PHYSICAL SECTION.

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On Convergents and Arithmetical Series, the Ratio of Whose Terms Approximate Successively the Value of π ; and on Their Application to the Construction of Computing Machines.

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Several years ago I had occasion to make use of a complicated reduction formula, the numerical application of which involved the frequent use of a value of π accurate to the sixth decimal place. In order to avoid the continued manipulation of the large number of figures which this necessitated, I substituted for the decimalized value a ratio between two whole numbers, P and Q , so chosen as to express this value to the required degree of accuracy. These numbers are easily found by the method of convergents. For the value of π required, *i. e.*, $\pi = 3.1415927$, this gives

$\frac{P}{Q} \doteq C_1 = \frac{3}{1}$	$= 3.0000000$	c	d
$\doteq C_2 = \frac{22}{7}$	$= 3.1428571$	3	141592.7×10^{-6}
$\doteq C_3 = \frac{333}{106}$	$= 3.1415094$	7	1264.4×10^{-6}
$\doteq C_4 = \frac{355}{113}$	$= 3.1415929$	15	83.3×10^{-6}
$\doteq C_5 = \frac{126003}{40108}$	$= 3.14159269 +$	1	$.2 \times 10^{-6}$
$\doteq C_6 = \frac{252361}{80329}$		354	$.01 \times 10^{-6}$
$\doteq C_7 = \frac{1640169}{522082}$		2	
		6	

	c	J
$\cong C_8 = \frac{1892530}{602411}$	1	
$\cong C_9 = \frac{9210289}{2931726}$	4	
$\cong C_{10} = \frac{11102819}{3534137}$	1	
$\cong C_{11} = \frac{31415927}{10000000} = 3.1415927$	2	0.0000×10^{-6}

The terms c and J of the above series denote respectively the "components" of the continued ratios and the differences between the successive convergents and the required value. The relations between the quantities $C_1, C_2, \dots, C_n$ are expressed by the well-known equation

$$C_n = \frac{P_n}{Q_n} = \frac{C_n P_{n-1} + P_{n-2}}{C_n Q_{n-1} + Q_{n-2}} \quad (1)$$

The remarkable characteristics of this particular series of convergents is the very rapid decrease in the value of J for the first four terms of the series. The third convergent, C_3 , of the series expresses the correct value of π to within one unit in the fourth decimal place, *i. e.*, to within 1 part in about 38,000; the fourth convergent, C_4 , expresses the required value to within 2 parts in the *seventh* decimal place, or to within 1 part in about 15,000,000. It is very unusual that an incommensurable number such as π can be expressed with this degree of accuracy by a ratio between two numbers, each consisting of only three figures.

The third convergent, C_3 , expresses the value of π to the degree of accuracy usually necessary in computations. For such purposes it is, however, just as easy to employ the fourth convergent, C_4 , which is of exactly the same order of magnitude as C_3 . The same statement applies to the use of these ratios in computing machines and mechanical solutions of equations involving this constant. In the latter operations, however, we can frequently employ with advantage a ratio which is less accurate and involves fewer numbers than C_3 , but which must be more accurate than the

next lower convergent, C_2 , whose error is about 1 part in about 2,400. This led me to search for other ratios intermediate both in order and accuracy to $C_2 - C_3$, and to the discovery of the arithmetical series which it is the first purpose of this paper to describe.

It will be noted that the "component" or approximate ratio, c_4 , of the fourth convergent is unity. From (1) it follows, therefore, at once that

$$P_4 = P_3 + P_2$$

and

$$Q_4 = Q_3 + Q_2$$

That is, both terms of the convergents C_3 and C_4 are members of arithmetical series whose common differences are the corresponding terms, 22 and 7, of the convergent C_2 .

Fig. 1.

This suggested to me the probability of the lower terms of the same series forming ratios which would be intermediate in value between the convergents C_3 and C_4 and the convergent C_2 . This was found to be the case. The two series are

$$S = 1 + 8 + 15 + 22 + \dots 106 + 113 + 120, \text{etc.}$$

$$S' = 3 + 25 + 47 + 69 + \dots 333 + 355 + 377, \text{etc.}$$

The successive values of the ratios

$$\frac{S'_m}{S_m}$$

are

			J_m
$m = 1$	$\frac{S'_1}{S_1} = \frac{3}{1} = 3.000000 = C_1$	$—$	141592.7×10^{-6}
$m = 2$	$\frac{S'_2}{S_2} = \frac{25}{8} = 3.125000$		16592.7×10^{-6}
$m = 9$	$\frac{S'_9}{S_9} = \frac{179}{57} = 3.1403509 \cong C_2$	$—$	1241.8×10^{-6}
$m = 10$	$\frac{S'_{10}}{S_{10}} = \frac{201}{64} = 3.1406250$	$—$	967.7×10^{-6}
$m = 11$	$\frac{S'_{11}}{S_{11}} = \frac{223}{71} = 3.1408451$	$—$	747.6×10^{-6}
$m = 12$	$\frac{S'_{12}}{S_{12}} = \frac{245}{78} = 3.1410256$	$—$	567.1×10^{-6}
$m = 13$	$\frac{S'_{13}}{S_{13}} = \frac{267}{85} = 3.1411765$	$—$	416.3×10^{-6}
$m = 14$	$\frac{S'_{14}}{S_{14}} = \frac{289}{92} = 3.1413043$	$—$	288.4×10^{-6}
$m = 15$	$\frac{S'_{15}}{S_{15}} = \frac{311}{99} = 3.1414141$	$—$	178.6×10^{-6}
$m = 16$	$\frac{S'_{16}}{S_{16}} = \frac{333}{106} = 3.1415094 = C_3$	$—$	83.3×10^{-6}
$m = 17$	$\frac{S'_{17}}{S_{17}} = \frac{355}{113} = 3.1415929 = C_4$	$+$	0.2×10^{-6}

The ratio

$$\frac{S'_9}{S_9}$$

is very nearly equal to C_2 . Any desired degree of approximation to the value of π between C_2 and C_4 may be secured by choice of a suitable value of m from the above table. Thus, for a degree of approximation of 1 part in 10,000 we select the ratio for which $m = 14$, *i. e.*,

$$\frac{S'_{14}}{S_{14}} = \frac{289}{92}$$

This ratio is sufficiently accurate for all ordinary computations and mechanical solutions, and involves one less figure than either the ratios C_3 or C_4 .

