation in the present case is clearly due to interference between the waves emitted by the several electrons of the ring. Each electron may be supposed to emit the same amount of energy as if it alone were present; but to absorb a great portion of the energy of the same type emitted by the remaining electrons of the ring. From this point of view, the reduction in the radiation from the electron is due to resonance. It is clear that the electron can in the same way absorb energy from any wave which passes it, provided the wave be properly attuned to the motion of the electron.

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The interference, and consequent reduction in radiation, is by no means confined to the case of uniform circular motion. For example, n electrons moving in the same ellipse will interfere, if their eccentric angles increase at the same uniform rate, and are in arithmetic progression.

§ 6. It is a very general assumption that the D lines of sodium can be attributed to steady motion radiation of an electron moving in a circle, as for instance in the elementary explanation of the Zeeman effect. We shall proceed to show that this is not so.

If possible suppose the D lines to be due to the steady motion of a circle of n electrons.

To account for the frequency of the D lines we must have $n\beta = \cdot 0010$ (§§ 3, 5). Hence $n\beta$ is so small that J. J. Thomson's approximation suffices. By §§ 3, 5,

$$R = \frac{2Ce^2}{\rho^2} \frac{n(n+1)(n\beta)^{2n+2}}{\underline{|2n+1}}$$

$$= 7\cdot 2 \,.\, 10^7 \,.\, 10^{-6(n+1)} \frac{n(n+1)}{\underline{|2n+1}}.$$

The kinetic energy, E, of the ring is $\frac{1}{2}C^2 mn\beta^2$. Hence

$$\frac{R}{E} = \frac{4e^2}{Cm\rho^2} \frac{n^2(n+1)(n\beta)^{2n}}{\underline{|2n+1}}$$

$$= 2\cdot 6 \,.\, 10^{14} \,.\, 10^{-6n} \frac{n^2(n+1)}{\underline{|2n+1}}$$

These formulæ give the following values :—

$n=1$,	2,	3,
$R=2\cdot 4\times 10^{-5}$,	$3\cdot 6\times 10^{-12}$,	$1\cdot 7\times 10^{-19}$,
$R/E=8\cdot 7\times 10^{+7}$,	26,	$1\cdot 7\times 10^{-6}$.

These numbers show that the whole energy would be dissipated by radiation for 1 electron in one millionth sec., for 2 in ·04 sec., for 3 in one-fifth year.

E. Wiedemann* has measured the radiation from the D lines and found it to be $13\cdot 45\times 10^{10}$ erg per sec. per gram. Assuming the number of molecules in 1 c.c. of hydrogen at 0° C. and 760 mm. to be 4×10^{19} (J. J. Thomson), and the sodium molecule to be monatomic, this gives $7\cdot 7\times 10^{-12}$ erg per sec. per atom, or $R=3\cdot 9\times 10^{-12}$ for each of the D lines. This is of the order of the second case above, $n=2$. In this case the atom can exist for at most ·04 sec., whatever the

* Drude, *Optik*, p. 487.

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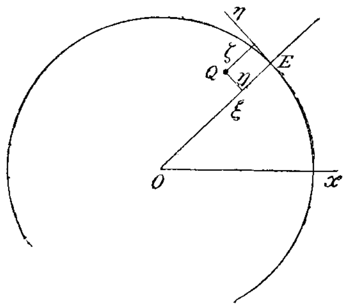
conditions, for the radiation now considered is continually taking place. This is obviously impossible.

Apart from this it has long been known that spectrum lines vary very much in intensity and width with the conditions under which they are produced, which in itself is a sufficient proof that their energy is derived from external sources.

§ 7. We shall now consider the disturbances possible in a ring of n equidistant electrons in uniform circular motion, and the vibrations which it emits in consequence. The kinematics has been very fully treated by Maxwell*.

Let (ξ, η, ζ) be the displacements from steady motion of the ith electron, with reference to the moving axes of § 5.



We may write

$$(\xi, \eta, \zeta) = (\mathbf{A}, \mathbf{B}\,\epsilon^{\sqrt{-1}\alpha}, \mathbf{C})\ \exp\left[-\kappa t + \sqrt{-1}\left(qt - k\frac{2\pi i}{n}\right)\right],$$

where real parts only are to be taken.

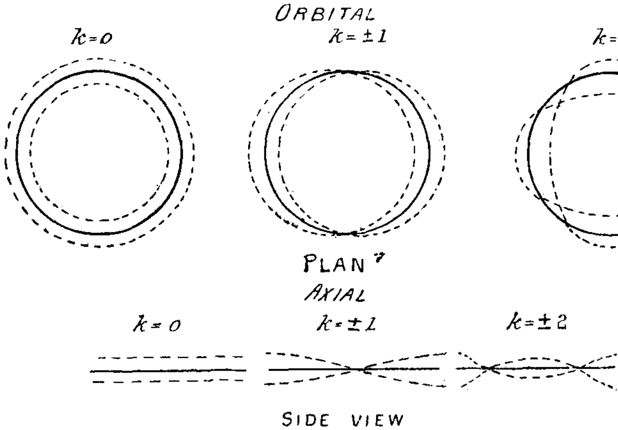
As we shall see, the axial vibration (ζ) is independent of the orbital vibration (ξ, η); hence, **A**, **B**, **C**, α may all be taken to be real. κ is a small quantity representing the damping; this we shall usually neglect. q is the relative frequency, as observed by an observer moving with the mean velocity ω of the ring. The electrons at any instant lie on a curve of k waves, with $2k$ nodes and loops; these waves travel round the ring with angular velocity q/k forward, their velocity relative to a stationary observer being $q/k + \omega$. Hence for the stationary observer the frequency is $q + k\omega$.

We may without loss of generality choose k to lie between $\pm\frac{n}{2}$. We shall call k the "class." The forms of the vibration for the chief classes are shown in the figures.

$k=0$ gives vibrations of the ring as a whole, with all the electrons in the same phase.

* Collected Papers, vol. i. pp. 321-323.

