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LV. *On the Self-Induction and on the Gravity-Potential of a Ring.* By W. M. HICKS*.

THE self-induction of a finite ring has been considered by Professor Minchin in the March number of the *Philosophical Magazine* on the assumption that it is the flux through a diaphragm composed of the aperture and half the surface of the ring. In reading this paper it occurred to me that the problem could more naturally be attacked by means of Toroidal Functions. In considering it from this point of view I was led to a conclusion which does not seem to have been heretofore noticed.

The expression for the self-induction of a thin circular ring as given by Maxwell is $\pi R(4 \log 8R/r - 8)$, and for a ring of finite size as given by Minchin is $\pi R(4 \log 8R/r - 8) +$ terms involving the radius (r) of the transverse section. In the finite ring the current-density varies inversely as the distance from the straight† axis. In Maxwell's wire the distribution of current across the transverse section is considered uniform, and it would therefore appear that his formula gives the self-induction so far as it is independent of the thickness, and that this part is independent of the distribution of current-density within the wire itself. As a matter of fact, however, the distribution of current-density has a deciding effect on the value of the second term, and this however small the section of the ring may be. The correct value of the self-induction, neglecting powers of r/R , is $\pi R(4 \log 8R/r - 7)$. The expression $\pi R(4 \log 8R/r - 8)$ gives the value of the flux through the *aperture* only, which to this degree of approximation is independent of the law of current-density. The difference is due to the fact that although the section across the ring may be exceedingly small, yet the forces are correspondingly large, and the interaction of the different parts of the current cannot be neglected even when the ring is extremely thin and of large aperture. *E. g.* in a ring of 1 metre radius and 1 millim. thick the usually accepted formula gives a value $3\frac{1}{2}$ per cent. too small.

In the present paper the question is treated by the method usually employed in dealing with problems of electric potential and fluid motion, but which, so far as I know, is new in its application to electromagnetic and gravity potentials. The

* Communicated by the Author.

† By *straight* axis is meant the line through the centre of the ring perpendicular to its plane. The *circular* axis is the locus of the centres of the transverse sections of the ring.

functions which naturally come in are Toroidal Functions, the properties of which are developed in two papers in the Transactions of the Royal Society*. These papers are referred to in the present pages as [T. F. i.] and [T. F. ii.].

The special problems considered are:—

- (1) The force-flux function and self-induction for a ring in which the current-density varies inversely as the distance from the straight axis. This is the case of current through a ring consisting of a single turn of round wire.
- (2) The same quantities when the current-density is constant. This leads to the case of a uniformly wound coil of circular cross section.
- (3) The method is then exemplified by finding the gravity-potential of a finite ring—a problem which has been recently very fully considered by Mr. Dyson†.

1. The force is symmetrical round the straight axis of the ring. Take this axis for the axis of z and the origin at the centre of the ring, and let ρ, z denote the cylindrical coordinates of any point. Further, let R denote the radius of the circular axis, r that of a cross section, and a that of the critical circle in the curvilinear coordinates used in toroidal functions—*i. e.* in the system of coaxial circles having Oz for common radical axis and having one circle of the system coinciding with the cross section. The length of a is equal to the length of a tangent line from the centre to the circle. It will be found convenient to express values later in terms of R and the angle subtended at the centre by a cross section: this angle will be denoted by α .

The flux function ψ at a point P will be taken as denoting

the total flux *up* through a circle, radius PN (see figure). It is the same as that through a diaphragm stretching from O

* Phil. Trans. 1881, Part iii. p. 609, and 1884, Part i. p. 161.

† Phil. Trans. 1893, A. pp. 43 & 1041.

and bounded by this circle. Let w, w' denote the velocities at P parallel to NP and Oz. Then

$$\begin{aligned}\frac{d\psi}{d\rho} d\rho &= \text{flux up through an annulus whose radii are} \\ &\quad \text{PN and PN} + d\rho, \\ &= 2\pi\rho d\rho w', \\ \frac{d\psi}{dz} dz &= \text{flux through cylindrical tyre (breadth } dz) \\ &\quad \text{towards N,} \\ &= -2\pi\rho dz w;\end{aligned}$$

whence

$$w = -\frac{1}{2\pi\rho} \frac{d\psi}{dz}, \quad w' = \frac{1}{2\pi\rho} \frac{d\psi}{d\rho}.$$

2. Consider a current flowing through an elementary ring whose cross section is $d\rho dz$ and in which the current-density is σ . Then the circulation taken round this

$$= 4\pi \times \text{current} = 4\pi\sigma d\rho dz.$$

But the circulation

$$= \left(\frac{dw}{dz} - \frac{dw'}{d\rho} \right) d\rho dz = 4\pi\sigma d\rho dz;$$

whence

$$\frac{1}{2\pi\rho} \frac{d^2\psi}{dz^2} + \frac{d}{d\rho} \left(\frac{1}{2\pi\rho} \frac{d\psi}{d\rho} \right) = -4\pi\sigma,$$

or

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\rho^2} - \frac{1}{\rho} \frac{d\psi}{d\rho} = -8\pi^2\rho\sigma. \quad \dots \quad (1)$$

At points where there is no current the right-hand side is of course zero.

