sides, their resultant will be a couple represented in sign and magnitude by double the area of the polygon. With the extended sense of the term area, it is easily seen that the same proposition is true for any polygon in space, the aspect of the area defining that of the couple.

Again, if a homogeneous incompressible fluid move in a tubular channel of small section, whose axis is a closed curve, and which it completely fills, it is plain that, as the centre of inertia of the fluid is at rest, the total momentum of the fluid is reducible to an angular momentum, which may easily be shown to be in the plane of "the area of the curve," and proportional to that area in intensity.

Lastly, to take an example from pure geometry, the area of a skew quadrilateral is the area of its projection on a plane parallel to both its diagonals. For, on any plane perpendicular to this, its diagonals will be projected into two parallel lines as AC, BD, and the quadrilateral into the plane autotomic quadrilateral ABCD whose area is evidently zero, so that on a plane parallel to both diagonals the projection is a maximum. Hence the area of a skew quadrilateral is the same as that of a plane quadrilateral with diagonals equal to those of the former, and inclined to one another at the same angle; and we may state generally that the area of *any* quadrilateral, skew or plane, is half that of the parallelogram contained by its diagonals. The same result might have been obtained by the vector-addition of the areas of the two triangles, into which the quadrilateral is divided by one of its diagonals.

On the Evaluation of a Class of Definite Integrals involving Circular Functions in the Numerator, and Powers of the Variable only in the Denominator. By J. W. L. GLAISHER, B.A., F.R.A.S., Fellow of Trinity College, Cambridge.

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The main purpose of the present paper will be best understood by commencing with an example. Suppose it were required to find the value of $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx$, which is clearly finite, then the most obvious mode of procedure would be to replace $\sin ax \sin bx$ by

u 2

the equivalent expression $\frac{1}{2} \{ \cos(a-b)x - \cos(a+b)x \}$, so as to obtain

$$\frac{1}{2} \cdot \int_0^\infty \frac{\cos(a-b)x}{x^{\frac{1}{2}}} dx - \frac{1}{2} \cdot \int_0^\infty \frac{\cos(a+b)x}{x^{\frac{1}{2}}} dx \dots\dots\dots (1).$$

Substituting, then, from the known formula

$$\int_0^\infty \frac{\cos bx}{x^n} dx = \frac{\pi}{2\Gamma(n)} \frac{b^{n-1}}{\cos \frac{n\pi}{2}} \dots\dots\dots (2),$$

we should obtain as the result

$$\frac{\sqrt{\pi}}{\sqrt{2}} \{ (a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}} \} \dots\dots\dots (3)$$

(a being supposed $> b$).

The value (3) is correct, although the manner in which it has been obtained is not sound; as, both the integrals in (1) being really infinite in value, we have no justification for replacing them by the evaluations in (3), which are derived from (2), a formula only true when the integral involved is finite.

It will be proved that the mode that has been pursued does, however, always give correct results; and a corresponding rule will be investigated for the case when the right-hand side of (2) becomes infinite (viz., when n is an odd integer), and for other similar cases of failure. For the sake both of convenience and greater generality, an exponential factor will be introduced under the integral sign throughout, and the integrals considered will be of the form $\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx$, $f(x)$ being composed of sines and cosines of x .

Before proceeding to the necessary developments, it is worth noticing, in the above simple example, how the accuracy of the result can be verified. This can be done in two ways: (i.) by integration by parts,

$$\begin{aligned} & \int_0^\infty \frac{\sin ax \sin bx}{x^{\frac{1}{2}}} dx \\ &= -2 \left[\frac{\sin ax \sin bx}{x^{\frac{1}{2}}} \right]_0^\infty - \int_0^\infty \{ (a-b) \sin(a-b)x - (a+b) \sin(a+b)x \} \frac{dx}{x^{\frac{1}{2}}} \\ &= \frac{\sqrt{\pi}}{\sqrt{2}} \{ (a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}} \}, \end{aligned}$$

as follows at once from the formula

$$\int_0^\infty \frac{\sin bx}{x^n} dx = \frac{\pi}{2\Gamma(n)} \frac{b^{n-1}}{\sin \frac{n\pi}{2}}.$$

the application of which is free from uncertainty, as both the integrals are finite.