$k=\pm 1$, orbital, are to and fro vibrations in the plane of the ring; $k=\pm 1$, axial, are vibrations with one diameter fixed. In all cases positive values of k mean waves travelling forward, negative, backward.



We are not at present concerned with the forces which produce these vibrations, but only with the waves emitted, which we now proceed to examine.

§ 8. At a great distance from the ring I find

$$P=\Lambda=0,$$

$$\begin{aligned}\Theta=N= & -\frac{ne\cot\theta}{r\rho^2}A\,\epsilon^{\sqrt{-1}q\left(t-\frac{r}{C}\right)}\sum_{s=-\infty}^{s=\infty} ml\beta J_m(l\beta\sin\theta)\,\epsilon^{\sqrt{-1}.m\psi} \\ & +\frac{ne}{r\rho^2}\cos\theta.\sqrt{-1}.B\,\epsilon^{\sqrt{-1}\left\{q\left(t-\frac{r}{C}\right)+\alpha\right\}}\sum_{s=-\infty}^{s=\infty} l^2\beta^2 J'_m(l\beta\sin\theta)\epsilon^{\sqrt{-1}.m\psi} \\ & +\frac{ne}{r\rho^2}C\epsilon^{\sqrt{-1}.q\left(t-\frac{r}{C}\right)}\sum_{s=-\infty}^{s=\infty}\left(\frac{m\beta}{\sin\theta}-l\beta\sin\theta\right)l\beta J_m(l\beta\sin\theta)\epsilon^{\sqrt{-1}.m\psi},\end{aligned}$$

$$\begin{aligned}\Pi=-M= & -\frac{ne}{r\rho^2}\sqrt{-1}.A\epsilon^{\sqrt{-1}.q\left(t-\frac{r}{C}\right)}\sum_{s=-\infty}^{s=\infty} l^2\beta^2 J'_m(l\beta\sin\theta)\epsilon^{\sqrt{-1}.m\psi} \\ & +\frac{ne}{r\rho^2}B\epsilon^{\sqrt{-1}\left\{q\left(t-\frac{r}{C}\right)+\alpha\right\}}\sum_{s=-\infty}^{s=\infty}\left(\frac{m\beta}{\sin\theta}-l\beta^2\sin\theta\right)l\beta J_m(l\beta\sin\theta)\epsilon^{\sqrt{-1}m.\psi} \\ & -\frac{ne\beta}{r\rho^2}\cos\theta\sqrt{-1}.C\epsilon^{\sqrt{-1}.q\left(t-\frac{r}{C}\right)}\sum_{s=-\infty}^{s=\infty} l^2\beta^2 J'_m(l\beta\sin\theta).\epsilon^{\sqrt{-1}.m\psi},\end{aligned}$$

where $m=k+sn$, $l=\frac{q}{\omega}+k+sn$, $\psi=\omega\left(t-\frac{r}{C}\right)+\delta+\frac{\pi}{2}-\phi.$

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By using Poynting's Theorem we find for the radiation

$$R=\frac{Ce^2n^2}{2\rho^4}\Big\{A^2\sum_{s=-\infty}^{s=\infty} l\beta[l^2\beta^2J'_{2m}(2l\beta)-(m^2-l^2\beta^2)l\int_0^\beta J_{2m}(2lx)dx]$$

$$+B^2\sum_{s=-\infty}^{s=\infty} l\beta\Big[\frac{1-\beta^2}{4}l^2\beta^2J'_{2m}(2l\beta)+\frac{3+\beta^2}{8}l\beta J_{2m}(2l\beta)$$

$$+\Big\{\frac{1+\beta^2}{2}(m^2+l^2\beta^2)-2ml\beta^2-\frac{3+\beta^2}{8}\Big\}l\int_0^\beta J_{2m}(2lx)dx\Big]$$

$$-2AB\sin\alpha\sum_{s=-\infty}^{s=\infty} l\beta\Big[\frac{m-l\beta^2}{2}l\beta J_{2m}(2l\beta)+\frac{m+l\beta^2}{2}l\int_0^\beta J_{2m}(2lx)dx\Big]$$

$$+C^2\sum_{s=-\infty}^{s=\infty} l\beta[-\frac{1-\beta^2}{4}l^2\beta^2J'_{2m}(2l\beta)+\frac{1+3\beta^2}{8}l\beta J_{2m}(2l\beta)$$

$$+\Big\{\frac{1+\beta^2}{2}(m^2+l^2\beta^2)-2ml\beta^2-\frac{1+3\beta^2}{8}\Big\}l\int_0^\beta J_{2m}(2lx)dx]\Big\}.$$

The expressions for the forces show that at a great distance from the ring the electric and magnetic forces are transverse and at right angles. They consist of an infinite series of harmonic terms, the frequency of the sth harmonic being $q+(k+sn)\omega$. Since negative values of s occur, there are in reality two distinct series, one with positive and the other with negative frequencies. The amplitude of the sth harmonic is of the order $J_m(l\beta)$, and the corresponding term in the radiation of order $J_{2m}(2l\beta)$. For small values of $l\beta$, such as correspond to waves of light, the orders are $(\frac{1}{2}l\beta)^\mu/\underline{|\mu}$, and $(l\beta)^{2\mu}/\underline{|2\mu}$, where $\mu=\text{Mod.}\, m$.

The fact that a single vibration of the ring, corresponding to one degree of freedom of the ring, gives rise to an infinite series of harmonics is due to the presence of the æther, which possesses an infinite number of degrees of freedom.

§ 9. Let λ be the wave-length of the vibration considered; then

$$l\beta=\frac{2\pi\rho}{\lambda}$$

For the extreme ultraviolet, $\lambda=10^{-5}$ cm., $l\beta=\cdot0063$; for the extreme red, $\lambda=8\times10^{-5}$ cm., $l\beta=\cdot0008$. In this case we need only retain the lowest power of $l\beta$, and the harmonic of greatest amplitude, given by $s=0$, $\mu=\text{Mod.}\ k$.

