The factor k^4 divides out, and omitting it we have

$$4Q - P^2 - 4(1 + k^2)R + 16k^2S + 2k^2PR - 4(k^2 + k^4)PS - k^4R^2 + 4k^4QS \\ = -2\sqrt{S}\{2 - (1 + k^2)P + 2k^2Q - (k^2 + k^4)R + 2k^4S\},$$

or, as this may also be written,

$$\{(-P^2 + 4Q - 4R) + k^2(-4R + 2PR + 16S - 4PS) \\ + k^4(-R^2 + 4QS - PS)\} \\ = -2\sqrt{S}\{2 - P + k^2(-P + 2Q - R) + k^4(-R + 2S)\},$$

which is the required rational equation involving the product of the sn's.

Coefficients of Induction and Capacity of Two Electrified Spheres.

By A. G. GREENHILL, M.A.

[Read January 9th, 1879.]

Let a, b be the radii of the spheres A and B , c the distance between the centres; and let A be charged to potential 1 and insulated, and let B be uninsulated, that is, at potential zero.

Then, with Thomson's notation (Papers on "Electrostatics and Magnetism," p. 87),

$$f_1 = 0, \quad p_1 = a; \quad f_n = \frac{b^2}{c - f_n}, \quad p'_n = -p_n \frac{f'_n}{b};$$

$$f_{n+1} = \frac{a}{c - f_n}, \quad p_{n+1} = -p'_n \frac{f_{n+1}}{a}.$$

Let
$$p_n = \frac{1}{P_n}, \quad p'_n = -\frac{1}{P'_n};$$

therefore
$$f'_n = b \frac{P_n}{P'_n}, \quad f_{n+1} = a \frac{P'_n}{P_{n+1}};$$

and therefore
$$b \frac{P_n}{P'_n} = \frac{b^3}{c - a \frac{P'_{n-1}}{P_n}},$$

or
$$cP_n = bP'_n + aP'_{n-1} \dots\dots\dots (1).$$

So also
$$a \frac{P'_n}{P_{n+1}} = \frac{a^3}{c - b \frac{P_n}{P'_n}},$$

or
$$cP'_n = aP_{n+1} + bP_n \dots\dots\dots (2).$$

From (1) and (2),

$$P_{n+1} - \frac{c^3 - a^3 - b^3}{ab} P_n + P_{n-1} = 0 \dots\dots\dots (3),$$

$$P'_{n+1} - \frac{c^3 - a^3 - b^3}{ab} P'_n + P'_{n-1} = 0 \dots\dots\dots (4).$$

Therefore
$$P_n = Ax^n + Bx^{-n},$$

$$P'_n = A'x^n + B'x^{-n},$$

where x is a root of the reciprocal quadratic equation

$$x^2 - \frac{c^3 - a^3 - b^3}{ab} x + 1 = 0.$$

If we put $x = e^\varpi$, then

$$e^\varpi + e^{-\varpi} = \frac{c^3 - a^3 - b^3}{ab},$$

or
$$\cosh \varpi = \frac{c^3 - a^3 - b^3}{2ab}.$$

If we put
$$\frac{b^3 + c^3 - a^3}{2bc} = \cosh \alpha$$

$$\frac{c^3 + a^3 - b^3}{2ca} = \cosh \beta,$$

then
$$\sinh \alpha = \frac{h}{b}, \quad \sinh \beta = \frac{h}{a}, \quad \sinh \varpi = \frac{hc}{ab};$$

where
$$h^2 = \frac{a^4 + b^4 + c^4 - 2b^3c^3 - 2c^3a^3 - 2a^3b^3}{4c^3},$$

and
$$\alpha + \beta = \varpi.$$

Determining the constants A, B, A', B' , we shall find

$$P_n = \frac{1}{h} \sinh (n\varpi - \alpha), \quad P'_n = \frac{1}{h} \sinh n\varpi,$$

and therefore $p_n = h \operatorname{cosech} (n\varpi - \alpha), \quad p'_n = -h \operatorname{cosech} n\varpi.$

Now q_{aa} , the coefficient of capacity of the sphere A , is the charge of the sphere A when charged to potential 1, the sphere B being at zero potential; and q_{ab} , the coefficient of induction of the spheres, is the charge of the sphere B .