Or (ii.) we may find the value of $\int_0^\infty e^{-cx} \frac{\sin ax \sin bx}{x^{\frac{1}{2}}} dx$ as follows, and then put $c=0$:

$$\begin{aligned} \int_0^\infty e^{-cx} \frac{\sin ax \sin bx}{x^{\frac{1}{2}}} dx &= \frac{1}{2} \cdot \int_0^\infty e^{-cx} \{ \cos(a-b)x - \cos(a+b)x \} \frac{dx}{x^{\frac{1}{2}}} \\ &= \frac{1}{2} \cdot \int_0^\infty dx \cdot \frac{e^{-cx}}{x^{\frac{1}{2}}} \left\{ 1 - \frac{a^2 x^2}{2!} + \frac{a^4 x^4}{4!} - \dots - 1 + \frac{\beta^2 x^2}{2!} - \frac{\beta^4 x^4}{4!} + \dots \right\}, \\ \text{writing } a \text{ and } \beta \text{ for } (a-b) \text{ and } (a+b) \text{ for brevity, and } n! \text{ for } 1.2 \dots n, \\ &= \frac{1}{2} \left\{ -\frac{\Gamma(\frac{3}{2})}{2!} \frac{a^2}{c^{\frac{3}{2}}} + \frac{\Gamma(\frac{5}{2})}{4!} \frac{a^4}{c^{\frac{5}{2}}} - \dots + \frac{\Gamma(\frac{3}{2})}{2!} \frac{\beta^2}{c^{\frac{3}{2}}} - \frac{\Gamma(\frac{5}{2})}{4!} \frac{\beta^4}{c^{\frac{5}{2}}} + \dots \right\} \\ &= \frac{\sqrt{\pi}}{2} c^{\frac{1}{2}} \left\{ -\left(1 + \frac{ai}{c}\right)^{\frac{1}{2}} - \left(1 - \frac{ai}{c}\right)^{\frac{1}{2}} + \left(1 + \frac{\beta i}{c}\right)^{\frac{1}{2}} + \left(1 - \frac{\beta i}{c}\right)^{\frac{1}{2}} \right\} \\ &= \frac{\sqrt{\pi}}{2} \left\{ \beta^{\frac{1}{2}} \left(\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} \right) - a^{\frac{1}{2}} \left(\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} \right) \right\} \quad \left(\begin{array}{l} \text{when } c \text{ is put equal} \\ \text{to zero} \end{array} \right) \\ &= \frac{\sqrt{\pi}}{\sqrt{2}} \left\{ (a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}} \right\}. \end{aligned}$$

The general cases can usually be treated in the same way; but it is clearly advantageous to be able to replace the long and tedious calculations that would then be necessary by a direct rule.

To resume, then, the general investigation, it is to be noticed that, for

$$\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx$$

to be of finite value, $f(x)$, when expanded in ascending powers of x , must contain no power of x less than x^{n-1} ; this is evident, for

$\int_0^\infty \frac{e^{-ax}}{x^r} dx$ is infinite, if r lie between 1 and ∞ inclusive:

First, suppose n fractional, and let m be the integer next below n .

Let $f(x)$, a function of $\sin x$ and $\cos x$, be expressed as a series of sines and cosines of multiple arcs, thus:

$$f(x) = A_1 \cos b_1 x + A_2 \cos b_2 x + \dots + A'_1 \sin b'_1 x + A'_2 \sin b'_2 x + \dots;$$

then we must have $\Sigma A b = 0$, $\Sigma A b^2 = 0$, $\Sigma A b^3 = 0$, ... $\Sigma A b^{m-1} = 0$; but $\Sigma A b^m$, and terms involving higher powers of b need not vanish [of course if $f(x)$ consists of cosines alone, there are no terms involving odd powers of b , and if $f(x)$ consists of sines alone, there are no terms involving even powers of b , so that in these cases some of the conditions are superfluous].

If r be intermediate to 0 and 1, it is known that

$$\begin{aligned}\int_0^\infty e^{-ax} \frac{\cos bx}{x^r} dx &= \Gamma(1-r) (a^2 + b^2)^{\frac{1}{2}(r-1)} \cos \left\{ (r-1) \tan^{-1} \frac{b}{a} \right\} \\ &= \frac{\pi}{\Gamma(r) \sin r\pi} (a^2 + b^2)^{\frac{1}{2}(r-1)} \cos \left\{ (r-1) \tan^{-1} \frac{b}{a} \right\}, \\ \int_0^\infty e^{-ax} \frac{\sin bx}{x^r} dx &= -\frac{\pi}{\Gamma(r) \sin r\pi} (a^2 + b^2)^{\frac{1}{2}(r-1)} \sin \left\{ (r-1) \tan^{-1} \frac{b}{a} \right\}.\end{aligned}$$