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We find

$\mu > 0$

$$R = \frac{n^2(\mathbf{A}^2+\mathbf{B}^2 \mp 2\mathbf{AB}\sin\alpha+\beta^2\mathbf{C}^2)}{\rho^2}\frac{2\pi^2 Ce^2}{\lambda^2}\frac{\mu(\mu+1)}{\lfloor 2\mu+1}\left(\frac{2\pi\rho}{\lambda}\right)^{2\mu},$$

where the sign $\mp$ refers to k $\begin{smallmatrix}\text{positive}\\\text{negative}\end{smallmatrix}$,

$\mu = 0$

$$R = \frac{n^2(\beta^2\mathbf{B}^2+\mathbf{C}^2)}{\rho^2}\frac{4\pi^2 Ce^2}{3\lambda^2}\left(\frac{2\pi\rho}{\lambda}\right)^2.$$

In this case the terms involving $\mathbf{A}^2$, $\mathbf{AB}$ are of the next higher order.

Throughout the investigation terms of the orders $\frac{\mathbf{A}^2}{\rho^2}$, $\frac{2\pi\mathbf{A}^2}{\lambda\rho}$ have been neglected. Hence $\frac{\mathbf{A}}{\rho}\ldots$ must all be small.

The distance between consecutive electrons of the ring is $\frac{2\pi\rho}{n}$; except in very violent disturbances $\mathbf{A}\ldots$ can hardly be greater than a small fraction of this; that is $\frac{n\mathbf{A}}{2\pi\rho}$—call it σ—can hardly be greater than a small fraction.

The table gives values of R/σ^2 for three wave-lengths calculated with the numerical values of § 3.

Class, k =		0.	$\pm$1.	$\pm$2.	$\pm$3.	$\pm$4.
$\frac{R}{\sigma^2}$ for λ =	$6 . 10^{-5}$ cm.	·00058	·00029	$4{\cdot}8 . 10^{-11}$	$2{\cdot}5 . 10^{-18}$	$6{\cdot}4 . 10^{-26}$
	$3{\cdot}5 . 10^{-5}$,,	·0051	·0025	$1{\cdot}2 . 10^{-9}$	$1{\cdot}9 . 10^{-16}$	$1{\cdot}2 . 10^{-23}$
	$2 . 10^{-5}$,,	·046	·024	$3{\cdot}6 . 10^{-8}$	$1{\cdot}7 . 10^{-14}$	$3{\cdot}8 . 10^{-21}$

§ 10. It is very frequently assumed that every disturbance of a system of electrons shows itself by a line in the spectrum of the system. We shall now show that for a single permanent ring of electrons this is not the case, because many of the vibrations emitted are far too weak to affect the photographic plate appreciably.

We must first form an estimate of the least amount of energy which will enable a vibration just to produce an impression on the plate. Bearing in mind that the photometric intensity of a spectrum-line is indicated by a scale-number, from 1 to 10, we see that we must find out how the energy of a weak line 1 compares with that of a strong line 10.

Each of the D lines is numbered 10, and their energy may for our purpose be taken as a measure of that of every other line 10.

Eder and Valenta * give the times of exposure required to produce distinct photographic impressions under the same conditions of certain ultraviolet lines; the following are examples:—

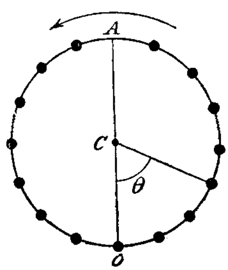
Mg lines, $\lambda=2936$, 2928, 2802, all 10, 15 sec.
Mg lines, $\lambda=3336-3329$, 2782, all 6, 90 sec.,
Zn lines, $\lambda=2063-2024$, all 1, 1200 sec.

Assuming the photographic effect to be proportional to the product of intensity by time of exposure, we see that the intensities are as $1:1/6:1/80$, practically as $1:(1{\cdot}6)^{-4}:(1{\cdot}6)^{-9}$.

Kayser † proposes to use a scale due to Jewell in which the intensities are as $1:2^{-4}:2^{-9}$. We shall not be far wrong if we take as our ratio, for 1 and 10, $1:400$. This makes the energy of the weakest observable line, 1, 10^{-14} erg per sec. per atom, the radiation in the D line being taken as 4.10^{-12} (§ 6).

In all probability this energy is underestimated, probably very much so for the ultraviolet; Pflüger ‡, from an investigation on the relative intensities of spectrum-lines by means of a thermopile, concludes that the extreme ultraviolet lines are enormously stronger than the visible lines, although their photographic effect is very much less. For example, he finds that the Mg line, $\lambda=280\,\mu\mu$, gives 970 scale-divisions deflexion, whilst the Mg line $\lambda=383\,\mu\mu$, of about the same photometric intensity, gives only 56 scale-divisions; this too although 280 is a single line, whilst 383 includes several neighbouring lines too near to separate.

§ 11. It is desirable that we should form some idea as to the values of σ to be expected for vibrations of different classes. For this purpose we shall consider the following problem:—



A ring of $n+1$ equidistant electrons is in steady orbital motion. An electron is suddenly expelled; required to find the subsequent motion.

To fix the ideas suppose n odd. After the expulsion of the electron we are left

* *Beiträge zur Photochemie*, p. 44.
† *Spectroscopie*, vol. i. p. 646.
‡ Drude, *Annalen*, xiii. p. 890.

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with a ring of n electrons, arranged as in the diagram, and revolving in the direction of the arrow.

This is not a possible state of steady motion for a ring of n electrons, because

(1) they are not equidistant;

(2) the angular velocity and radius correspond, not to n electrons, but to $n+1$.

When n is at all large, the angular velocity and radius are practically the same for n electrons as for $n+1$. The deviation (2) produces slight oscillations in velocity and radius about the values corresponding to n electrons; but they are very small compared with the oscillations due to (1). We shall for simplicity neglect them. Thus we treat the ring as one of n electrons initially displaced from the state of steady motion, with zero radial displacement and zero velocity.

