ON SOME OF THE SYMBOLS OF ELEMENTARY MATHEMATICS.

BY G. A. MILLER,

University of Illinois.

The symbols $+$, $-$, $=$ for plus, minus, and equals respectively are a part of the universal mathematical language. Not only do they have the same meaning throughout the civilized world, but they are also the only symbols in common use to convey the ideas for which they stand. The symbol $\times$ for multiplied by is also in universal use, but it is not the only symbol in common use to express this concept as the period is very frequently used to convey the same thought.

The position of the latter symbol in mathematical notation is far from an ideal one. It has two meanings, being used both for a decimal point and to indicate multiplication. When it is used to represent either of these ideas its position is not fixed, since the English generally write it before the middle of a figure to represent a decimal point and near the bottom to represent multiplication, while in America the reverse is commonly done. Moreover, it is not the only symbol that is commonly used as a decimal point, since the French, German and Italians use the comma to represent this concept. In fact, in earlier times the English generally employed the comma for the same purpose.

While the uses of the period do not exhibit the worst side of the notation in elementary mathematics, yet they are sufficiently objectionable to invite discussion with a view to improvement. If a symbol has two very distinct meanings in the same subject, if people using the same language employ different methods to distinguish between these meanings, and if each of its meanings is shared with another symbol which is in more general use, there seems to be considerable reason for inquiry into possible improvement, especially since this symbol is met so early in the student's career. It should not be inferred that the given methods of using the period as a symbol are without exceptions. In Williamson's *Differential Calculus*, for instance, the decimal point is written in exactly the same way as the symbol for multiplication, while in Carr's *Synopsis of elementary results in pure mathematics* the decimal point is sometimes used as in America, but more generally according to the English custom.

As the comma was in general use as a decimal point long before the period¹ and as it is now used for this purpose by a majority of the leading mathematicians, it would appear that the English speaking people had considerable reason for considering the desirability of adopting the comma as the decimal point. In fact, this would only be a re-adoption since the early English writers such as Briggs were among the earliest to employ the comma as a decimal point. Briggs used it in this way in his table of logarithms in 1624, while the first known instance of its use is in the Italian trigonometry of Pitiscus of 1608.

The common symbols for division present a slightly better situation than those for multiplication. The oldest of these is the line between two numbers written one above the other. This symbol is in universal use and has only one meaning in the elementary mathematics at the present time. The other symbols in common use to represent division are $\div$ and $:$. The former of these has also the ideal properties that it stands for only one concept in modern elementary literature and that it is used extensively. The latter, however, does not have the first of these two properties as it is sometimes used to represent a ratio between two numbers of the same denomination and at other times to represent ordinary division. It is true that the ratio between two abstract numbers is identical with division and hence there is no theoretical objection to the use of the same symbol to represent division of abstract numbers and the ratio between two denominate numbers. The symbol $:$ is used in this way by some of the best writers.

While there are strong reasons in favor of the use of the symbol $:$ for division in elementary text-books in view of the extensive use of this symbol in European modern mathematical literature, yet the arguments against its use are so serious that Tannery and Molk suggest that the symbol should only be used to represent a ratio, in their excellent article on the fundamental principles of arithmetic published in volume 1 of the *Encyclopedie des Sciences Mathematiques*, 1904, page 44. It is extremely important that a student should realize that we may speak of the ratio of two things when division would have no meaning, and, vice versa, we may be able to divide when there is no ratio between the dividend and the divisor. Since division and ratio may be so distinct it is a question whether the same

¹ Tropicke, Geschichte der Elementar-Mathematik, 1902, vol. 2, p. 158.

symbol should be used to represent these two concepts even if this can be done logically within the proper limits.²

The symbol $\simeq$ is very extensively used in elementary geometry to denote equivalent to. As it is used very little outside of the elementary text-books and is difficult to make, it seems very desirable that its use as well as the use of the term for which it stands should not be continued. The term equal and the symbol $=$ should be used in its place as is already done in some of the best text-books. On the other hand the term congruent, and the extensively used symbol $\cong$ should be employed in all text-books on elementary geometry.

It seems unfortunate that this symbol, introduced by Leibnitz with the present meaning, should have two different forms, viz., $\cong$ and $\simeq$. In fact the upper part of this symbol $\sim$ which indicates similarity is also written in the form $\smile$. Both of these forms are very old and were used during the latter part of the seventeenth century to represent equality.³ Some authors seem inclined to regard them as two distinct symbols but they are so nearly alike that they are very apt to be confused and the mathematical notation should be chosen with a view to avoid all possible sources of errors.

The symbol $\sim$ is extensively used to represent many different concepts in mathematics. For instance, it is used to represent the difference between two numbers without any reference to the relative values of the numbers. Webster's dictionary defines the symbol in this way, while the Century dictionary says it signifies "is proportional to." It is extensively used in the theory of aggregates to represent equivalence, and in Pierpont's recent work "The theory of functions of a real variable" it is used to represent the same order of infinite. Its use in group theory, in representing approximate values, and in the theory of infinite series are well known.⁴

Although $\sim$ is used to represent so many different concepts, yet it appears to us desirable that it should be universally used in elementary geometry to represent similarity—the meaning

² The symbol $\div$ was used by Oughtred to represent a ratio at least as early as 1657; Beman, *Intermediaire*, vol. 9, (1902) p. 229; as a symbol of division it was employed by Leibniz in 1684.