Suppose $n=m+r$, and let m be an integer, and r as above, then since

$$\begin{aligned}\int_0^\infty e^{-ax} \frac{f(x)}{x^r} dx &= \frac{\pi}{\Gamma(r) \sin r\pi} \Sigma \left[A (a^2 + b^2)^{\frac{1}{2}(r-1)} \cos \left\{ (r-1) \tan^{-1} \frac{b}{a} \right\} \right. \\ &\quad \left. - A' (a^2 + b'^2)^{\frac{1}{2}(r-1)} \sin \left\{ (r-1) \tan^{-1} \frac{b'}{a} \right\} \right] \dots\dots\dots (4),\end{aligned}$$

integrate m times with regard to a between the limits ∞ and a , and it will be shown that there results

$$\begin{aligned}\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx &= \frac{\pi}{\Gamma(n) \sin n\pi} \Sigma \left[A (a^2 + b^2)^{\frac{1}{2}(n-1)} \cos \left\{ (n-1) \tan^{-1} \frac{b}{a} \right\} \right. \\ &\quad \left. - A' (a^2 + b'^2)^{\frac{1}{2}(n-1)} \sin \left\{ (n-1) \tan^{-1} \frac{b'}{a} \right\} \right] \dots\dots\dots (5).\end{aligned}$$

To demonstrate this, it is necessary to establish two results, viz. (i.) that the indefinite integral with regard to a of

$$\frac{1}{\Gamma(p) \sin p\pi} (a^2 + b^2)^{\frac{1}{2}(p-1)} \cos \left\{ (p-1) \tan^{-1} \frac{b}{a} \right\},$$

$$\text{is} \quad -\frac{1}{\Gamma(p+1) \sin (p+1)\pi} (a^2 + b^2)^{\frac{1}{2}p} \cos \left\{ p \tan^{-1} \frac{b}{a} \right\},$$

and that a similar formula is true for the corresponding expression involving the sine; (ii.) that at the limit ∞ the expression on the right-hand side of (5) always vanishes after each of the integrations.

The truth of the first result is evident on differentiation, and the second may be proved as follows:—It will be found to be sufficient to show that the right-hand side of (5) vanishes as it stands when $a = \infty$, as this includes the cases of the inferior integrations.

By De Moivre's theorem,

$$\begin{aligned}(a^2 + b^2)^{\frac{1}{2}n} \cos \left(n \tan^{-1} \frac{b}{a} \right) &= a^n \left\{ 1 - \frac{n(n-1)}{2!} \frac{b^2}{a^2} \right. \\ &\quad \left. + \frac{n(n-1) \dots (n-4)}{4!} \frac{b^4}{a^4} - \dots\dots \right\},\end{aligned}$$

and

$$(a^2 + b^2)^{\frac{1}{2}n} \sin \left(n \tan^{-1} \frac{b}{a} \right) = a^n \left\{ n \frac{b}{a} - \frac{n(n-1)(n-2)}{3!} \frac{b^3}{a^3} + \dots\dots \right\};$$

whence the right-hand side of (5), leaving out the constant factor, is

$$\Sigma \left[A a^{n-1} \left\{ 1 - \frac{(n-1)(n-2)}{2!} \frac{b^2}{a^2} + \dots \right\} \right. \\ \left. - A' a^{n-1} \left\{ n \frac{b'}{a} - \frac{(n-1)(n-2)(n-3)}{3!} \frac{b'^3}{a^3} + \dots \right\} \right]$$

but in virtue of the relations $\Sigma A = 0$, $\Sigma A b = 0$, ... $\Sigma A b^{m-1}$, &c., all the terms involving a raised to a positive power vanish (or do not occur at all), and the first term that can remain is of the form

$$P a^{n-1} \cdot \frac{b^m}{a^m} = P a^{m+n-1} \cdot \frac{b^m}{a^m} = P \frac{b^m}{a^{1-r}},$$

which, with all the succeeding terms, vanishes for $a = \infty$.

The same reasoning clearly shows that all the expressions obtained by the previous integrations vanish also for $a = \infty$.

The case of n = an integer, presents rather more difficulty.

In order that $\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx$ may then be finite, it is necessary that all terms up to and including x^{n-1} in the expansion of $f(x)$ should vanish, so that we have $\Sigma A = 0$, $\Sigma A b = 0$, ..., $\Sigma A b^{n-1} = 0$ (subject to the condition that if a term in b' does not occur in $f(x)$, the condition $\Sigma A b' = 0$ need not be satisfied); but neither $\Sigma A b^n$, nor terms involving higher powers, need vanish.

