XXIX. On the range of Stokes's deep-water waves

R.F. Gwyther

To cite this article: R.F. Gwyther (1906) XXIX. On the range of Stokes's deep-water waves , Philosophical Magazine Series 6, 11:63, 374-378, DOI: [10.1080/14786440609463452](https://doi.org/10.1080/14786440609463452)

To link to this article: <http://dx.doi.org/10.1080/14786440609463452>

Published online: 16 Apr 2009.

Submit your article to this journal [↗](#)

Article views: 2

View related articles [↗](#)

Hence the term to be added to $-EI\gamma$ is

$$\frac{4}{3} \frac{I'}{l} \left(\frac{x^2}{2} - lx + \frac{l^2}{2} \right);$$

and for a first approximation,

$$-\frac{\ddot{\gamma}_1}{\gamma_1} = \frac{EI}{\cdot 08194 \rho \omega l^4 + \cdot 3 ml^3 + \cdot 6 I'l}.$$

This example is sufficient to indicate the procedure to be adopted in all such cases.

University College, Bristol,
October 1905.

XXIX. *On the Range of Stokes's Deep-Water Waves.*
By R. F. GWYTHER*.

THE waves with which I propose to deal are the waves of finite amplitude of which the investigation was initiated by Sir George Stokes† in his paper on the Theory of Oscillatory Waves, and continued in the Supplement to that paper. The object of the paper is to establish the correctness of the opinion expressed in the paper quoted (p. 227) "After careful consideration I feel satisfied ... that we may approach as near as we please to the form in which the curvature at the vertex becomes infinite, and the vertex becomes a multiple point where the two branches ... enclose an angle of 120° ." The method of the Supplement is adhered to as closely as is convenient. The result is to establish that the velocity in all waves of the series, small as well as great, is represented by a function which possesses poles which as the amplitude increases approach nearer to the fluid surface, and in the limiting form is identical with that investigated by Mr. Michell‡ in his paper on the "Highest Wave in Water," in which the poles lie on the water-surface.

§ 1. Making a change in the notation used by Stokes in the Supplement, in order to make the analytical form of the velocity more obvious, I write

$$x + iy = \frac{\phi + i\psi}{c} - i \sum \frac{h_n}{nk} e^{\frac{ink(\phi + i\psi)}{c}},$$

where n is an integer, and k is a constant defining the wavelength. Also let $\psi=0$ define the free surface of the water.

* Communicated by the Author.

† Mathematical and Physical Papers, vol. i. p. 197, and Supplement, p. 314.

‡ Phil. Mag. November 1893.

Then the velocity is given by

$$qe^{i\omega} = u + iv = \frac{1}{\frac{d(x-iy)}{d(\phi-i\psi)}} = \frac{c}{f\left(\frac{\phi-i\psi}{c}\right)}$$

where

$$f\left(\frac{\phi+i\psi}{c}\right) = 1 + \sum h_n e^{\frac{ink(\phi+i\psi)}{c}}.$$

At the free surface we have $\psi=0$, and it will be convenient in this case to write, in place of $f\left(\frac{\phi+i\psi}{c}\right)$,

$$F\left(\frac{ik\phi}{c}\right) = 1 + \sum h_n e^{\frac{ink\phi}{c}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The condition for uniform pressure along the free surface requires that, when $\psi=0$,

$$gy - \frac{1}{2}q^2 = \text{constant}.$$

The determination of the successive values of h_n resulting from this condition is only a matter of labour. I have carried the determination to the sixth order, as an extended number of terms is important as the basis of the future argument.

Stating results only, I find

$$\begin{aligned} kc^2/g &= 1 + h^2 + \frac{7}{2}h^4 + \frac{229}{12}h^6, \\ h_2 &= 2h^2 + h^4 + \frac{29}{6}h^6, \\ h_3 &= \frac{9}{2}h^3 + \frac{19}{4}h^5, \\ h_4 &= \frac{32}{3}h^4 + \frac{313}{18}h^6, \\ h_5 &= \frac{625}{24}h^5, \\ h_6 &= \frac{324}{5}h^6, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

where I have written h in place of h_1 .