3. To find the force-flux we shall require two functions, one (ψ') for space outside the ring, and the other (ψ) for space inside. The conditions which they have to fulfil are as follows:—

ψ' must be finite at all points outside the ring, be zero at the point O, and satisfy the equation

$$\frac{d^2\psi'}{dz^2} + \frac{d^2\psi'}{d\rho^2} - \frac{1}{\rho} \frac{d\psi'}{d\rho} = 0.$$

Using toroidal functions, these conditions are satisfied by

$$\psi' = \frac{1}{\sqrt{(C-c)}} \sum R_n (A_n \cos nv + A'_n \sin nv).$$

ψ must be finite at all points inside the ring, and must satisfy

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\rho^2} - \frac{1}{\rho} \frac{d\psi}{d\rho} = -8\pi^2\rho\sigma. \quad (1)$$

These conditions are satisfied by

$$\psi = f + \frac{1}{\sqrt{C-c}} \Sigma T_n (B_n \cos nv + B'_n \sin nv),$$

where f is any particular solution of the equation (1).

Further, at the boundary of the ring ψ and the force are continuous. That is, when $u = u_0$ then for all values of v ,

$$\psi = \psi' \quad \text{and} \quad \frac{d\psi}{du} = \frac{d\psi'}{du}, \quad (2)$$

where u_0 is the value of u corresponding to the boundary. These conditions will serve to find completely ψ and ψ' in all cases.

In order to apply these last conditions it will be necessary to expand f in a series of cosines and sines of multiple angles. In the cases here considered it will be seen later that the sine terms do not enter, and that f is of the form

$$f = \frac{1}{\sqrt{C-c}} \Sigma F_n \cos nv,$$

where F_n is a function of u only.

The conditions (2) now give

$$A'_n = 0, \quad B'_n = 0.$$

$$A_n R_n = B_n T_n + F_n,$$

$$A_n R'_n = B_n T'_n + F'_n;$$

where dashed letters denote differentiation with respect to u , and the functions are now to be regarded as *functions of u_0 only*.

From these,

$$A_n (R'_n T_n - R_n T'_n) = T_n F'_n - T'_n F_n,$$

$$B_n (R'_n T_n - R_n T'_n) = R_n F'_n - R'_n F_n.$$

Now

$$R'_n T_n - R_n T'_n = (n^2 - \frac{1}{4}) S^2 (P'_n Q_n - P_n Q'_n) \quad [\text{T. F. ii. 2.}]$$

$$= (n^2 - \frac{1}{4}) \pi S. \quad [\text{T. F. i. 24.}]$$

Hence

$$\left. \begin{aligned} A_n &= \frac{4}{(4n^2-1)\pi S} (T_n F'_n - T'_n F_n), \\ B_n &= \frac{4}{(4n^2-1)\pi S} (R_n F'_n - R'_n F_n). \end{aligned} \right\} \quad \dots \quad (3)$$

This is the general solution for any case in terms of F .

4. We proceed to apply the foregoing to special cases. Consider, first, the case of a current flowing in a uniform circular ring of any size. The parts of the current all flow in circles, and the equipotential surfaces will be planes through the straight axis. The lengths of conductor which a current uses between two such surfaces will be proportional to the arc between—i. e., to ρ . Hence, in order that each element of current may produce equal fall of potential between the two planes, the current-density must vary inversely as ρ , i. e. $\sigma = b/\rho$, where b is some constant. Let I denote the total current; then, integrating over a cross section, it will be found that

$$I = 2\pi b \{R - \sqrt{(R^2 - r^2)}\}.$$

$$\text{Now} \quad r = R \sin \alpha, \quad a = R \cos \alpha; \quad \dots \quad (4)$$

$$\text{whence} \quad I = 2\pi b R (1 - \cos \alpha),$$

$$\text{or} \quad \sigma = \frac{I}{2\pi \rho R (1 - \cos \alpha)}. \quad \dots \quad (5)$$

Equation (1) now becomes

$$\frac{d^2 \psi}{dz^2} + \frac{d^2 \psi}{d\rho^2} - \frac{1}{\rho} \frac{d\psi}{d\rho} = -8\pi^2 b,$$

a particular solution of which is

$$\begin{aligned} f &= -4\pi^2 b z^2 \\ &= -4\pi^2 b a^2 \frac{\sin^2 v}{(C-c)^2} \quad [\text{T. F. i. 3}]. \end{aligned}$$

This has to be expanded in a series of the form

$$-4\pi^2 b a^2 \frac{\sin^2 v}{(C-c)^2} = \frac{1}{\sqrt{(C-c)}} \Sigma F_n \cos nv.$$

It is clear that there are no terms in $\sin nv$. Hence

$$\begin{aligned} \frac{\pi}{2} F_n &= -4\pi^2 b a^2 \int_0^\pi \frac{\sin^2 v \cos nv}{(C-c)^{\frac{3}{2}}} dv \\ &= \frac{4\pi^2 b a^2}{S} \frac{d}{du} \int_0^\pi \frac{\cos nv - \frac{1}{2}(\cos n+2v + \cos n-2v)}{\sqrt{(C-c)}} dv \\ &= \frac{4\pi^2 b a^2 \sqrt{2}}{S} \frac{d}{du} \{Q_n - \frac{1}{2}(Q_{n+2} + Q_{n-2})\} \quad [\text{T. F. ii. 1}]; \end{aligned}$$

therefore

$$\begin{aligned} F_n &= \frac{4\pi ba^2 \sqrt{2}}{S} \{ (Q'_n - Q'_{n-2}) - (Q'_{n+2} - Q'_n) \} \\ &= 8\pi ba^2 \sqrt{2} \{ (n-1)Q_{n-1} - (n+1)Q_{n+1} \}; \end{aligned}$$