³ *Intermediaire des Mathematiciens*, vol. 2, (1895) p. 1.1 and vol. 4, (1897) p. 60.

⁴ Sander's in his *Plane and Solid Geometry*, 1903, p. 14 uses $\sim$ to represent is (or are) measured by, and Dupuis' excellent *Elementary Synthetic Geometry* uses $\simeq$ to represent similarity.

which Leibnitz assigned to it—even if such good works as Henrici und Treutlein, "Lehrbuch der Elementar-Geometrie," 1897, and Beman and Smith, "Plane and Solid Geometry," 1895, use $\sphericalangle$ to represent this concept. As one of the latter authors uses $\sim$ to represent similarity in his well known work, "The elementary teaching of mathematics," it appears that the latter of the two geometries mentioned above does not express a strong preference in favor of the symbol $\sphericalangle$. Among the most recent text-books which employ $\sim$ for similarity we may mention Bush and Clarke, "The Elements of Geometry," 1905, and Halsted, "Rational Geometry," 1904.

The symbol $\odot$ which stands for circle has two very distinct meanings in elementary mathematics just as the term circle itself has. Some authors use it for area, while others use it to represent the curve which encloses this area. As the latter is the more common meaning in mathematical literature it is to be hoped that this use will become universal and that the term circumference will soon find no place in elementary text-books. One of the most grotesque types of mathematical symbols is represented by $\sin^{-1}x$ for the angle whose sine is x . While $\sin ax$ means the a power of $\sin x$ for every value of a which differs from unity, it assumes an entirely different meaning for this special value of a . Various other symbols have been employed to represent this concept. One of the most desirable is $\sin \widehat{x}$ although this is not as commonly used as $\arcsin a$ introduced by Euler in 1737.

The main object of this paper is to call attention to several points with respect to symbols in very common use at the present time without entering upon the history of the ordinary symbols. The latter can be gathered with a fair degree of completeness from the histories of mathematics which are fortunately becoming more common. Teachers and students who use only a few books are apt to conclude that the symbols which they meet are the only ones in common use for these ideas. It is hoped that the present note will inspire caution and will aid to bring about a better condition in the course of time. It has contact with the "Elementary Notes" by Professor Oliver, published in *Annals of Mathematics*, Vol. 4 (1888), p. 186. As mathematics is preëminently cosmopolitan and eternal it is very important that its symbols should be world symbols. All national distinctions along this line should be obliterated as rapidly as possible.

The main points which the preceding arguments tend to establish may be expressed as follows: The decimal period should be replaced by the decimal comma. The term circle and the symbol $\odot$ should be used exclusively to represent the curve, similarity should be represented by $\sim$ and if the symbol $\sphericalangle$ is used it should represent the same thing as $\sim$ the symbol $\sphericalangle$ and the term equivalent should be abolished from elementary geometry; such symbols as $\sin^{-1}x$ should be replaced by $\arcsin x$ or by $\widehat{\sin x}$, and the symbol $:$ should not be defined to indicate division in general, but only to express the ratio.

METROLOGY NOTES.

Great Firm Adopts Metric System.

The Birmingham, England, firm of Kynochs, Ltd., which makes all sorts of goods from gunpowder to candles and soap, is introducing the metric system, in place of the old English, into its extensive manufacturing works. After a preparation of some two or three months, the change was effected in January of this year. Manufacturers who hesitate about the utility of such a change, the cost, or the great saving, should read the statements of Arthur Chamberlain, Chairman of Kynochs: He says, in part: "The metric system has been made legal, but permissive, and it has been expected that our Parliament would make it compulsory, giving two or three years of grace, at the end of which time only metric weights and measures would be legal. But Parliament continues to do nothing, and refuses to take the most simple and most valuable step that could be taken for the benefit of our commerce. * * * The matter became imperative with us because of the enormous mass of figures we have to deal with in our cost office, and we came to the conclusion that we would not wait any longer. * * * We have prepared tables to introduce the metric system everywhere throughout the works, and we have arranged to translate our results into English weights and measures for the benefit of the benighted English. * * * All our catalogues will be printed in the metric system, but for home use we may still translate them into English."

Mr. Chamberlain speaks in no uncertain terms concerning foreign commerce: "The English trade suffers immensely more in its foreign business from its weights and measures and its pounds, shillings and pence, than it does from the language. It is much easier for a foreigner with a small acquaintance with our language to understand our catalogues in English than it is to deal with pages of sizes and weights. Buyers are deterred ten times more by this confusion than by reason of the catalogues being printed in English."

Of the trouble and expense of changing from old to new: "A business man who objects to taking the trouble involved in improving his system had better retire at once. There is no room today for business men