For the angular displacement I find

$$\frac{2\pi}{n(n+1)}\Sigma(-1)^{k-1}\frac{q'\sin\left(qt-k\frac{2\pi i}{n}\right)}{(q+q')\sin(k\pi/n)},$$

and therefore for the angular velocity,

$$\frac{2\pi}{n(n+1)}\Sigma(-1)^{k-1}\frac{qq'\cos\left(qt-k\frac{2\pi i}{n}\right)}{(q+q')\sin(k\pi/n)},$$

where the summation is for all values of k between $\pm\frac{n}{2}$, excluding zero, and q, q' are the frequencies corresponding to classes k, $-k$. The effect of damping has been neglected, so that the amplitudes are approximate only for the initial stage of the disturbance.

It is to be remarked that, as in the case of Saturn's rings considered by Maxwell *, there are four frequencies for disturbances in the plane of the ring; for two of these the tangential displacement is a large multiple of the radial displacement, for the remaining two it is about twice as great; the values of q, q' belong to the first pair, because in our case the radial disturbance is negligible.

For vibrations of classes $\pm k$ we have initially

$$\sigma_0=\frac{q'}{(n+1)(q+q')\sin(k\pi/n)} \text{ for } k$$

$$=\frac{q}{(n+1)(q+q')\sin(k\pi n)} \text{ for } -k;$$

* Collected Papers, vol. i. p. 321.

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maximum kinetic energy

$$=\frac{\pi^2 m\rho^2 \cdot q^2 q'^2}{n(n+1)^2(q+q')^2 \sin^2(k\pi/n)} \text{ for } \pm k;$$

that is, $\pi^2 m\rho^2 q^2 \sigma_0{}^2/n$ for k, and $\pi^2 m\rho^2 q'^2 \sigma_0{}^2/n$ for $-k$. The value of σ_0 is clearly less than $\frac{1}{n\sin(k\pi/n)}$ for $\pm k$.

§ 12. The ideal problem of § 11 differs in two respects from an actual case of ionization of a ring. The expulsion of the electron cannot really be instantaneous, but must be more or less gradual; the disturbance produced will in reality be less violent than in the ideal case, the values of σ will be less, and probably the vibrations of higher classes will be weaker, not merely absolutely, but also relatively to those of lower classes. Again, the effect of damping on account of radiation has been neglected; in consequence the values of σ diminish with the time according to an exponential law, those of lower classes very rapidly, those of higher classes much more slowly. We must now consider the effect of damping on the radiation.

It has long been thought that the radiation from a gas, in a vacuum-tube or in a flame, is connected with the ionization of its atoms. Stark*, from his experiments on the Doppler effect of canal rays, concludes that the line-spectra are emitted by positive ions produced from atoms by the expulsion of a negative electron, and that band-spectra are emitted in the recombination of the positive ion and negative electron. We may compare a radiating gas to a system of a very large number of rings like that of § 11. Suppose that on the average a ring remains ionized for T seconds. Further, suppose that the energy of an ionized ring due to a vibration of class k and amplitude σ is $E=A\sigma^2$, and the corresponding radiation $R=B\sigma^2$. We have

$$\frac{dE}{dt}=-R,$$

whence
$$\frac{1}{E}\frac{dE}{dt}=-\frac{R}{E}=-\frac{B}{A}, \text{ a constant.}$$

Write $B/A=\gamma$; then

$$E=E_0\epsilon^{-\gamma t}, \quad R=R_0\epsilon^{-\gamma t},$$

where t is reckoned from the instant of ionization.

* *Physikalische Zeitschrift*, vi. p. 892; 'Nature,' lxxiii. pp. 78, 389, 533.

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The mean radiation from the gas per second per free ion is

$$\frac{1}{T}\int_0^T R\,dt = R_0 \frac{1-\epsilon^{-\gamma T}}{\gamma T}.$$

When the damping γ is very small, as for the vibrations of high class, the last factor is practically unity and the radiation is R_0. When the damping is large, as for vibrations of low class, the last factor is nearly $\frac{1}{\gamma T}$, and the radiation is nearly $R_0/\gamma T$, that is E_0/T. Thus the relative intensity of waves of high class and small radiation depends mainly on the relative radiation, calculated in § 9. The relative intensity of waves of low class and large radiation depends mainly on their relative initial energy, calculated in § 11.

The damping, γ, is given by $\gamma = \frac{R_0}{E_0}$; for E_0 we may take the maximum kinetic energy of § 11, which is $\frac{\pi^2 m\rho^2 q^2}{n}\sigma_0^2$; thus $\gamma = \frac{nR_0/\sigma_0^2}{\pi^2 m\rho^2 q^2}$. This may also be written

$$\gamma = \frac{nR_0/\sigma^2}{\pi^2 C^2 m(q\beta/\omega)^2}.$$

§ 13. We shall now try to form an estimate of the values of γT for vibrations of different classes, in order to decide for which of these classes the intensity of the waves emitted is determined by the initial radiation, or by the initial energy.

In § 8 we saw that a vibration of class k produces a series of harmonic waves, whose frequencies are $q + k\omega + sn\omega$. The strongest of these, for k between $\pm\frac{n}{2}$, is given by $s=0$; this alone will be considered, because the others are extremely weak in proportion. If λ be the corresponding wave-length, we have

$$q + k\omega = \frac{2\pi C}{\lambda};$$

hence

$$\frac{q\beta}{\omega} + k\beta = \frac{2\pi\rho}{\lambda}.$$

For frequencies within the limits of the spectrum $\frac{2\pi\rho}{\lambda}$ is very small; hence $\frac{q\beta}{\omega}$ is very nearly equal to $-k\beta$. Therefore for waves of corresponding frequency we find

$$\gamma T = \frac{nT}{\pi^2 C^2 m\beta^2}\frac{R_0}{k^2\sigma_0^2}.$$

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The first factor depends only on the nature of the system of rings. The second factor depends on the class k of the vibration considered; its value can be found by means of the table of § 9.