$$\text{Therefore} \quad q_{aa} = \Sigma p_n = h \sum_{n=1}^{\infty} \operatorname{cosech} (n\varpi - \alpha),$$

$$q_{ab} = \Sigma p'_n = -h \sum_{n=1}^{\infty} \operatorname{cosech} n\varpi.$$

If the sphere B had been insulated and charged to potential 1, and the sphere A had been uninsulated, we should have obtained the same value of q_{ab} , and we should find

$$q_{bb} = h \sum_{n=1}^{\infty} \operatorname{cosech} (n\varpi - \beta).$$

These are equivalent to the expressions given in Maxwell's *Electricity*, p. 216.

Put $\alpha - \beta = x$; also $\alpha + \beta = \varpi$, and put $q = e^{-\varpi}$, then

$$q_{aa} = h \sum_{n=1}^{\infty} \operatorname{cosech} \frac{1}{2} \{ (2n-1) \varpi - x \},$$

$$q_{bb} = h \sum_{n=1}^{\infty} \operatorname{cosech} \frac{1}{2} \{ (2n-1) \varpi + x \};$$

and, by a formula of elliptic functions,

$$q_{aa} - q_{bb} = hk \frac{K'}{\varpi} \operatorname{tanam} \left(K' \frac{x}{\varpi}, k' \right),$$

where $q = e^{-\varpi} = e^{-\pi \frac{K'}{K}}, \text{ or } \varpi = \pi \frac{K'}{K}.$

This is the formula for any two spheres not in contact, analogous to the formula for two spheres in contact,

$$E_a - E_b = \frac{\pi ab}{a+b} \tan \frac{\pi}{2} \frac{a-b}{a+b},$$

given on page 220 of Maxwell's *Electricity*.

When the spheres are in contact, $c = a+b$, $h = 0$, $\alpha = 0$, $\beta = 0$, $\varpi = 0$; but

$$lt \frac{h}{\varpi} = \frac{ab}{c} = \frac{ab}{a+b}; \quad lt \frac{\alpha}{\varpi} = \frac{a}{a+b}, \quad lt \frac{\beta}{\varpi} = \frac{b}{a+b};$$

and

$$q_{aa} = \frac{ab}{a+b} \sum_{n=1}^{\infty} \frac{1}{n - \frac{a}{a+b}} = \frac{ab}{a+b} \sum_{n=0}^{\infty} \frac{1}{n + \frac{b}{a+b}},$$

$$q_{ab} = -\frac{ab}{a+b} \sum_{n=1}^{\infty} \frac{1}{n},$$

$$q_{bb} = \frac{ab}{a+b} \sum_{n=1}^{\infty} \frac{1}{n - \frac{b}{a+b}} = \frac{ab}{a+b} \sum_{n=0}^{\infty} \frac{1}{n + \frac{a}{a+b}}.$$

The series for q_{aa} , q_{ab} , q_{bb} are now infinite, but E_a and E_b are finite, for, the potential of each sphere being unity,

$$E_a = q_{aa} + q_{ab} = \frac{ab}{a+b} \sum_{n=1}^{\infty} \left(\frac{1}{n - \frac{a}{a+b}} - \frac{1}{n} \right),$$

$$E_b = q_{ab} + q_{bb} = \frac{ab}{a+b} \sum_{n=0}^{\infty} \left(\frac{1}{n - \frac{b}{a+b}} - \frac{1}{n} \right).$$

Now $\Gamma(x) = \lim_{\mu \rightarrow \infty} \mu^x \frac{1 \cdot 2 \cdot 3 \dots \mu}{x(x+1)(x+2) \dots (x+\mu)};$

and therefore, denoting $\frac{d}{dx} \log \Gamma(x)$ by $\psi(x)$,

$$\psi(x) = \log \mu - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \dots - \frac{1}{x+\mu}, \quad \mu = \infty.$$

Therefore $E_a = \frac{ab}{a+b} \left\{ \psi(1) - \psi\left(\frac{b}{a+b}\right) \right\},$

$$E_b = \frac{ab}{a+b} \left\{ \psi(1) - \psi\left(\frac{a}{a+b}\right) \right\}.$$