also

$$F_0 = -8\pi ba^2 \sqrt{2} Q_1.$$

It is now only necessary to substitute these values in equations (3). Then

$$\begin{aligned} T_n F'_n - T'_n F_n &= 8\pi ba^2 \sqrt{2} [-SQ'_n \{ (n-1)Q'_{n-1} - (n+1)Q'_{n+1} \} \\ &\quad + (n^2 - \frac{1}{4})SQ_n \{ (n-1)Q_{n-1} - (n+1)Q_{n+1} \}]; \end{aligned}$$

since

$$T_n = -SQ'_n \text{ and } T'_n = -(n^2 - \frac{1}{4})SQ_n.$$

Now

$$\begin{aligned} &(n^2 - \frac{1}{4})Q_n Q_{n-1} - Q'_n Q'_{n-1} \\ &= \frac{2n+1}{4} Q_n \{ 4n C Q_n - (2n+1)Q_{n+1} \} - Q'_n (Q'_{n+1} - 2n S Q_n); \end{aligned}$$

since

$$\left. \begin{aligned} (2n+1)Q_{n+1} - 4n C Q_n + (2n-1)Q_{n-1} &= 0 \\ Q'_{n+1} - Q'_{n-1} &= 2n S Q_n. \end{aligned} \right\}$$

Therefore

$$\begin{aligned} &\left(n^2 - \frac{1}{4} \right) Q_n Q_{n-1} - Q'_n Q'_{n-1} \\ &= - \frac{(2n+1)^2}{4} Q_n Q_{n+1} - Q'_n Q'_{n+1} + n Q_n \{ (2n+1) C Q_n + 2 S Q'_n \} \\ &= \left\{ n(2n+1) - \frac{(2n+1)^2}{4} \right\} Q_n Q_{n+1} - Q'_n Q'_{n+1} \\ &= \left(n^2 - \frac{1}{4} \right) Q_n Q_{n+1} - Q'_n Q'_{n+1}; \end{aligned}$$

therefore

$$T_n F'_n - T'_n F_n = -16\pi ba^2 \sqrt{2} \left\{ \left(n^2 - \frac{1}{4} \right) Q_n Q_{n+1} - Q'_n Q'_{n+1} \right\};$$

whence

$$A_n = - \frac{16ba^2 \sqrt{2}}{n^2 - \frac{1}{4}} \left\{ \left(n^2 - \frac{1}{4} \right) Q_n Q_{n+1} - Q'_n Q'_{n+1} \right\} \left. \begin{aligned} &= 16ba^2 \sqrt{2} \alpha_n \text{ (say)} \\ &A_0 = -8ba^2 \sqrt{2} (Q_0 Q_1 + 4Q'_0 Q'_1). \end{aligned} \right\} \quad (6)$$

with

$$A_0 = -8ba^2 \sqrt{2} (Q_0 Q_1 + 4Q'_0 Q'_1).$$

If $(n^2 - \frac{1}{4})Q_n Q_{n+1} - Q'_n Q'_{n+1}$ be written X_n , it can be shown that

$$X_n = X_{n-1} + (2n-1)Q_n Q_{n-1},$$

which is more convenient for calculation.

So also

$$\begin{aligned} R_n F'_n - R'_n F_n &= 8\pi b a^2 \sqrt{2} [SP'_n \{ (n-1)Q'_{n-1} - (n+1)Q'_{n+1} \} \\ &\quad - (n^2 - \frac{1}{4})SP_n \{ (n-1)Q_{n-1} - (n+1)Q_{n+1} \}]. \end{aligned}$$

As in the former case, it may be shown that

$$(n^2 - \frac{1}{4})P_n Q_{n-1} - P'_n Q'_{n-1} = 2n\pi + (n^2 - \frac{1}{4})P_n Q_{n+1} - P'_n Q'_{n+1},$$

and

$$\left. \begin{aligned} B_n &= \frac{-16ba^2 \sqrt{2}}{n^2 - \frac{1}{4}} \left\{ n(n-1)\pi - \left(n^2 - \frac{1}{4} \right) P_n Q_{n+1} + P'_n Q'_{n+1} \right\} \\ &= 16ba^2 \sqrt{2} \beta_n \text{ (say)} \end{aligned} \right\} \quad (7)$$

with

$$B_0 = 8ba^2 \sqrt{2} (P_0 Q_1 + 4P'_0 Q'_1).$$

Also, if the expression in the bracket for B_n be denoted by Y_n ,

$$Y_n = Y_{n-1} + 2(n-1)\pi + (2n-1)P_{n-1}Q_n.$$

Thus ψ and ψ' are now completely determined. It must be remembered that the functions P , Q in formulæ (6) and (7) are the values at the surface of the tore u_0 .

5. In dealing with the self-induction of such a ring as we are now considering, it is necessary to particularize. Its value will differ for alternating currents of different periods. The time-constant for a constant E.M.F. applied to the ring will not be the same as on the assumption here made as to distribution of current-density. In order to be definite we will find the coefficient for the energy on the assumption of steady current—in other words, the value of L in the expression $\frac{1}{2}LI^2$. This will very approximately give the self-induction except for rapid alternations. The energy is given by

$$E = \frac{1}{2} \iiint \psi \sigma \, d\rho \, dz,$$

the integral being taken over a cross section.