We start with the well known formulæ

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} = \frac{1}{2} \left\{ \frac{1}{a-bi} + \frac{1}{a+bi} \right\}, \\ \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} = \frac{1}{2i} \left\{ \frac{1}{a-bi} - \frac{1}{a+bi} \right\};$$

so that, in order to perform the integrations that will be required, it is clearly necessary to find the n th integral of $\log x$. This can be obtained as follows:

By a well known theorem, if u_m denote the m th integral of u , then

$$m! \cdot u_{m+1} = x^m \int u \, dx - m x^{m-1} \int x u \, dx + \frac{m(m-1)}{2!} \int x^2 u \, dx \dots \\ \dots + (-)^m \int x^m u \, dx,$$

$$\text{and} \quad \int x^m \log x \, dx = \frac{x^{m+1}}{m+1} \left\{ \log x - \frac{1}{m+1} \right\};$$

whence the $(n+1)$ th integral of $\log x$

$$= \frac{x^{n+1}}{n!} \log x \left\{ 1 - n \cdot \frac{1}{2} + \frac{n(n-1)}{2!} \cdot \frac{1}{3} - \dots \pm \frac{1}{n+1} \right\} \\ - \frac{x^{n+1}}{n!} \left\{ 1 - n \cdot \frac{1}{2^2} + \frac{n(n-1)}{2!} \cdot \frac{1}{3^2} - \dots \pm \frac{1}{(n+1)^2} \right\} \\ = \frac{x^{n+1}}{(n+1)!} \log x - \frac{x^{n+1}}{n!} \psi(n),$$

putting for the first series its equivalent, viz. $\frac{1}{n+1}$, and denoting the second by $\psi(n)$.

The n th integral of $\frac{1}{2} \left\{ \frac{1}{a-bi} + \frac{1}{a+bi} \right\}$ with regard to a is therefore equal to one-half the real part of

$$\begin{aligned} & \frac{\rho^{n-1}}{(n-1)!} \{ \cos(n-1)\theta - i \sin(n-1)\theta \} (\log \rho - \theta i) \\ & - \frac{\rho^{n-1}}{(n-2)!} \psi(n-2) \{ \cos(n-1)\theta - i \sin(n-1)\theta \} \\ & + \text{a similar expression with } -i \text{ written for } i; \end{aligned}$$

$$\begin{aligned} \text{viz. is equal to } & \frac{\rho^{n-1}}{(n-1)!} \{ \log \rho \cos(n-1)\theta - \theta \sin(n-1)\theta \} \\ & - \frac{\psi(n-2)}{(n-2)!} \rho^{n-1} \cos(n-1)\theta, \end{aligned}$$

where $\rho = (a^2 + b^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1} \frac{b}{a}$.

Similarly the n th integral of $\frac{1}{2i} \left(\frac{1}{a-bi} - \frac{1}{a+bi} \right)$ is

$$\begin{aligned} & - \frac{\rho^{n-1}}{(n-1)!} \{ \log \rho \sin(n-1)\theta + \theta \cos(n-1)\theta \} \\ & + \frac{\psi(n-2)}{(n-2)!} \rho^{n-1} \sin(n-1)\theta. \end{aligned}$$

We thus have

$$\begin{aligned} \int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx &= (-)^n \cdot \Sigma \left[\frac{A \rho^{n-1}}{(n-1)!} \{ \log \rho \cos(n-1)\theta - \theta \sin(n-1)\theta \} \right. \\ & - A \frac{\psi(n-2)}{(n-2)!} \rho^{n-1} \cos(n-1)\theta - A' \frac{\rho'^{n-1}}{(n-1)!} \{ \log \rho' \sin(n-1)\theta' \\ & \quad \left. + \theta' \cos(n-1)\theta' \} \right. \\ & \quad \left. + A' \frac{\psi(n-2)}{(n-2)!} \rho'^{n-1} \sin(n-1)\theta' \right] \dots \dots \dots (6); \end{aligned}$$

ρ' being $= (a^2 + b'^2)^{\frac{1}{2}}$ and $\theta' = \tan^{-1} \frac{b'}{a}$; it will now be demonstrated, as before, that the right-hand side of (6) vanishes when $a = \infty$.