Since $\Gamma\left(\frac{1-x}{2}\right) \Gamma\left(\frac{1+x}{2}\right) = \frac{\pi}{\cos \frac{1}{2}\pi x},$

therefore $\psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{1-x}{2}\right) = \pi \tan \frac{1}{2}\pi x,$

and, putting $x = \frac{a-b}{a+b},$

$$\psi\left(\frac{a}{a+b}\right) - \psi\left(\frac{b}{a+b}\right) = \pi \tan\left(\frac{\pi}{2} \frac{a-b}{a+b}\right),$$

leading to the result

$$E_a - E_b = \frac{\pi ab}{a+b} \tan\left(\frac{\pi}{2} \frac{a-b}{a+b}\right).$$

[$\psi(1)$ is Euler's constant; these expressions for E_a and E_b in terms of the function ψ are given in Mr. Niven's paper read before this Society Dec. 1876, Vol. VIII., p. 78.]

Now $\frac{d}{dx} \log \coth \frac{1}{2}(n\pi - x) = \operatorname{cosech}(n\pi - x),$

and this suggests that we should put

$$\begin{aligned}\phi(x, q) &= \prod_{n=1}^{\infty} \coth \frac{1}{2} (n\varpi - x) \\ &= \prod_{n=1}^{\infty} \frac{1 + e^{-n\varpi + x}}{1 - e^{-n\varpi + x}} \\ &= \prod_{n=1}^{\infty} \frac{1 + q^n e^x}{1 - q^n e^x}, \text{ where } q = e^{-\varpi},\end{aligned}$$

and that we should also put

$$\begin{aligned}\psi(x, q) &= \frac{d}{dx} \log \phi(x, q) \\ &= \sum_{n=1}^{\infty} \operatorname{cosech} (n\varpi - x),\end{aligned}$$

so that

$$\begin{aligned}q_{aa} &= h\psi(\alpha, q) = h\psi\left(\frac{\varpi+x}{2}, q\right), \\ q_{bb} &= h\psi(\beta, q) = h\psi\left(\frac{\varpi-x}{2}, q\right).\end{aligned}$$

The expression $\phi(x, q)$ is one of Heine's generalized hyper-geometric series, discussed on p. 97 of the second edition of Heine's *Kugelfunctionen*.

Analogous to the formula

$$\Gamma\left(\frac{1+x}{2}\right) \Gamma\left(\frac{1-x}{2}\right) = \frac{\pi}{\cos \frac{\pi}{2} x},$$

we have
$$\begin{aligned}\phi\left(\frac{\varpi+x}{2}, q\right) &= \prod_{n=1}^{\infty} \coth \frac{1}{2} (n\varpi - \frac{\varpi+x}{2}) \\ &= \prod_{n=0}^{\infty} \coth \frac{1}{2} \{(2n+1)\varpi - x\},\end{aligned}$$

and
$$\phi\left(\frac{\varpi-x}{2}, q\right) = \prod_{n=0}^{\infty} \coth \frac{1}{2} \{(2n+1)\varpi + x\};$$

and therefore
$$\begin{aligned}&\phi\left(\frac{\varpi+x}{2}, q\right) \phi\left(\frac{\varpi-x}{2}, q\right) \\ &= \prod_{n=0}^{\infty} \coth \frac{1}{2} \{(2n+1)\varpi - x\} \coth \frac{1}{2} \{(2n+1)\varpi + x\} \\ &= \prod_{n=-\infty}^{\infty} \prod_{m=-\infty}^{\infty} \frac{x + (2n+1)\varpi + (2m+1)2i\pi}{x + (2n+1)\varpi + 4mi\pi} \\ &= \prod_{n=-\infty}^{\infty} \prod_{m=-\infty}^{\infty} \frac{K_1' \frac{x}{\varpi} + (2n+1)K_1' + (2m+1)iK_1}{K_1' \frac{x}{\varpi} + (2n+1)K_1' + 2miK_1},\end{aligned}$$

where
$$\frac{K'_1}{K_1} = \frac{\varpi}{2\pi} = \frac{K'}{2K}.$$

Therefore
$$\phi\left(\frac{\varpi+x}{2}, q\right) \phi\left(\frac{\varpi-x}{2}, q\right)$$