With the numerical values of § 3 we get

$$\frac{1}{\pi^2 C^2 m} = 1{\cdot}9 \, . \, 10^5.$$

The value of $\frac{n}{\beta^2}$ can be calculated if we assume that the value of β is fixed by the condition of § 2, in which case it depends largely on the value adopted for the rate of expansion of the electron. If a be the radius of the electron, the table given previously (*loc. cit.* p. 24) shows that with $\frac{\dot{a}}{a} = 10^{-16}$, $\beta = {\cdot}023$ for $n=10$, ${\cdot}566$ for $n=100$, while with $\frac{\dot{a}}{a} = 10^{-46}$, $\beta = {\cdot}00074$ for $n=10$, ${\cdot}398$ for $n=100$. In these cases we have for n/β^2 the values $1{\cdot}9 \, . \, 10^3$ for $n=10$, $2{\cdot}8 \, . \, 10^2$ for $n=100$, with $\frac{\dot{a}}{a} = 10^{-16}$, and $1{\cdot}9 \, . \, 10^7$ for $n=10$, $6{\cdot}3 \, . \, 10^2$ for $n=100$, with $\frac{\dot{a}}{a} = 10^{-46}$. Except for small values of n, n/β^2 does not vary greatly, even for a small value of $\frac{\dot{a}}{a}$; we may take it to be 1000; with this value $\frac{n}{\pi^2 C^2 m \beta^2}$ is of order $2 \, . \, 10^8$.

The value of T, the average life of a free ion, is unknown for the gas in a flame or a vacuum-tube, but it has been measured for other cases, *e. g.* gases ionized by Röntgen rays, where it has been found to be of the order of 1/2 second *. In default of a better estimate we shall assume T to be of this order in the present case, and therefore $\frac{nT}{\pi^2 C^2 m \beta^2}$ of order 10^8.

The table of values of R_0/σ_0^2 of § 9 now shows that for $k = \pm 3$ or more, γT is of order 10^{-7} or less; for $k = \pm 2$, γT is of order 1 for $\lambda = 2 \, . \, 10^{-5}$ cm., and of order ${\cdot}001$ for $\lambda = 6 \, . \, 10^{-5}$ cm.; for $k = 0$, ± 1, γT is very large.

Thus the intensities of waves of classes ± 3 and upwards are determined by radiation, those of waves of classes 0 and ± 1 by the initial energy, those of classes ± 2 mainly, if not entirely, by the initial radiation.

In the former case the energy radiated per free ion per sec. by the gas is R_0, in the latter it is $R_0/\gamma T = E_0/T$.

* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 18.

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§ 14. We shall now apply these results to decide whether our system of rings can account for spectrum-lines of the intensities of those found in nature. In § 6 we found that for a line of photometric intensity 10, *e. g.* the D lines of sodium, the energy radiated is of the order $4 \,.\, 10^{-12}$ erg per sec. per atom; and in § 10 we estimated that for a line of photometric intensity 1, that is just observable, the radiation is of the order 10^{-14} erg per sec. per atom. These estimates are based on the assumption that every atom in the sodium flame at every instant shares in the radiation. If we assume that the radiation is due only to the ions, the radiation per sec. per ion is greater in proportion.

The ionization in a flame is difficult to estimate; Stark * calculates the ionization in the unstriated positive column of a vacuum-tube to be of the order 10^{-5}, and states that for a flame it is smaller. This value makes the radiation for a line 1 at least 10^{-9} erg per sec. per ion.

This is the least value that R_0 can have for any vibration of class ± 3 or upwards, if that vibration is to give an observable spectrum-line.

In order to calculate R_0 we use the table of values of R/σ^2 given in § 9. The values of σ_0 have been found in § 11; for class k σ_0 is of order $\frac{1}{k\pi}$, for small values of k and moderately large values of n.

Let us apply our results to the case $k = \pm 3$.

The value of R/σ^2 is $1{\cdot}7 \,.\, 10^{-14}$ for $\lambda = 2 \,.\, 10^{-5}$ cm. Hence the value of R is $2 \,.\, 10^{-16}$.

This is only one five-millionth part of the value required for the faintest observable line.

For greater wave-lengths the value of R is less still.

Even if we had assumed all the atoms to be ionized the value of R would have been 50 times too small.

We conclude that a vibration of class ± 3 or upwards is far too weak to produce an observable spectrum-line.

Again, consider the vibrations of class ± 2.

The values of R/σ^2 are for

$$\lambda = 2 \,.\, 10^{-5} \text{ cm.}, \quad 3{\cdot}6 \,.\, 10^{-8},$$
$$\lambda = 3{\cdot}5 \,.\, 10^{-5} \text{ cm.}, \quad 1{\cdot}2 \,.\, 10^{-9},$$
$$\lambda = 6 \,.\, 10^{-5} \text{ cm.}, \quad 4{\cdot}8 \,.\, 10^{-11}.$$

The corresponding values of γT are ·9, ·03, ·001 respectively, those of $(1-\epsilon^{-\gamma T})/\gamma T$ are ·66, 1, 1 respectively. The corresponding radiations are respectively $6 \,.\, 10^{-10}$, $3 \,.\, 10^{-11}$, $1{\cdot}2 \,.\, 10^{-12}$ erg per sec. per ion.

* *Die Elektrizität in Gasen*, p. 269.

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We conclude that with an ionization 10^{-5} our system might give an extremely faint line in the ultraviolet, but could hardly give one in the visible spectrum. On the other hand, if all the atoms were ionized it could give a strong line in any part of the spectrum.

It is noteworthy that for these vibrations of classes ± 2 and upwards the brightest lines occur in the ultraviolet, where the value of R/σ^2 is greatest. This is because the initial radiation R_0 is here of most importance.

§ 15. From what has been said, it follows that observable spectrum-lines can only be produced by vibrations of classes $0, \pm 1$, and under certain circumstances by those of classes ± 2. We shall now consider the relative intensities of these lines.