Also if we write 2α to denote the height of the wave from trough to crest, I find

$$k\alpha = h + \frac{3}{2}h^3 + \frac{163}{24}h^5 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we write out in full the resulting expression for $F\left(\frac{ik\phi}{c}\right)$, the successive terms will be seen to show a marked resemblance to those in the expansion of

$$\left(1 - 3he^{\frac{ik\phi}{c}}\right)^{-\frac{1}{3}}.$$

In order to examine this resemblance more closely, I expand

$$\left\{F\left(\frac{ik\phi}{c}\right)\right\}^{-3} \text{ on the assumption of convergence, and find}$$

$$\left\{F\left(\frac{ik\phi}{c}\right)\right\}^{-3} = 1 - 3he^{\frac{ik\phi}{c}} - \left(3h^4 + \frac{29}{2}h^6\right)e^{\frac{2ik\phi}{c}}$$

$$+ \left(\frac{1}{2}h^3 - \frac{9}{4}h^5\right)e^{\frac{3ik\phi}{c}} + \left(h^4 - \frac{7}{6}h^6\right)e^{\frac{4ik\phi}{c}} + \frac{15}{8}h^5e^{\frac{5ik\phi}{c}}$$

$$+ \frac{18}{5}h^6e^{\frac{6ik\phi}{c}} (4)$$

It is on the greatly increased convergency produced in the series by this step that the argument depends. It is clear that there is a value of μ , not very widely different from $3h$,

for which $1 - \mu e^{\frac{ik\phi}{c}}$ is a factor of the right-hand side of (4).

For more precision, write

$$3h = \alpha\mu + \beta\mu^3 + \gamma\mu^5$$

and determine α, β, γ so that the right-hand side of (4) may

vanish approximately when $1/\mu$ is put in place of $e^{\frac{ik\phi}{c}}$, i. e. so as to satisfy

$$1 - \frac{3h}{\mu} - \frac{3h^4 + \frac{29}{2}h^6}{\mu^2} + \dots = 0.$$

This leads to the equations

$$1 - \alpha + \frac{1}{54}\alpha^3 + \frac{1}{81}\alpha^4 + \frac{5}{648}\alpha^5 + \frac{2}{402}\alpha^6 = 0,$$

$$\beta\left(1 - \frac{1}{18}\alpha^2 - \frac{4}{81}\alpha^3\right) = -\left(\frac{1}{27}\alpha^4 + \frac{1}{108}\alpha^5 + \frac{7}{4574}\alpha^6\right),$$

$$\gamma = -\left(\frac{4}{27}\alpha^3\beta + \frac{29}{1458}\alpha^6\right).$$

$\mu=1$, this method leads to expressions identical with those of Mr. Michell, I return to the general free-surface condition, and, for convenience, write ξ in place of ϕ/c .

On differentiating the free-surface condition in respect to ξ , we get an equation of the form

$$\frac{g}{i} \{ F(ik\xi) - F(-ik\xi) \} = c^2 \frac{d}{d\xi} \{ F(ik\xi) F(-ik\xi) \} - 1;$$

and in this I write

$$\begin{aligned} F(ik\xi) &= (1 - \mu e^{ik\xi})^{-\frac{1}{2}} U - 1, \\ F(-ik\xi) &= (1 - \mu e^{-ik\xi})^{-\frac{1}{2}} V - 1, \end{aligned}$$

where U and V are rapidly converging series of the type

$$1 + C_1 e^{ik\xi} + C_2 e^{2ik\xi} + \dots,$$

so that

$$\begin{aligned} \frac{g}{i} \left\{ \frac{1}{(1 - \mu e^{ik\xi})^{\frac{1}{2}} U} - \frac{1}{(1 - \mu e^{-ik\xi})^{\frac{1}{2}} V} \right\} \\ = c^2 \frac{d}{d\xi} \{ (1 - \mu e^{ik\xi})^{\frac{1}{2}} (1 - \mu e^{-ik\xi})^{\frac{1}{2}} UV \}. \quad (9) \end{aligned}$$

This may be put in the form

$$\begin{aligned} - \frac{g}{i} \left\{ \left(\frac{1 - \mu e^{ik\xi}}{1 - \mu e^{-ik\xi}} \right)^{\frac{1}{2}} U - \left(\frac{1 - \mu e^{-ik\xi}}{1 - \mu e^{ik\xi}} \right)^{\frac{1}{2}} V \right\} \\ = c^2 \left\{ \frac{(1 - \mu e^{ik\xi})^{\frac{1}{2}} (1 - \mu e^{-ik\xi})^{\frac{1}{2}}}{2} \frac{d}{d\xi} (U^2 V^2) \right. \\ \left. + \frac{2k\mu}{3} \frac{\sin k\xi}{(1 - \mu e^{ik\xi})^{\frac{1}{2}} (1 - \mu e^{-ik\xi})^{\frac{1}{2}}} U^2 V^2 \right\} \quad (10) \end{aligned}$$

which becomes, when $\mu=1$, identical, except for the change in units, with the equation which Mr. Michell has employed to obtain the form of the Highest Wave in Water.

If, however, the left-hand side of (9) is rationalized the right-hand side is also found to be rational, and this form may be found more convenient whether μ is less than or equal to unity. The values of the constants in U and V are, of course, different in these two cases.

The simplification on which I have relied would appear to be applicable to all cases of irrotational waves whether in shallow or in deep water, and it will probably be found that such waves in either case will be related to poles, as I have shown to be the case with deep-water waves.