Putting in the value of ψ and transferring to curvilinear coordinates,

$$\begin{aligned} E &= -2\pi^2 b^2 \iint \frac{z^2}{\rho} \, d\rho \, dz + \frac{1}{2} b \Sigma \iint \frac{B_n T_n \cos nv}{\sqrt{(C-c)}} \frac{1}{\rho} \frac{dn}{du} \cdot \frac{dn'}{dv} \, du \, dv \\ &= -\frac{2}{3} \pi^2 b^2 R^2 (1 - \cos \alpha)^2 (1 + 2 \cos \alpha) + ab \Sigma B_n \int_{u_0}^{\infty} \int_0^{\pi} \frac{T_n \cos nv}{S(C-c)^{\frac{3}{2}}} \, du \, dv. \end{aligned}$$

Now

$$\int_0^{\pi} \frac{\cos nv}{(C-c)^{\frac{3}{2}}} \, dv = -\frac{2\sqrt{2}}{S} Q'_n;$$

therefore

$$E = -\frac{2}{3}\pi^3 b^2 R^2 (1 - \cos \alpha)^2 (1 + 2 \cos \alpha) + 2\sqrt{2ab}\Sigma B_n \int_{u_0}^{\infty} \frac{1}{S} (Q'_n)^2 du.$$

The integral may be evaluated as follows :

$$Q'_{n+1} - Q'_{n-1} = 2n S Q_n.$$

Also it may easily be proved that

$$Q'_{n+1} + Q'_{n-1} = 2CQ'_n + SQ_n;$$

Therefore

$$\begin{aligned} (Q'_{n+1})^2 - (Q'_{n-1})^2 &= 2nS(2CQ_n Q'_n + SQ_n^2) \\ &= 2nS \frac{d}{du} (CQ_n^2); \end{aligned}$$

therefore

$$\begin{aligned} \int_{u_0}^{\infty} \frac{1}{S} (Q'_{n+1})^2 du - \int_{u_0}^{\infty} \frac{1}{S} (Q'_{n-1})^2 du &= [2nCQ_n^2]_{u_0}^{\infty} \\ &= -2nCQ_n^2, \end{aligned}$$

or, say,

$$X_{n+1} - X_{n-1} = -2nCQ_n^2;$$

therefore

$$X_n = 2C\{(n+1)Q_{n+1}^2 + (n+3)Q_{n+3}^2 + \dots\},$$

an infinite and rapidly convergent series.

Substituting this and the value of b from (5),

$$E = -\frac{\pi R I^2}{6} (1 + 2 \cos \alpha) + \frac{R\sqrt{2}}{2} \cdot \frac{I^2 \cos \alpha}{(1 - \cos \alpha)^2} \Sigma \frac{B_n X_n}{\pi^2 R^2 b},$$

and the self-induction,

$$= -\frac{\pi R}{3} (1 + 2 \cos \alpha) + 32\pi R \cdot \frac{\cos^3 \alpha}{(1 - \cos \alpha)^2} \Sigma \frac{\beta_n X_n}{\pi^3}. \quad (8)$$

Now

$$P_n = \alpha_n E' + \beta_n F',$$

$$Q_n = \alpha_n (F - E) + \beta_n F,$$

where E, F are elliptic integrals to modulus k ; where

$$k^2 = \frac{1 - \cos \alpha}{1 + \cos \alpha},$$

E', F' are the complementary integrals.

Also α_n, β_n are algebraical expressions in k whose values are given up to $n=5$ in [T. F. ii. 8]. Their general values are also given in a paper by Mr. Basset*. Consequently the

* "On Toroidal Functions," Amer. Journ. Math. xv. p. 301.

flux functions (and therefore forces) and self-induction of large tores of any size may be calculated to any desired degree of approximation directly from the formulæ obtained. When, however, the rings are of smaller cross section, and k small, it is more convenient to have the formulæ expressed directly in terms of k .

6. Putting in the values of P , Q given in [T. F. ii. 9, 10], and remembering that $d/du = -kd/dk$, it will be found that, L denoting $\log_e 4/k$,

$$\frac{\beta_0}{\pi} = \frac{1}{2} k^2 \left\{ 4L - 6 + \left(7L - \frac{43}{4} \right) k^2 + \left(\frac{81}{8} L - \frac{489}{32} \right) k^4 \right\},$$

$$\frac{\beta_1}{\pi} = 2k^2 \left\{ 1 - \left(\frac{3}{4} L - \frac{13}{8} \right) k^2 \right\},$$

$$\frac{\beta_2}{\pi} = -\frac{8}{15} \left(1 - \frac{15}{8} k^2 \right),$$

$$\frac{\beta_3}{\pi} = -\frac{24}{35};$$

and that the self-induction

$$\begin{aligned} &= -\frac{\pi R}{3} (1 + 2 \cos \alpha) + \frac{4\pi R \cos^3 \alpha}{(1 + \cos \alpha)^2} \left\{ 4L - 6 + (14L - 18)k^2 \right\} \\ &= \pi R \left[\frac{4 \cos^3 \alpha}{(1 + \cos \alpha)^2} \left\{ 4L - 6 + (14L - 18)k^2 \right\} - \frac{1 + 2 \cos \alpha}{3} \right]; \end{aligned}$$

when α is very small this is $\pi R(4L - 7)$, and not $\pi R(4L - 8)$ as usually assumed.

Taking Professor Minchin's example, $R = 1$ cm., $r = \cdot 1$ cm.,

$$\sin \alpha = \cdot 1, \quad \cos \alpha = \cdot 995, \quad k^2 = 10025,$$

and self-induction = $33\cdot 36$ C.G.S. units. The formula $4\pi R(L - 2)$ gives $29\cdot 9$. The numerical values given by Prof. Minchin appear to be doubled, as if he had used the diameter instead of the radius.