Since $\log \rho = \frac{1}{2} \log(a^2 + b^2) = \log a + \frac{b^2}{a^2} - \frac{1}{2} \frac{b^4}{a^4} + \dots$,

it is clear that the expression

$$\begin{aligned} & a^{n-1} \left(1 + \alpha_1 \frac{b}{a} + \alpha_2 \frac{b^2}{a^2} + \dots \right) \left(\beta \log a + \beta_1 \frac{b}{a} + \beta_2 \frac{b^2}{a^2} + \dots \right) \\ & \times \left(1 + \gamma_1 \frac{b}{a} + \gamma_2 \frac{b^2}{a^2} + \dots \right) \end{aligned}$$

$\alpha_1 \alpha_2 \dots, \beta \beta_1 \dots, \gamma_1 \gamma_2 \dots$ being coefficients, any of which may be zero, includes all the cases that can arise by the expansion of terms in (6); and in virtue of the relations $\Sigma A = 0, \Sigma A b = 0, \dots, \Sigma A b^{n-1} = 0$, the

only terms that remain are of the form $\frac{b^n}{a}$, $\frac{b^{n+1}}{a^2}$, $\frac{b^{n+2}}{a^3}$, ..., or $\frac{b^n}{a} \log a$, $\frac{b^n}{a^2} \log a$, ..., all of which vanish when a is infinite.

The formulæ will assume a somewhat less confused appearance if we write $\phi(n)$ for $n\psi(n-1)$, so that

$$\phi(n) = n - \frac{\pi(n-1)}{2!} \frac{1}{2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{3} \dots \pm \frac{1}{n};$$

$\frac{\psi(n-2)}{(n-2)!}$ then becomes $\frac{\phi(n-1)}{(n-1)!}$.

We have therefore the general rule that whenever $\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx$ is finite, its value may be obtained by expressing $f(x)$ as a series of sines and cosines of multiple arcs, and substituting for the separate integrals that occur from the formulæ,

$$\int_0^\infty e^{-ax} \frac{\cos bx}{x^n} dx = \frac{\pi}{\Gamma(n) \sin n\pi} \rho^{n-1} \cos(n-1)\theta \dots (7),$$

$$\int_0^\infty e^{-ax} \frac{\sin bx}{x^n} dx = -\frac{\pi}{\Gamma(n) \sin n\pi} \rho^{n-1} \sin(n-1)\theta \dots (8),$$

when n is fractional; and

$$\int_0^\infty e^{-ax} \frac{\cos bx}{x^n} dx = (-1)^n \frac{\rho^{n-1}}{(n-1)!} \{ \log \rho \cos(n-1)\theta - \theta \sin(n-1)\theta \\ - \phi(n-1) \cos(n-1)\theta \} \dots (9),$$

$$\int_0^\infty e^{-ax} \frac{\sin bx}{x^n} dx = (-1)^{n-1} \frac{\rho^{n-1}}{(n-1)!} \{ \log \rho \sin(n-1)\theta + \theta \cos(n-1)\theta \\ - \phi(n-1) \sin(n-1)\theta \} \dots (10),$$

n being an integer, $\rho = (a^2 + b^2)^{\frac{1}{2}}$, $\tan \theta = \frac{b}{a}$,

and $\phi(n-1) = n-1 - \frac{(n-1)(n-2)}{2!} \cdot \frac{1}{2} + \dots$, as above.

The most important applications of these formulæ take place when $a=0$, and they then become much simpler. Thus, in the case of an integral of the form $\int_0^\infty \frac{f(x)}{x^n} dx$, we may use, just as if they were always true, the formulæ

$$\int_0^\infty \frac{\cos bx}{x^n} dx = \frac{\pi b^{n-1}}{2\Gamma(n) \cos \frac{n\pi}{2}} \dots (11),$$

$$\int_0^\infty \frac{\sin bx}{x^n} dx = \frac{\pi b^{n-1}}{2\Gamma(n) \sin \frac{n\pi}{2}} \dots (12),$$

when n is fractional; these follow from (7) and (8), since when $a=0$, $\theta = \frac{\pi}{2}$ and $\rho = b$.

When n is an integer (positive), it is necessary to separate the cases of n even or odd.

$$\int_0^\infty \frac{\cos bx}{x^n} dx = (-)^{1^n} \frac{b^{n-1}}{(n-1)!} \frac{\pi}{2}, \text{ if } n \text{ be even} \dots\dots\dots (13),$$

$$= (-)^{1^{(n+1)}} \frac{b^{n-1}}{(n-1)!} \{\log b - \phi(n-1)\}, \text{ if } n \text{ be odd} \dots (14),$$

$$\int_0^\infty \frac{\sin bx}{x^n} dx = (-)^{1^n} \frac{b^{n-1}}{(n-1)!} \{\log b - \phi(n-1)\}, \text{ if } n \text{ be even} \dots\dots (15),$$

$$= (-)^{1^{(n-1)}} \frac{b^{n-1}}{(n-1)!} \frac{\pi}{2}, \text{ if } n \text{ be odd} \dots\dots\dots (16).$$

But two of these formulæ admit of still further simplification, as it will be shown that the terms involving $\phi(n-1)$ will always vanish, and may therefore be omitted.