For classes $0, \pm 1$ the damping is large; hence the radiation is approximately E_0/T (§ 13).

For E_0 we may take the maximum kinetic energy. For $k = \pm 1, \pm 2$ this is equal to $\frac{\pi^2 m\rho^2 q^2}{n}\sigma_0^2$, where

$$\sigma_0 = \frac{q'}{(q+q')(n+1)\sin(k\pi/n)} \text{ for } k,$$

and

$$\frac{q}{(q+q')(n+1)\sin(k\pi/n)} \text{ for } -k.$$

The radiation becomes $\frac{\pi^2 m\rho^2 q^2}{nT}\sigma_0^2$, which for a spectrum-line reduces to

$$\frac{\pi^2 C^2 m\beta^2}{nT} k^2\sigma_0^2 = k^2\sigma_0^2 \,.\, 10^{-8} \text{ erg per sec. per ion (§ 14).}$$

For a moderately large value of n this becomes

$$10^{-9}. \left\{ \frac{q'}{q+q'} \frac{k\pi/n}{\sin(k\pi/n)} \right\}^2 \text{ erg per sec. per ion for } k,$$

and

$$10^{-9}. \left\{ \frac{q}{q+q'} \frac{k\pi/n}{\sin(k\pi/n)} \right\}^2 \text{ erg per sec. per ion for } -k.$$

These are of the order (10^{-9}) of the radiation for an observable line with the ionization (10^{-5}) calculated by Stark for a vacuum-tube. Their relative intensities depend mainly on the corresponding free frequencies q, q'.

§ 16. We have hitherto neglected to take into account the slight disturbance due to the fact that the initial velocity and radius of our ring are not exactly those necessary for the steady motion of a ring of n equidistant electrons. The differences are the same for each electron of the ring,

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and consequently during the readjustment each electron moves in the same way, so far as these disturbances are concerned. In other words, the vibrations due to these differences are vibrations of class 0. Their amplitudes are of the order of the differences in velocity and radius, and therefore small; nevertheless they may be comparable with the mean amplitudes for classes ± 1, ± 2, because these are much reduced by radiation. We must therefore enquire whether these vibrations of class 0 are powerful enough to produce observable spectrum-lines.

Let $\Delta\beta$ be the excess of the value of β for $n+1$ above that for n electrons in steady motion. Then the energy E_0 is of the order $\frac{1}{4}C^2mn(\Delta\beta)^2$, and the radiation is of the order $\frac{C^2mn}{4T}(\Delta\beta)^2$, that is, of order $10^{-9}.\left(\frac{n\Delta\beta}{2\beta}\right)^2$ erg per sec. per ion.

The velocity β is given by an equation of the form

$$\frac{nU}{\beta}=\frac{\rho^2\psi(\beta)}{C}\frac{\dot{a}}{a^2}\text{*},$$

where $U=2\sum\limits_{s=1}^{s=\infty}\left[sn\beta^2J'_{2sn}(2sn\beta)-s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)dx\right]$.

We find, as in § 6,

$$nU/\beta=\sqrt{\frac{n\gamma}{2\pi}}.\exp.\, n\left(\gamma-\tfrac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right),\quad \gamma=\sqrt{1-\beta^2}.$$

Hence β is given by

$$\sqrt{n\gamma}\,.\exp.\, n\left(\gamma-\tfrac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right)=\sqrt{2\pi}\frac{\rho^2\psi(\beta)}{C}\frac{\dot{a}}{a^2}.$$

The right-hand member may also be written

$$\sqrt{2\pi}\,.\frac{Cm\rho^2}{e}\frac{\dot{a}}{a},\text{ where } m \text{ is the mass of the electron.}$$

The value of $\sqrt{2\pi}\frac{Cm\rho^2}{e^2}$ varies but slightly with β, that is with n; its value is $4\,.\,10^{-12}$. Thus the right-hand member is nearly constant and small when $\dot{a}/a$ is small. I find

$$\frac{n\Delta\beta}{2\beta}=\frac{-\log\left(10^{-12}.\frac{\dot{a}}{a}\right)}{2n\gamma^2}\text{ very approximately.}$$

* Schott, *loc. cit.* p. 23.

This gives the following results :—

	n	10.	20.	50.	100.	1000.
$\frac{n\Delta\beta}{2\beta}$.	$\frac{\dot{a}}{a}=10^{-16}$	·60	·31	·15	·09	·03
	$\frac{\dot{a}}{a}=10^{-46}$	1·25	·62	·26	·19	·04

These values show that the vibrations of class 0 for small values of n are sufficiently powerful to give an observable spectrum-line, even when the ionization is as small as that in a vacuum-tube. For rings of 20 electrons or so, these lines are about as bright as those of classes ± 1, for larger rings they are probably weaker, for smaller rings stronger. The ratios of the intensities are not generally very large, but are comparable with those of Jewell's scale.

§ 17. We have found that of all the vibrations of our ring only those of classes 0, ± 1, and occasionally ± 2, can produce lines sufficiently strong to be observable, but it may happen that the frequency of any one of these lines is outside the limits of the spectrum. In § 13 we saw that

$$\frac{q\beta}{\omega}+k\beta=\frac{2\pi\rho}{\lambda},$$

where λ is the wave-length of the line. For the extreme red $\frac{2\pi\rho}{\lambda}\leq\cdot 0008$, for the extreme ultraviolet $\frac{2\pi\rho}{\lambda}\leq\cdot 0063$; these are upper limits, since ρ cannot exceed the radius of the atom, that is, 10^{-8} cm. For rings of from 10 to 20 or more electrons $\beta>\cdot 02$; hence for such rings

$$\frac{q}{\omega}+k<\cdot 04 \text{ for a line in the extreme red,}$$

$$\frac{q}{\omega}+k<\cdot 3 \text{ for a line in the extreme ultraviolet.}$$

Negative frequencies are possible ; they represent lines of the same absolute frequency. As usual let $\left|\frac{q}{\omega}\right|$ represent the absolute value of $\frac{q}{\omega}$. Then we must have

$$\text{for positive } q : \left|\frac{q}{\omega}\right|<\cdot 04-k,\ \cdot 3-k,$$

$$\text{for negative } q : \left|\frac{q}{\omega}\right|>k-\cdot 04,\ k-\cdot 3,$$

for a red, ultraviolet line respectively.