Similarly,

$$\alpha_0 = -\pi^2 k^2 \left(1 + \frac{7}{4} k^2 + \frac{81}{32} k^4 \right),$$

$$\alpha_1 = \frac{3}{4} \pi^2 k^4 \left(1 + \frac{11}{6} k^2 \right),$$

$$\alpha_2 = \frac{5}{32} \pi^2 k^6.$$

Hence, for space outside the ring,

$$\begin{aligned}\psi &= \frac{16\pi^2 a^2 b \sqrt{2k^2}}{\sqrt{(C-c)}} \left\{ - \left(1 + \frac{7}{4}k^2 + \frac{81}{32}k^4 \right) R_0 \right. \\ &\quad + \frac{3}{4}k^2 \left(1 + \frac{11}{6}k^2 \right) R_1 \cos v \\ &\quad \left. + \frac{5}{32}k^4 R_2 \cos 2v + \dots \right\} \\ &= \frac{8\pi R I \sqrt{2 \cos^2 \alpha}}{1 + \cos \alpha} \frac{1}{\sqrt{(C-c)}} \left\{ \&c. \right\}.\end{aligned}$$

The total flux through the aperture is the value of ψ at the inside edge, *i. e.* at $u=u_0$, $v=\pi$.

Then

$$\begin{aligned}\psi &= \frac{16\pi R I \cos^2 \alpha}{1 + \cos \alpha} \frac{\sqrt{k}}{1+k} \left\{ - \left(1 + \frac{7}{4}k^2 + \frac{81}{32}k^4 \right) R_0 \right. \\ &\quad - \frac{3}{4}k^2 \left(1 + \frac{11}{6}k^2 \right) R_1 \\ &\quad \left. + \frac{5}{32}k^4 R_2 \right\}.\end{aligned}$$

Substituting for the R it will be found that

$$\psi/I = \frac{8\pi R \cos^2 \alpha}{(1 + \cos \alpha)(1+k)} \left\{ L - 2 - \frac{3}{4}k + \frac{1}{2} \left(L + \frac{1}{8} \right) k^2 + \frac{3}{8} \left(3L - \frac{31}{6} \right) k^3 \right\},$$

which for very small rings gives the self-induction

$$= 4\pi R(L-2),$$

the value usually accepted for the true value.

The flux as taken by Prof. Minchin is found by putting $v=0$, when

$$\psi/I = \frac{8\pi R \cos^2 \alpha}{(1 + \cos \alpha)(1-k)} \left\{ L - 2 + \frac{3}{4}k + 2 \left(L - \frac{23}{32} \right) k^2 - \frac{3}{8} \left(3L - \frac{31}{6} \right) k^3 \right\}.$$

This agrees with his result up to the term in k .

The force at the centre of the ring is

$$\frac{1}{2\pi\rho} \frac{d\psi}{du} \cdot \frac{du}{dn} \quad \text{when } u=0 \text{ and } v=\pi.$$

This

$$\begin{aligned}&= \frac{1}{2\pi S} \left(\frac{C-c}{a} \right)^2 \left[- \frac{S\psi}{2(C-c)} + \frac{1}{\sqrt{(C-c)}} \Sigma \left(n^2 - \frac{1}{4} \right) A_n S P_n \cos nv \right] \\ &= \frac{2\frac{1}{2}}{2\pi a^2} \pi \Sigma (-1)^n \left(n^2 - \frac{1}{4} \right) A_n \\ &= 32b \left\{ \frac{1}{2} (Q_0 Q_1 + 4 Q'_0 Q'_1) + \Sigma_1 (-1)^n \left(Q'_n Q'_{n+1} - \overline{n^2 - \frac{1}{4}} Q_n Q_{n+1} \right) \right\}.\end{aligned}$$

When small this becomes

$$= \frac{2\pi I}{R} \left(1 + \frac{1}{2} k^2 + \frac{1}{4} k^4 \right).$$

7. The case of uniform current-density is of more practical importance, as it leads to the case of a uniformly-wound circular coil whose cross section is also circular. The mathematical treatment, however, does not lead to such a complete solution as in the former case.

As the current-density is uniform, $\sigma = I/\pi r^2$, and the right-hand side of equation (1) is $-8\pi I\rho/r^2$.

A particular integral is

$$f = -\frac{8}{3}\pi \frac{I}{r^2} \cdot \rho^3 = -\frac{1}{\sqrt{(C-c)}} \times \frac{8\pi I a^3 S^3}{3r^2(C-c)^{\frac{3}{2}}},$$

whence

$$F_n = -\frac{16 I a^3 S^3}{3r^2} \int_0^\pi \frac{\cos nv}{(C-c)^{\frac{3}{2}}} dv.$$

Now

$$Q_n \sqrt{2} = \int_0^\pi \frac{\cos nv}{(C-c)^{\frac{3}{2}}} dv;$$

whence it is easy to show that

$$\left. \begin{aligned} F_n &= -\frac{64\sqrt{2} I a^3}{9r^2} (SQ_n'' - CQ_n'), \\ F_0 &= -\frac{32\sqrt{2} I a^3}{9r^2} (SQ_0'' - CQ_0'); \end{aligned} \right\} \dots \quad (9)$$