To prove this, we need only remark, that $f(x)$, when expressed as a series of sines and cosines of multiple arcs, consists of two portions, viz., an even portion

$$f_2(x) = A_0 + A_1 \cos b_1 x + A_2 \cos b_2 x \dots\dots + A_r \cos b_r x,$$

which does not change sign with x , and an odd portion

$$f_1(x) = A'_1 \sin b'_1 x + A'_2 \sin b'_2 x \dots\dots + A'_s \sin b'_s x,$$

which does change sign with x ; and it is evident that for $\int_0^\infty \frac{f(x)}{x^n} dx$

to be finite, we must have $\int_0^\infty \frac{f_2(x)}{x^n} dx$ and $\int_0^\infty \frac{f_1(x)}{x^n} dx$ each finite also,

as $f_2(x)$ contains only even powers of x , and $f_1(x)$ only odd powers. We may therefore, without loss of generality, consider separately the cases of $f(x)$ wholly even or wholly odd (and perhaps it would have been better to have done so throughout).

(i.) If $f(x)$ be even, and n be even, then x^{n-1} being an odd power does not occur in $f(x)$, so that, although we must have $\Sigma A b^{n-2}$ zero, there is no reason why $\Sigma A b^{n-1}$ should be zero, so that (13) needs no alteration; but if n be odd, then x^{n-1} is an even power, and we must have $\Sigma A b^{n-1} = 0$, else the integral would be infinite, thus in (14) the term involving $\phi(n-1)$ may be omitted.

(ii.) If $f(x)$ be odd, and n be even, x^{n-1} is odd, and we must have $\Sigma A' b'^{n-1} = 0$, so that, in (15), the term involving $\phi(n-1)$ may be omitted; but if n be odd, there is no necessity for the vanishing of $\Sigma A' b'^{n-1}$, and the integral is as written in (16).

The formulæ of substitution are therefore

$$\int_0^\infty \frac{\cos bx}{x^n} dx = (-)^{1^n} \frac{b^{n-1}}{(n-1)!} \frac{\pi}{2} \text{ if } n \text{ be even} \dots\dots\dots (17),$$

$$= (-)^{1^{(n+1)}} \frac{b^{n-1}}{(n-1)!} \log b, \text{ if } n \text{ be odd} \dots\dots (18),$$

$$\int_0^\infty \frac{\sin bx}{x^n} dx = (-)^{1^n} \frac{b^{n-1}}{(n-1)!} \log b, \text{ if } n \text{ be even} \dots\dots\dots (19),$$

$$= (-)^{1^{(n-1)}} \frac{b^{n-1}}{(n-1)!} \frac{\pi}{2}, \text{ if } n \text{ be odd} \dots\dots\dots (20).$$

For example, since $\sin^4 x = \frac{1}{8} (\cos 4x - 4 \cos 2x + 3)$, we have, from (18),

$$\int_0^\infty \frac{\sin^4 x}{x^3} dx = \frac{1}{8 \cdot 2} \{4^2 \log 4 - 4 \cdot 2^2 \log 2\} = \log 2,$$

and from (17), $\int_0^\infty \frac{\sin^4 x}{x^2} dx = -\frac{1}{8 \cdot 1} \{4 - 4 \cdot 2\} \frac{\pi}{2} = \frac{\pi}{4} \dots\dots\dots (21).$

These can be verified as follows:—

By integration by parts,

$$\int \frac{\cos bx}{x^3} dx = -\frac{1}{2} \frac{\cos bx}{x^2} + \frac{b \sin bx}{2x} - \frac{b^2}{2} \int \frac{\cos bx}{x} dx,$$

whence

$$\int_0^\infty \frac{\sin^4 x}{x^3} dx = -\frac{1}{8 \cdot 2} \left\{ 4^2 \int_0^\infty \frac{\cos 4x}{x} dx - 4 \cdot 2^2 \int_0^\infty \frac{\cos 2x}{x} dx \right\} \dots (22).$$