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Thus the only vibrations, which can give rise to observable spectrum-lines, are

for positive q : those of classes 0, -1, and occasionally -2.
for negative q : those of classes 0, $+1$, and occasionally $+2$.

That is, for a single ring of electrons, if we divide all the frequencies into groups, each group corresponding to vibrations of the same type, *e. g.* axial vibrations, orbital vibrations and so on, but of various classes, n in number, each group of frequencies can give rise only to two, or at most three, observable spectrum-lines.

§ 18. For example, Nagaoka* has discussed a system of particles illustrating spectra; he compares the free oscillations of a rotating ring with the vibrations giving rise to spectrum-lines, and finds certain analogies between the grouping of the lines in bands and series and that of the oscillations of the ring, when their frequencies are compared. The frequencies for the axial oscillations are given by a quadratic, different for each class: these two groups he considers analogous to the vibrations producing bands. The frequencies for the orbital oscillations are given by a quartic with a pair of real and a pair of imaginary roots; the two groups belonging to the real roots he considers analogous to the vibrations giving series. Each line of a band or series corresponds to a value of k (h in Nagaoka's notation), that is, to an oscillation of that class.

But we have just seen that any one group gives rise to at most three spectrum-lines. On the other hand, the Balmer series of hydrogen has 29 lines, and few series are known with less than 10. Even if we suppose all Nagaoka's 4 groups to combine to give a single series we can only get 12 lines at most; this is obviously inadequate. For bands the difficulty is still greater.

When we consider the whole series of waves emitted in one group, we find that the intensities differ little for the first 2 or 3 members, given by $k=0$, ± 1, ± 2 as the case may be, but after that diminish with very great rapidity, faster than the terms of a geometric progression whose ratio is one to one million. Nothing like this is found for series or for bands.

I think this difficulty is conclusive against Nagaoka's view; but apart from that the frequencies of the oscillations used by Nagaoka are the frequencies *relative to the ring* (n in his notation); the frequencies of the *waves* emitted by the ring and received by a *stationary* observer are different ($n+h\omega$ in his notation). The two sets only agree when the

* Phil. Mag. ser. 6, vol. vii. p. 445.

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frequencies n are very large compared with the angular velocity ω, which is not the case. If it be true that the values of n crowd together when h becomes large, as Nagaoka supposes, it follows that the frequencies of the lines produced approach to coincidence with an arithmetic progression, with difference equal to ω. This corresponds neither to a band nor to a series, but to a set of lines of constant frequency difference.

§ 19. Nagaoka's model may be modified in two ways. (1) The controlling field, in which the ring moves, may be altered; for instance, the ring may be made stable as in J. J. Thomson's model *. A comparison of Nagaoka's equations (9), (12) with Thomson's equations (4), (3), and indeed with Maxwell's equations (14) (22) †, shows that in all these cases the frequency equations are fundamentally the same; we have the same six groups of vibrations, each of the same number n of classes, but the values of the frequencies are altered; this is true however we modify the controlling field. Hence in all these cases the system can give rise to observable spectrum-lines for at most three classes in each group, that is 18 lines in all.

(2) In the cases just considered the velocity has been supposed negligible compared with that of light, and therefore higher powers of β have been omitted. Let us remove this restriction.

The frequency equations reduce as before to a separate equation for axial and orbital vibrations, but each equation is transcendental and therefore has an infinite number of roots for each of the n classes. For stability all these must be real, if damping be neglected; if damping be taken into account they must be complex with positive imaginary part. Can each of these give rise to 3 spectrum-lines? If so, we have a sufficient number to account for series and bands.

§ 20. I have found the complete frequency equations for a ring rotating with large velocity in a given controlling field, but the results are very complicated; as an example I give the frequency equation for axial vibrations; in the notation of the present paper Nagaoka's equation (9) may be written

$$\nu - \mathrm{J} = \frac{\mathrm{C}^2 m\rho}{e^2}\left(\frac{q\beta}{\omega}\right)^2,$$

where the central positive charge is equal to ν of the negative charges, and

$$\mathrm{J} = \sum_{i=1}^{i=n-1} \frac{\sin^2 (k\pi i/n)}{4 \sin^3 (\pi i/n)}.$$

* J. J. Thomson, Phil. Mag. ser. 6, vol. vii. p. 237.

† Collected Papers, vol. i. pp. 315, 316.

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The corresponding complete equation is

$$\nu-(1-\beta^2)^2\mathrm{J}-\tfrac{1}{2}\beta^2(5-\beta^2)\mathrm{H}$$
$$=\left\{\frac{C^2m\rho}{e^2}+(1+\beta^2)(\mathrm{K}-2\mathrm{H})\right\}\left(\frac{q\beta}{\omega}\right)^2-2\beta(1-\beta^2)\mathrm{M}\frac{q\beta}{\omega}+f(0,0)-f(k,q),$$

where

$$\mathrm{K}=\sum_{i=1}^{i=n-1}\frac{1}{4\sin\pi i/n},\quad \mathrm{H}=\sum_{i=1}^{i=n-1}\frac{\sin^2(k\pi i/n)}{4\sin(\pi i/n)},$$

$$\mathrm{M}=\sum_{i=1}^{i=n-1}\frac{\sin(2k\pi i/n)\cos(\pi i/n)}{8\sin^2(\pi i/n)},$$

$$f(k,q)=\sum_{s=-\infty}^{s=\infty}\frac{1}{16}\cot\frac{(2s-2k+1)\pi}{2n}\Big[-(1-\beta^2)l^2\beta^2\mathrm{J}'_m(l\beta)$$
$$+(1+3\beta^2)l\beta\mathrm{J}_m(l\beta)+\{(1+\beta^2)(m^2+l^2\beta^2)$$
$$-4ml\beta^2-1-3\beta^2\}\int_0^{l\beta}\mathrm{J}_m(y)dy\Big],$$

with $m=2s+1$, $l=\dfrac{2q}{\omega}+2s+1$.