also

$$\therefore F_n' = -\frac{64\sqrt{2} I a^3}{9r^2} S(Q_n''' - Q_n').$$

Substituting in $R_n F_n' - R_n' F_n$, and remembering that

$$SQ_n'' + CQ_n' - \left(n^2 - \frac{1}{4}\right) SQ_n = 0,$$

it will be found that

$$\begin{aligned} B_n &= -\frac{64\sqrt{2} I a^3}{9\pi r^2} \left\{ C(P_n Q_n' - P_n' Q_n) + S(P_n' Q_n' - P_n Q_n'') \right. \\ &\quad \left. - \frac{2P_n'}{n^2 - \frac{1}{4}} \left(S - \frac{C^2}{S} \right) Q_n' \right\} \\ &= -\frac{64\sqrt{2} I a^3}{9\pi r^2} \left\{ -\frac{\pi C}{S} + S(P_n' Q_n' - P_n Q_n'') + \frac{2P_n' Q_n'}{(n^2 - \frac{1}{4})S} \right\} \\ &= \frac{64\sqrt{2} I a^3}{9r^2} \beta_n \text{ (say),} \end{aligned}$$

and

$$\beta_0 = \frac{1}{2} \left\{ \frac{C}{S} - \frac{S}{\pi} (P'_0 Q'_0 - P_0 Q''_0) + \frac{8P'_0 Q'_0}{\pi S} \right\}.$$

So also, in a similar manner, $T_n F'_n - T'_n F_n$ gives

$$\begin{aligned} A_n &= \frac{64\sqrt{2}Ia^3}{9\pi r^2} \left\{ S(Q_n'^2 - Q_n Q_n'') + \frac{2Q_n'^2}{(n^2 - \frac{1}{4})S} \right\} \\ &= \frac{64\sqrt{2}Ia^3}{9r^2} \alpha_n \text{ (say),} \end{aligned}$$

with

$$\alpha_0 = \frac{1}{2\pi} \left\{ S(Q_0'^2 - Q_0 Q_0'') - \frac{8Q_0'^2}{S} \right\}.$$

The energy is

$$E = \frac{1}{2} \iint \psi \sigma \, dn \, dn' = \frac{Ia^2}{2\pi r^2} \int_{u_0}^{\infty} \int_0^{2\pi} \frac{\psi}{(C-c)^2} \, du \, dv.$$

The part of E depending on f is found by an easy integration to be

$$- \frac{4\pi R I^2}{3} \left(\frac{3}{4} + \operatorname{cosec}^2 \alpha \right).$$

The part depending on the infinite series is

$$\frac{Ia^2}{2\pi r^2} \Sigma B_n \int_{u_0}^{\infty} \int_0^{2\pi} \frac{T_n \cos nv}{(C-c)^{\frac{3}{2}}} \, du \, dv.$$

But from (9)

$$\begin{aligned} \int_0^{\pi} \frac{\cos nv}{(C-c)^{\frac{3}{2}}} &= \frac{4\sqrt{2}}{3S^{\frac{3}{2}}} (SQ_n'' - CQ_n') \\ &= \frac{4\sqrt{2}}{3} \frac{1}{S} \frac{d}{du} \left(\frac{Q_n'}{S} \right). \end{aligned}$$

Therefore the above

$$= - \frac{4\sqrt{2}Ia^2}{3\pi r^2} \Sigma B_n \int_{u_0}^{\infty} Q_n' \frac{d}{du} \left(\frac{Q_n'}{S} \right) \, du.$$

Now,

$$\int Q_n' \frac{d}{du} \left(\frac{Q_n'}{S} \right) = \frac{1}{2} \frac{Q_n'^2}{S} - \frac{1}{2} \int \frac{CQ_n'^2}{S^2} \, du;$$

also

$$\begin{aligned} \int \frac{CQ_n'^2}{S^2} \, du &= \int \frac{Q_n'}{S} \left(-Q_n'' + \left(n^2 - \frac{1}{4} \right) Q_n \right) \, du \\ &= -\frac{1}{2} \frac{Q_n'^2}{S} + \frac{1}{2} \left(n^2 - \frac{1}{4} \right) \frac{Q_n^2}{S} \\ &\quad - \frac{1}{2} \int \left\{ \frac{CQ_n'^2}{S^2} - \left(n^2 - \frac{1}{4} \right) \frac{CQ_n^2}{S^2} \right\} \, du; \end{aligned}$$

$$\begin{aligned}\therefore \int \frac{CQ_n^2}{S^2} du &= -\frac{1}{3} \frac{Q_n^2}{S} + \frac{1}{3} \left(n^2 - \frac{1}{4} \right) \left\{ \frac{Q_n^2}{S} + \int \frac{C}{S^2} Q_n^2 du \right\}; \\ \therefore \int_{u_0}^{\infty} Q_n' \frac{d}{du} \left(\frac{Q_n'}{S} \right) du &= \left[\frac{2}{3} \frac{Q_n^2}{S} - \frac{1}{6} \left(n^2 - \frac{1}{4} \right) \frac{Q_n^2}{S} \right]_{u_0}^{\infty} - \frac{1}{6} \left(n^2 - \frac{1}{4} \right) \int_{u_0}^{\infty} \frac{CQ_n^2}{S^2} du \\ &= -\frac{2}{3} \frac{Q_n^2}{S} + \frac{1}{6} \left(n^2 - \frac{1}{4} \right) \left\{ \frac{Q_n^2}{S} - \int_{u_0}^{\infty} \frac{CQ_n^2}{S^2} du \right\}\end{aligned}$$

where the functions outside the integral are functions of u_0 .