$$\begin{aligned} &= \frac{\Theta_1\left(K'_1 \frac{x}{\varpi}, k'_1\right)}{H_1\left(K'_1 \frac{x}{\varpi}, k'_1\right)} = \frac{1}{\sqrt{k_1}} \frac{\operatorname{dn} K'_1 \frac{x}{\varpi}}{\operatorname{cn} K'_1 \frac{x}{\varpi}} \\ &= \frac{1}{\sqrt{k_1} \operatorname{sn}\left(K'_1 \frac{\varpi-x}{\varpi}, k'_1\right)}, \end{aligned}$$

analogous to
$$\Gamma\left(\frac{1+x}{2}\right) \Gamma\left(\frac{1-x}{2}\right) = \frac{\pi}{\sin \frac{\pi}{2}(1-x)},$$

Therefore
$$\begin{aligned} q_{aa} - q_{bb} &= h\psi\left(\frac{\varpi+x}{2}, q\right) - h\psi\left(\frac{\varpi-x}{2}, q\right) \\ &= \frac{1}{2}h \frac{d}{dx} \log \frac{\operatorname{dn}\left(K'_1 \frac{x}{\varpi}, k'_1\right)}{\operatorname{cn}\left(K'_1 \frac{x}{\varpi}, k'_1\right)} \\ &= \frac{1}{2}h \frac{K'_1}{\varpi} \frac{k_1^2 \operatorname{sn}\left(K'_1 \frac{x}{\varpi}, k'_1\right)}{\operatorname{cn}\left(K'_1 \frac{x}{\varpi}, k'_1\right) \operatorname{dn}\left(K'_1 \frac{x}{\varpi}, k'_1\right)} \\ &= \frac{1}{2i} h \frac{K'_1}{\varpi} \frac{k_1^2 \operatorname{sn}\left(iK'_1 \frac{x}{\varpi}, k_1\right) \operatorname{cn}\left(iK'_1 \frac{x}{\varpi}, k_1\right)}{\operatorname{dn}\left(iK'_1 \frac{x}{\varpi}, k_1\right)}, \end{aligned}$$

which, by Landen's transformation,

$$\begin{aligned} &= \frac{1}{2i} h \frac{K'_1}{\varpi} \frac{k_1^3}{1+k_1} \operatorname{sn}\left(iK' \frac{x}{\varpi}, k\right) \\ &= h \frac{K'}{\varpi} \frac{1-k_1}{1+k_1} \operatorname{tanam}\left(K' \frac{x}{\varpi}, k'\right) \\ &= hk \frac{K'}{\varpi} \operatorname{tanam}\left(K' \frac{x}{\varpi}, k'\right), \end{aligned}$$

as before.

The expression for q_{ab} is Lambert's series, and is given at the end of the "Fundamenta Nova."

54 *Coefficients of Induction of Two Electrified Spheres.* [Jan. 9,

With Heine's notation for the generalized hypergeometric series,

$$\phi(\alpha, \beta, \gamma, q, \xi) = 1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} q^\xi + \dots,$$

we shall have
$$\phi(x, q) = \phi\left(i \frac{\pi}{\omega}, 1, 1, q, x\right).$$

The electrification at any point Q of the sphere A when insulated and charged to potential 1, the sphere B being uninsulated, is that due to the successive images $P_1, P_2, P_3, \dots$

The electrification due to the image P_n

$$= p_n \frac{a^2 - f_n^2}{4\pi a} \frac{1}{P_n Q^3},$$

and this expanded in a series of zonal harmonics, of which the axis is the line drawn from the centre of the sphere A to the centre of the sphere B ,

$$\begin{aligned} &= \frac{p_n}{4\pi a^3} \sum_{i=0}^{\infty} (2i+1) \left(\frac{f_n}{a}\right)^i Q_i \\ &= \frac{h}{4\pi a^3} \operatorname{cosech}(n\omega - \alpha) \sum_{i=0}^{\infty} (2i+1) \left\{ \frac{\sinh(n\omega - \alpha)}{\sinh(n\omega - \alpha)} \right\}^i Q_i; \end{aligned}$$

and therefore the electrification of the sphere A

$$= \frac{h}{4\pi a^3} \sum_{n=1}^{\infty} \operatorname{cosech}(n\omega - \alpha) \sum_{i=0}^{\infty} (2i+1) \left\{ \frac{\sinh(n\omega - \alpha)}{\sinh(n\omega - \alpha)} \right\}^i Q_i$$

Similarly, the electrification at any point of the sphere B

$$= -\frac{h}{4\pi a^3} \sum_{n=1}^{\infty} \operatorname{cosech} n\omega \sum_{i=0}^{\infty} (2i+1) \left\{ \frac{\sinh(n\omega - \alpha)}{\sinh n\omega} \right\}^i Q_i,$$

the axis of the zonal harmonics being now the line drawn from the centre of the sphere B to the centre of the sphere A .