With the numerical values of § 3 we find $\dfrac{\mathrm{C}^2m\rho}{e^2}=47,000$ approximately. The constants ν, K, H, M are of order n; for small values of k J is of the same order; small values of k alone are of importance for our purpose. Thus ν, K, H, M, and J are all small compared with $\dfrac{\mathrm{C}^2m\rho}{e^2}$ unless n be larger than is physically possible, since the distance between consecutive electrons must be a large multiple of their radius.

The largest terms of $f(k.q)$ are generally given by $m=\pm1$, that is, by $s=0$, $s=-1$; these are of the orders

$$\tfrac{1}{8}\cot\frac{(2k\mp1)\pi}{2n}\left(\frac{q}{\omega}\pm\frac{1}{2}\right)^2\beta^2 \text{ respectively.}$$

Unless $\dfrac{q\beta}{\omega}$ be very large, $f(k,q)$ is at most of the order $\mathrm{K}\left(\dfrac{q\beta}{\omega}\right)^2$.

If it be expanded in a series of ascending powers of $\dfrac{q\beta}{\omega}$, the frequency equation takes the form

$$\nu-(1-\beta^2)^2\mathrm{J}=\frac{\mathrm{C}^2m\rho}{e^2}\left(\frac{q\beta}{\omega}\right)^2+a_0+a_1\frac{q\beta}{\omega}+a_2\left(\frac{q\beta}{\omega}\right)^2+a_3\left(\frac{q\beta}{\omega}\right)^3+\ldots,$$

where a, a_1, a_2, $a_3\ldots$ are functions of n and β, which

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diminish with increasing index, and are small compared with $\frac{C^2 m\rho}{e^2}$. The equation is transcendental, and thus has an infinite number of roots; for stability all must be real. We shall, however, not concern ourselves with the question of stability.

When $\frac{q\beta}{\omega}$ is small enough we may neglect the higher powers in the series; we then obtain a quadratic, which is not very different from Nagaoka's equation, and has two roots of the same order of magnitude. For values of $k=0, \pm 1, \pm 2$, these correspond to observable spectrum-lines, six in number at most.

§ 21. When $\frac{q\beta}{\omega}$ is large, the equation reduces approximately to

$$0=\frac{C^2 m\rho}{e^2}+a_3\frac{q\beta}{\omega}+a_4\left(\frac{q\beta}{\omega}\right)^2+\dots$$

This equation has an infinite number of large roots. Can any of these give rise to observable spectrum-lines?

In order that they may do so the frequencies must fall within the proper limits, that is, $\frac{q\beta}{\omega}+k\beta$, which is equal to $\frac{2\pi\rho}{\lambda}$, must lie between ·0063 and ·0008. We find

$$a_3=\frac{1}{6}\sum_{s=-\infty}^{s=\infty}\cot\frac{(2k-2s-1)\pi}{2n}\left[2J_{2s+1}\{(2s+1)\beta\}\right.$$
$$\left.+(1-\beta^2)(2s+1)\beta J'_{2s+1}(2s+1)\beta\}\right],$$
$$<\frac{\beta(3+\beta^2)}{6(1-\beta^2)}\cot\frac{\pi}{2n}.$$

This is exceedingly small compared with $\frac{C^2 m\rho}{e^2}$; hence $\frac{q\beta}{\omega}$ is a very large number; it follows that $\frac{q\beta}{\omega}+k\beta$ cannot be small, for k is less than $\pm\frac{n}{2}$, and β is less than unity. The only exception might occur for n very large and β nearly unity. For $n=1000$, $\beta=·9$, we find

$$a_3<1800,\ \left|\frac{q\beta}{\omega}\right|>26.$$

In this case the distance between consecutive electrons is only 50 times the radius of an electron. Even when the

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electrons are as crowded as this, the value of $k\beta$ is as great as 26, and that of k as great as 30, so that the line is far too weak to be observable.

We conclude that the large roots of the frequency equations of the ring cannot give observable spectrum-lines; so that the most general type of single ring cannot account either for spectrum-series or for bands.

§ 22. The argument of this paper may be summarized briefly as follows. After some preliminary work (§§ 1–10) an ideal radiating system is considered, consisting of a large number of mutually independent similar rings of electrons, each in orbital motion in a suitable controlling field, of which a fraction are ionized owing to the previous expulsion of an electron (§§ 11–12).

Each ion executes a number of vibrations; these can be arranged in six groups according to the frequency (q) relative to the rotating ring; each group includes n classes of vibrations, n being the number of electrons in the ionized ring (ion), and the class (k) giving the number of segments in which the ring vibrates (§ 7).

Each vibrating ion emits a corresponding number of waves (§ 8), of which the frequencies to a stationary observer are $(q+k\omega)$, ω being the angular velocity of rotation of the ring.

The intensities of these waves after the first two or three classes fall off with such great rapidity (§§ 9, 14), that waves of classes ± 3 and upwards are far too weak to give rise to observable spectrum-lines; waves of classes ± 1 and 0 give rise to lines whose intensities are of the same order of magnitude as those of spectrum-lines (§§ 14, 16); waves of classes ± 2 do so when the ionization is large (§ 14).

When the velocity of the electrons in steady motion is very small compared with that of light, there are six frequencies (q) for each class (k); hence the ring gives at most 18 observable lines (§§ 18, 19). When the velocity is comparable with that of light, the frequency equations are transcendental, and there are an infinite number of frequencies (q) for each class (k); but of these frequencies only six can give rise to observable lines, so that the maximum number for the ring is only 18 as before (§§ 20, 21), that is, too small to account for series or bands.

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