It results that

$$\begin{aligned}E &= -\frac{4\pi R I^2}{3} \left(\frac{3}{4} + \operatorname{cosec}^2 \alpha \right) - \frac{512 R \cos^5 \alpha}{81 \pi \sin^4 \alpha} \Sigma \beta_n \left\{ -2 \frac{Q_n^2}{S} \right. \\ &\quad \left. + \frac{1}{2} \left(n^2 - \frac{1}{4} \right) \left(\frac{Q_n^2}{S} - \int \frac{CQ_n^2}{S^2} du \right) \right\},\end{aligned}$$

and that the self-induction

$$\begin{aligned}&= -\frac{8\pi R}{3} \left(\frac{3}{4} + \operatorname{cosec}^2 \alpha \right) + \frac{512 R \cos^5 \alpha}{81 \pi \sin^4 \alpha} \Sigma \beta_n \left\{ 4 \frac{Q_n^2}{S} \right. \\ &\quad \left. - \left(n^2 - \frac{1}{4} \right) \left(\frac{Q_n^2}{S} - \int_{u_0}^{\infty} \frac{CQ_n^2}{S^2} du \right) \right\}.\end{aligned}$$

It is seen that the question is not completely solved, as the integral $\int_{u_0}^{\infty} \frac{CQ_n^2}{S^2} du$ has not been evaluated, as in the preceding case. Its value can, however, be approximated to when k is small.

If it be denoted by X_n , it can be shown that

$$(2n+1)^2 X_{n+1} - 16n X_n - (2n-1)^2 X_{n-1} = 8n \frac{C^2}{S} Q_n^2,$$

a difference equation, which if solved in a series ascending from Q_{n+1} would give X_n .

If we call the self-induction L , the energy is $\frac{1}{2} L I^2$.

If there are N turns of wire with a current i ,

$$I = N i,$$

$$\text{and energy} = \frac{1}{2} N^2 L i^2,$$

whence the self-induction of such a coil is $N^2 L$.

8. For such a coil the values of the flux function are:—

Outside,

$$\psi' = \frac{64 \sqrt{2} R \cos^3 \alpha N i}{9 \sin^2 \alpha} \frac{1}{\sqrt{(C-c)}} \Sigma_0 \alpha_n R_n \cos n v;$$

Inside,

$$\psi = -\frac{8\pi R \cos^3 \alpha N i}{3 \sin^2 \alpha} \cdot \frac{S^3}{(C-c)^3} + \frac{64 \sqrt{2} R \cos^3 \alpha N i}{9 \sin^2 \alpha} \frac{1}{\sqrt{C-c}} \sum \beta_n T_n \cos n v,$$

where α_n, β_n are the expressions given above.

When α (or k) is small, the values can be expressed in a few terms of a series of powers of k as in the former case. These values are as follows :—

$$\alpha_0 = -\frac{9\pi k^2}{4} \left(1 + \frac{13}{4} k^2 + \frac{221}{32} k^4\right),$$

$$\alpha_1 = \frac{45}{16} \pi k^4 \left(1 + \frac{17}{6} k^2\right),$$

$$\alpha_2 = \frac{105}{128} \pi k^6;$$

$$\beta_0 = \frac{3}{4} \left\{ 1 + (6L-9)k^2 + \frac{1}{2} \left(39L - \frac{255}{4}\right)k^4 + \left(\frac{663}{16}L - 66\frac{35}{64}\right)k^6 \right\},$$

$$\beta_1 = \frac{5}{2} \left\{ 1 + 3k^2 - \left(\frac{9}{4}L - \frac{57}{8}\right)k^4 \right\},$$

$$\beta_2 = \frac{7}{2} \left(1 + \frac{3}{2} k^2\right),$$

$$\beta_3 = \frac{9}{2}.$$

When k is not large, the integrals in the expression for the self-induction may be found approximately in a series of powers of k . If this be done, after some rather tedious calculations the value is found to be

$$4\pi R \left[-\frac{7}{6} - \frac{2}{3} \frac{\cos^2 \alpha + 3 \cos^3 \alpha + 4 \cos^4 \alpha}{(1 + \cos \alpha)^3} + \frac{8 \cos^5 \alpha}{3(1 + \cos \alpha)^4} \left\{ 6L + \frac{1}{2} + (39L-5)k^2 \left(\frac{2115}{16}L - \frac{981}{64} \right) k^4 \right\} \right].$$

When k is very small this is $\pi R(4L-7)$.

The value of ψ' will give the coefficient of mutual induction between the circular coil and a thin circular wire with the same straight axis and with its plane parallel to the coil.

9. The same method of attack can be employed in dealing with gravitation potentials. Here

$$-\nabla^2\phi = -4\pi\sigma,$$

where σ is the density at the point (x, y, z) .

If f be a particular solution of this, and ϕ denote the general solution of $\nabla^2\phi=0$, then at all points within the attracting body the potential $=f+\phi_1$, and at all points outside $=\phi_2$. Also ϕ_1 and ϕ_2 must be finite and continuous in their respective regions, and at the surface of the body the ϕ_1 and ϕ_2 and their first differential coefficients are continuous. These conditions suffice to determine ϕ_1 and ϕ_2 .

Take, for instance, the case of a uniform sphere. Inside σ is constant, and

$$\frac{d^2V}{dr^2} + \frac{2}{r} \frac{dV}{dr} = -4\pi\sigma.$$

A particular integral is

$$V = -\frac{2}{3}\pi\sigma r^2;$$

whilst the general solution is

$$V = A + \frac{B}{r}.$$

Hence:—

$$\text{Inside, } \phi = -\frac{2}{3}\pi\sigma r^2 + A + \frac{B}{r};$$

$$\text{Outside, } \phi' = A' + \frac{B'}{r}.$$

Inside, ϕ is finite, $\therefore B=0$.