Now $\int_0^\infty \frac{\cos x}{x} dx$ is the cosine-integral, usually written $\text{Ci } x$, and we have the known formulæ

$$\text{Ci } x = \gamma + \frac{1}{2} \log(x^2) - \frac{1}{2} \frac{x^2}{2!} + \frac{1}{4} \frac{x^4}{4!} - \frac{1}{6} \frac{x^6}{6!} + \dots\dots,$$

where γ is Euler's constant .5772156..., and

$$\text{Ci } x = \sin x \left\{ \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \dots \right\} - \cos x \left\{ \frac{1}{x^2} - \frac{3!}{x^4} + \frac{5!}{x^6} - \dots \right\},$$

the former showing (what, however, is clear from inspection of the integral) that $\text{Ci } \infty = 0$; thus, in (22), taking the lower limit of the integral, an evanescent quantity ϵ , instead of 0, we have, since

$$\int_\epsilon^x \frac{\cos nx}{x} dx = \text{Ci } nx,$$

$$\int_0^\infty \frac{\sin^4 x}{x^3} dx = \text{Ci } 4\epsilon - \text{Ci } 2\epsilon$$

$$= \gamma + \log 4\epsilon - \frac{1}{2} \frac{4^2 \epsilon^2}{2!} + \dots\dots - \left(\gamma + \log 2\epsilon - \frac{1}{2} \frac{2^2 \epsilon^2}{2!} + \dots\dots \right)$$

$$= \log 2, \text{ when } \epsilon = 0.$$

The verification of (21) by integration by parts presents no difficulty, the integral being reduced to dependence on $\int_0^\infty \frac{\sin ax}{x} dx$,

which, when a is positive, $= \frac{\pi}{2}$. I may here remark that throughout, in applying the formulæ (17)–(20), we must make b positive before substituting it in b^{n-1} , e.g. in (17), we must substitute $(-)^n \frac{2^{n-1}}{(n-1)!}$ for $\int_0^\infty \frac{\cos 2x}{x^n} dx$ and not $(-)^n \frac{(-2)^{n-1}}{(n-1)!}$.

The general formulæ can be proved by integration by parts by means of the cosine-integral exactly as in the example above; for, n being odd,

$$\begin{aligned} \int_0^\infty \frac{\Sigma A \cos bx}{x^n} dx &= \text{terms that vanish between the limits} \\ &\quad \pm \frac{1}{(n-1)!} \int_0^\infty \frac{\Sigma A b^{n-1} \cos bx}{x} dx \\ &= \mp \frac{1}{(n-1)!} \Sigma A b^{n-1} \text{Ci}(b\epsilon) \\ &= \mp \frac{1}{(n-1)!} \Sigma A b^{n-1} \left\{ \gamma + \log b\epsilon - \frac{1}{2} \cdot \frac{(b\epsilon)^2}{2!} + \dots \right\} \\ &= \mp \frac{1}{(n-1)!} \Sigma A b^{n-1} \left\{ \gamma + \log b + \log \epsilon - \frac{1}{2} \cdot \frac{(b\epsilon)^2}{2!} + \dots \right\} \\ &= \mp \frac{\Sigma A b^{n-1} \log b}{(n-1)!}, \end{aligned}$$

since $\Sigma A b^{n-1} = 0$ absolutely, and we take $\epsilon=0$ ultimately.

There is one special case that deserves notice, viz., when $n=1$ and $f(x)$ is an even function; for should $f(x)$, when expanded as a series of cosines, involve a constant term, the value of the integral is infinite, as is seen by making $b=0$ in (18). The same thing appears by integration by parts; thus, for example,

$$\begin{aligned} \int_0^\infty \frac{\sin^2 x}{x} dx &= \frac{1}{2} \int_0^\infty \frac{1 - \cos 2x}{x} dx \\ &= \frac{1}{2} \left\{ \log \infty - \log \epsilon + \gamma + \log 2\epsilon - \frac{1}{2} \frac{(2\epsilon)^2}{2!} + \dots \right\} \\ &= \infty. \end{aligned}$$

In the same way we see that $\int_0^\infty \frac{\sin^m x}{x} dx$ is infinite whenever m is an even integer, though its value is finite for m odd. The reason for this is clear, for the whole area of the curve $y = \frac{\sin^m x}{x}$ (m even) between $x=0$ and ∞ is positive, while there are positive and negative portions, when n is odd, which help to cancel one another. The infinite term occurs

only when $n=1$, for $b^{n-1} \log b$ vanishes with b in every other case. If we evaluate the integral by integration by parts, when $n > 1$ we have to perform at least one differentiation upon the series of cosines before the integral is reduced to depend on cosine-integrals, and the constant term is thus removed.