To obtain the expressions for q_{aa} , q_{ab} , and q_{bb} as definite integrals, as given by Poisson,

$$\begin{aligned} q_{aa} &= h \sum_{n=1}^{\infty} \operatorname{cosech}(n\omega - \alpha) \\ &= 2h \left(\frac{e^{-\beta}}{1 - e^{-2\beta}} + \frac{e^{-\omega - \beta}}{1 - e^{-2\omega - 2\beta}} + \dots \right) \\ &= 2h (e^{-\beta} + e^{-3\beta} + e^{-5\beta} + \dots + e^{-\omega - \beta} + e^{-3\omega - 3\beta} + e^{-5\omega - 5\beta} + \dots) + \dots \end{aligned}$$

and, adding vertical columns,

$$\begin{aligned}
 q_{aa} &= 2h \left(\frac{e^{-\beta}}{1-e^{-\varpi}} + \frac{e^{-3\beta}}{1-e^{-3\varpi}} + \dots \right) \\
 &= 2h \sum_{n=0}^{h-\infty} \frac{e^{-(2n+1)\beta}}{1-e^{-(2n+1)\varpi}} \\
 &= h \sum_{n=0}^{h-\infty} e^{-(2n+1)\beta} \left\{ 1 + \frac{1+e^{-(2n+1)\varpi}}{1-e^{-(2n+1)\varpi}} \right\} \\
 &= h \frac{e^{-\beta}}{1-e^{-2\beta}} + h \sum e^{-(2n+1)\beta} \coth \frac{1}{2}(2n+1)\varpi.
 \end{aligned}$$

Now
$$2 \int_0^\infty \coth \pi x \sin mx \, dx = \coth \frac{1}{2}m,$$

therefore
$$\begin{aligned}
 q_{aa} &= \frac{1}{2}h \operatorname{cosech} \beta + 2h \int_0^\infty \coth \pi x \sum_{n=0}^{h-\infty} e^{-(2n+1)\beta} \sin (2n+1)\varpi x \, dx \\
 &= \frac{1}{2}h \operatorname{cosech} \beta + 2h \int_0^\infty \coth \pi x \frac{e^{-\beta}(1+e^{-2\beta}) \sin \varpi x \, dx}{1-2e^{-2\beta} \cos 2\varpi x + e^{-4\beta}} \\
 &= \frac{1}{2}h \operatorname{cosech} \beta + 2h \cosh \beta \int_0^\infty \frac{\coth \pi x \sin \varpi x \, dx}{\cosh 2\beta - \cos 2\varpi x}.
 \end{aligned}$$

Similarly
$$q_{bb} = \frac{1}{2}h \operatorname{cosech} \alpha + 2h \cosh \alpha \int_0^\infty \frac{\coth \pi x \sin \varpi x \, dx}{\cosh 2\alpha - \cos 2\varpi x},$$

and
$$q_{ab} = \frac{1}{2} \operatorname{cosech} \varpi + 2h \cosh \varpi \int_0^\infty \frac{\coth \pi x \sin \varpi x \, dx}{\cosh 2\varpi - \cos 2\varpi x};$$

equivalent to the expressions given by Poisson.

On Certain Systems of Partial Differential Equations of the First Order with several Dependent Variables. By H. W. LLOYD TANNER, M.A.

[Read January 9th, 1879.]

The simplest systems considered in this paper are composed of equations such as

$$\frac{dz_1}{dx_1} + \frac{dz_2}{dx_2} + \dots + \frac{dz_n}{dx_n} = 0.$$

In the general case there are n independent variables, and $m(n-m)+1$ dependent variables. For the sake of symmetry, however, we assume