At the surface, when $r=a$,

$$\phi = \phi', \quad \text{and} \quad \frac{d\phi}{dr} = \frac{d\phi'}{dr}.$$

Therefore

$$\left. \begin{aligned} A - \frac{2}{3}\pi\sigma a^2 &= A' + \frac{B'}{a} \\ -\frac{4}{3}\pi\sigma a &= -\frac{B'}{a^2} \end{aligned} \right\};$$

whence

$$A' = A - 2\pi\sigma a^2,$$

$$B' = \frac{4}{3}\pi\sigma a^3,$$

and

$$\phi = A - \frac{2}{3}\pi\sigma r^2,$$

$$\phi' = A - 2\pi\sigma a^2 + \frac{4}{3}\frac{\pi\sigma a^3}{r}.$$

If the density of the sphere is variable, the only difficulty is that of finding the particular integral f . As an example of variable density, take the case of a sphere in which the density varies as the square of the distance from a diametral plane. Here

$$\begin{aligned} -\nabla^2\phi &= -4\pi cz^2, \\ f &= -\frac{1}{3}\pi cz^4 = -\frac{1}{3}\pi cr^4 \cos^4\theta, \end{aligned}$$

and the general solution involves zonal harmonics (P_n).

Hence, inside,

$$\phi' = -\frac{1}{3}\pi cr^4 \cos^4\theta + \Sigma B_n r^n P_n,$$

and outside,

$$\phi = \Sigma \frac{A_n}{r^{n+1}} P_n.$$

Now,

$$\cos^4\theta = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}.$$

The surface conditions give at once

$$A_n = 0 = B_n \text{ for all } n > 4,$$

and

$$A_1 = A_3 = 0 = B_1 = B_3.$$

Also

$$\left. \begin{aligned} -\frac{1}{15}\pi ca^4 + B_0 &= \frac{A_0}{a} \\ -\frac{4}{21}\pi ca^4 + B_2a^2 &= \frac{A_2}{a^3} \\ -\frac{8}{105}\pi ca^4 + B_4a^4 &= \frac{A_4}{a^5} \end{aligned} \right\} \phi = \phi' \text{ when } r=a,$$

and

$$\left. \begin{aligned} -\frac{4}{15}\pi ca^3 &= -\frac{A_0}{a^2} \\ -\frac{16}{21}\pi ca^3 + 2B_2a &= -3\frac{A_2}{a^4} \\ -\frac{32}{105}\pi ca^3 + 4B_4a^3 &= -5\frac{A_4}{a^6} \end{aligned} \right\} \frac{d\phi}{dr} = \frac{d\phi'}{dr} \text{ when } r=a,$$

giving

$$\phi/\pi c = \frac{4}{15}\frac{a^5}{r} + \frac{8}{105}\frac{a^7}{r^3}P_2;$$

$$\phi'/\pi c = \frac{1}{15}(5a^4 - r^4) + \frac{4}{105}(7a^2r^2 - 5r^4)P_2.$$

10. The gravitation potential of a tore has been recently

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considered by Mr. Dyson * by means of special functions and integration. It may be of use to indicate how the problem would be solved by toroidal functions and the differentiation method illustrated in this paper.

Inside the tore $-\nabla^2\phi = -4\pi\sigma$ with σ constant. A particular solution is

$$f = -\pi\sigma\rho^2 = -\pi\sigma a^2 S^2 (U-c)^{-2}.$$

Hence:—

Outside, $\phi = \sqrt{(C-c)} \Sigma A_n P_n \cos nv,$

Inside, $\phi' = -\pi\sigma a^2 S^2 / (C-c)^2 + \sqrt{(C-c)} \Sigma B_n Q_n \cos nv.$

f must be expanded in a series of the form

$$f = \sqrt{(C-c)} \Sigma F_n \cos nv;$$

whence

$$\frac{\pi}{2} F_n = -\pi\sigma a^2 S^2 \int \frac{\cos nv}{(C-c)^{\frac{3}{2}}} dv.$$

This has been found above (9), viz.

$$F_n = -\frac{8\sqrt{2}}{3} \cdot \frac{\sigma a^2}{S} (S Q_n'' - C Q_n'),$$

$$F_0 = -\frac{4\sqrt{2}}{3} \cdot \frac{\sigma a^2}{S} (S Q_0'' - C Q_0').$$

Writing P_n for R_n and Q_n for T_n in § 3, we find

$$\left. \begin{aligned} A_n (P_n' Q_n - P_n Q_n') &= Q_n F_n' - Q_n' F_n \\ B_n (P_n' Q_n - P_n Q_n') &= P_n F_n' - P_n' F_n \end{aligned} \right\}.$$

But

$$P_n' Q_n - P_n Q_n' = \frac{\pi}{S}.$$

Hence

$$A_n = \frac{S}{\pi} (Q_n F_n' - Q_n' F_n),$$

$$B_n = \frac{S}{\pi} (P_n F_n' - P_n' F_n).$$

Inserting the values of F_n and F_n' , we get

$$A_n = \frac{16\sqrt{2}}{3} \sigma a^2 C \left\{ \left(n^2 - \frac{1}{4} \right) Q_n^2 - \frac{C^2 + 1}{CS} Q_n Q_n' - Q_n'^2 \right\}$$

$$B_n = \frac{16\sqrt{2}}{3} \sigma a^2 C \left\{ \left(n^2 - \frac{1}{4} \right) \frac{\pi}{2C} + \left(n^2 - \frac{1}{4} \right) P_n Q_n - \frac{C^2 + 1}{CS} P_n Q_n' - P_n' Q_n \right\}$$

The external and internal potentials are thus completely found.

* "The Potential of an Anchor Ring," Phil. Trans. 1893, pp. 43, 1041.