GRAPHICAL AND MECHANICAL COMPUTATIONS.

The very accurate representation of the value of π by the simple ratios just found enables us to readily obtain the circumference of any circle whose diameter is given, or, *vice versa*, by means of an ordinary slide rule, or, better still, by a simple gearing. If we gear together two wheels which contain respectively 355 and 113 teeth, the angular motion of the second will be π times that of the first. If we add two dials divided into the same number of parts and two

Fig. 2.

revolution counters to register the whole turns, we can obtain the product of any number of figures by π correct to within two units in the eighth place.

A plan and front view of a machine constructed for this purpose is shown in *Figs. 1 and 2*.^{*} The two gear wheels are 48 pitch, and are therefore 7.396" and 2.354" pitch diameter respectively. They are provided with dials, each about 4" in diameter, divided into 500 parts graduated from 0 to 100. The revolution counters are two worm

^{*} I have been unable to find the original drawings, and as the machine itself is not now in my possession, the general details of the device are drawn from memory and may not exactly correspond to those of the original construction. They are, however, essentially the same.—AUTHOR.

wheels, each having 100 teeth driven by worms on the axes of the two gears. With well-cut gears the error due to irregular spacing, roughness in the teeth, etc., etc., does not exceed 2 per cent. of the tooth space. The accuracy of the ratio between the readings of the two dials can therefore be relied upon to within $\frac{1}{10}$ of a division, *i. e.*, to $\frac{1}{5000}$ of a revolution. If, therefore, we demand an accuracy of 1 part in the seventh decimal place, *i. e.*, $\frac{1}{3000000}$ of the value of π , we must revolve the driving gear *A* at least 200 times. Thus, if the number x , whose product by π is desired is 10,254, $10\frac{254}{1000}$ turns of the dial and gear *A* will give the product πx on the dials of gear *B* to 1 part in 50,000; $102\frac{54}{100}$ turns of *A* will give the product πx to 1 part in 500,000; and $1,025\frac{4}{10}$ turns, the product πx to 1 part in 5,000,000. Similarly, if the number x is 495687·3, we must give *A* $495\frac{687\frac{3}{10}}{1000}$ turns if we wish to obtain πx to the same number of significant figures. For operations involving this number of fig-

Fig. 3.

ures it is inconvenient to turn the wheels by hand, and a small motor belted to the axis of *B*, as shown by the dotted lines, is provided for this purpose. The motor is, of course, stopped just before the final reading is reached and the operation completed by hand. In order to return the readings to zero for another operation, the worms *C* and *D* may be lifted out of engagement with their respective axes, as shown by the dotted lines of *Fig. 1*, and the index mark of wheel *B* adjusted and clamped at any point on the circumference of its respective dial. The index mark of *D* is correspondingly adjustable over a space of 1 division, so that there may never be any uncertainty as to the whole number of revolutions indicated by *D*.

With this machine it is possible to obtain the product of any given number of 8 places or less by π much more easily and quickly than it can be done by the use of logarithms or by any of the usual forms of calculating machines.

For less accurate work a much smaller instrument can be constructed by using the ratio S_{14} , making the gears of finer diametrical pitch and omitting the revolution counters C and D . Greater compactness can also be secured by using, instead of external gears, two concentric internal and external gears of 289 and 92 teeth respectively, connected by idler gears. The reading dials can be placed on opposite sides of the small circular case containing the whole. In this way, using gears of 128 pitch, an instrument of about $2\frac{1}{2}$ inches diameter and $\frac{1}{2}$ inch thick, or about the size of an old-fashioned watch, can be made. This is small enough to be readily carried in the pocket and will enable a machinist to mechanically rectify, or, if desired, "square" a circle of any given diameter with an accuracy of 1 part in 10,000.

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POWER-TRANSMISSION POSSIBILITIES IN THE SOUTHERN APPALACHIANS.

The present commercial possibilities in the electrical transmission of power over long distances with small loss, and the opportunities thus offered for the combination of a number of small plants into one of considerable magnitude, serve to direct attention to a number of water powers in the Southern States which are now totally ignored.

The popular idea of an available water-power is a moderate fall on a stream of considerable size, located within a few yards of a railroad. Like many popular ideas, it is only partly correct. The power which can be obtained at any power site on a stream is the product of the weight of the water passing the site per unit of time multiplied by the height of the perpendicular fall obtainable, and for the same total power these quantities vary inversely; that is, if the head be doubled, the quantity of water may be diminished by a half and the resultant power will be the same; if the head be multiplied by ten, then one-tenth the quantity of water will yield the power, and so on, in similar ratio.

Given a sufficient fall, therefore, a small amount of water may be made to furnish a very large amount of power, and, in point of fact, many of the large power developments of the Western States are of this type, the quantity of water used being small, but the head being measured by hundreds of feet. The distance between the points where the power is developed and where it is to be utilized is annihilated by electrical transmission.

A large number of streams in that part of the Southern States in which the Appalachian Mountain system attains its maximum development have their sources far up on the mountain slopes, in regions of great rainfall and of large