The investigations contained in this communication were suggested by, and are in part an extension of, a paper by Leslie Ellis, in vol. iii. of the Cambridge Mathematical Journal.* The formulæ (17) and (20) are there obtained and applied to several very interesting questions. Ellis did not, however, discuss the cases when $f(x)$ and x^n were not both even or both odd; nor did he consider (except in one special instance) what the formulæ became when n was fractional, or when the factor e^{-ax} was introduced under the integral sign. A desire to extend the investigation so as to include these cases led me to examine Ellis's paper carefully, and I found that, although his results were correct, and the principle involved was so similar to that used above as to suggest it at once, the method employed was not rigorous. He has, in fact, considered the integrals of the terms that arise by expanding $f(x)$ in sines or cosines separately, and has thus obtained his results by the method which it is one of the main objects of this communication to justify. Starting with the integral

$$\int_0^\infty e^{-ax} \frac{\sin rx}{x} dx = \tan^{-1} \frac{r}{a},$$

and integrating $(n-1)$ series with regard to r , there results

$$\int_0^\infty e^{-ax} \frac{\left. \begin{matrix} \sin \\ \cos \end{matrix} \right\} rx}{x^n} dx = \pm \frac{r^{n-1}}{(n-1)!} \tan^{-1} \frac{r}{a} + aF(ra) + Cr^{n-2} + C_1 r^{n-4} + \&c. \dots\dots\dots(23);$$

the sine or cosine being taken on the left-hand side according as r is odd or even, and $F(ra)$ being a function that does not become infinite when $a=0$. In virtue of the conditions $\Sigma Aa^{n-2}=0$, $\Sigma Aa^{n-4}=0$, ... (which are equivalent to the ΣAb^{n-1} , &c., in the notation of this paper) by putting $a_1, a_2, \dots$ for r in (23), multiplying by $A_1, A_2, \dots$ and adding, there results

$$\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx = \pm \Sigma \frac{\Sigma Aa^{n-1}}{(n-1)!} \tan^{-1} \frac{a}{a} + a \Sigma AF(aa),$$

which, taking $a=0$, gives

$$\int_0^\infty \frac{f(x)}{x^n} dx = \pm \frac{\Sigma Aa^{n-1}}{(n-1)!} \frac{\pi}{2}.$$

It thus appears that the course of the investigation requires us to assign a finite value to $\int_0^\infty e^{-ax} \frac{\cos rx}{x^n} dx$ and $\int_0^\infty e^{-ax} \frac{\sin rx}{x^n} dx$, while in

* Evaluation of Certain Definite Integrals, p. 185.

reality they are infinite, so that the method cannot be regarded as rigorous.

The concluding paragraph in Ellis's paper is,—“The method of this paper leads to some elegant results when applied to the definite integral $\int_0^\infty e^{-ax} dx$, but it is enough to point out this application, which

involves no difficulty whatever.” I do not see what application is here referred to, unless integrals of the form $\int_0^\infty \frac{e^{-ax}}{x^n} dx$ or $\int_0^\infty \frac{\Sigma A e^{-ax}}{x^n} dx$ are meant; but the former is infinite for $n=1$, and the discussion of the latter is of little value, as $\Sigma A e^{-ax}$, subject to the required conditions for the integral to be finite, includes no functions of any interest.

In connexion with Ellis's paper I may refer to one by Bronwin in vol. xxiv., p. 491, of the *Philosophical Magazine* (1844), in which several integrals are evaluated, to which Ellis's results, and those given in this communication, are at once applicable.

Note on Normals and the Surface of Centres of an Algebraical Surface. By SAMUEL ROBERTS, M.A.

[Read March 13th, 1873.]

I.

There are several methods by which we can determine, in ordinary cases, the number of normals to a surface which can be drawn from an arbitrary point. We may, for instance, consider a system of concentric spheres, and find the number of these which touch the given surface.

It must be assumed, however, that this has no special relation to the plane at infinity, or the imaginary circle there.

Let the order of the surface, its class, and the degree of the tangent cone be m, n, a . Then, since one sphere of the system can be drawn through any point ($\mu=1$), one sphere touches any plane ($\nu=1$), and one sphere touches any line ($\rho=1$), we have, by a formula of De Jonquières

$$\mu n + \nu m + \rho a = n + m + a.$$

For a surface, order m , without singularities,

$$n = m(m-1)^2, \quad a = m(m-1), \quad m+n+a = m^3 - m^2 + m.$$

2. If, however, the surface is specially related to the plane at infinity, the above formula fails. We will now endeavour to determine the number sought by finding how many normals can be drawn to the surface from a point in the plane at infinity. In general, the normals to a surface at points in the plane at infinity lie altogether in that plane. We have therefore altogether at infinity $m+a$ normals, and of course there are n finite parallel normals, so that we get, as before,

$$m+n+a$$