This article was downloaded by: [UQ Library]

On: 13 March 2015, At: 23:34

Publisher: Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK

Transactions of the Royal Society of South Africa

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/ttrs20>

NOTE ON THE PRODUCT OF A SPECIAL N-LINE DETERMINANT BY ITS CENTRAL MINOR OF THE (N-4)TH ORDER

Thomas Muir LL.D.

Published online: 26 Mar 2010.

To cite this article: Thomas Muir LL.D. (1914) NOTE ON THE PRODUCT OF A SPECIAL N-LINE DETERMINANT BY ITS CENTRAL MINOR OF THE (N-4)TH ORDER, Transactions of the Royal Society of South Africa, 4:1, 273-277, DOI: [10.1080/00359191409519537](https://doi.org/10.1080/00359191409519537)

To link to this article: <http://dx.doi.org/10.1080/00359191409519537>

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at <http://www.tandfonline.com/page/terms-and-conditions>

NOTE ON THE PRODUCT OF A SPECIAL N -LINE DETERMINANT BY ITS CENTRAL MINOR OF THE $(N-4)$ TH ORDER.

By THOMAS MUIR, LL.D.

(Read August 19, 1914.)

1. The main attraction of Mr. Roseveare's paper centres in the very interesting equivalent given by him for the product of a special n -line determinant by its central minor of the $(n-4)$ th order, this equivalent taking the form of an aggregate of products of pairs of minors of the $(n-2)$ th order.

For such a product an outwardly similar expression, perfectly general in character, has been known since 1879; * and, not unnaturally, one has some expectation that on putting into this general result the very special elements occurring in Mr. Roseveare's paper there would be evolved by degeneration the new theorem under consideration. This, however, is far from being the case. For example, when n is 7, the general theorem gives

$$\begin{aligned} \begin{vmatrix} 1234567 \\ 1234567 \end{vmatrix} \cdot \begin{vmatrix} 345 \\ 345 \end{vmatrix} &= \begin{vmatrix} 12345 \\ 12345 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 34567 \end{vmatrix} - \begin{vmatrix} 12345 \\ 13456 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 23457 \end{vmatrix} \\ &+ \begin{vmatrix} 12345 \\ 13457 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 23456 \end{vmatrix} + \begin{vmatrix} 12345 \\ 23456 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 13457 \end{vmatrix} \\ &- \begin{vmatrix} 12345 \\ 23457 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 13456 \end{vmatrix} + \begin{vmatrix} 12345 \\ 34567 \end{vmatrix} \cdot \begin{vmatrix} 34567 \\ 12345 \end{vmatrix}; \end{aligned}$$

whereas the new theorem gives

$$\begin{vmatrix} 1234567 \\ 1234567 \end{vmatrix} \cdot \begin{vmatrix} 345 \\ 345 \end{vmatrix} = \begin{vmatrix} 23456 \\ 23456 \end{vmatrix}^2 - \begin{vmatrix} 23456 \\ 23457 \end{vmatrix}^2 \lambda + \begin{vmatrix} 23456 \\ 23456 \end{vmatrix} \cdot \begin{vmatrix} 23456 \\ 23467 \end{vmatrix} \lambda$$

* *Trans. R. Soc. of Edinburgh*, xxix., pp. 47-51.

for the case where the involved elements are those of the special determinant whose matrix is the sum of the two matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_6 \\ . & a_0 & a_1 & \dots & a_5 \\ . & . & a_0 & \dots & a_4 \\ \dots & \dots & \dots & \dots & \dots \\ . & . & . & \dots & a_0 \end{pmatrix}, \begin{pmatrix} . & \dots & . & . & -a_8\lambda \\ . & \dots & . & -a_8\lambda^2 & -a_7\lambda \\ . & \dots & -a_8\lambda^3 & -a_7\lambda^2 & -a_6\lambda \\ \dots & \dots & \dots & \dots & \dots \\ -a_8\lambda^7 & \dots & -a_4\lambda^3 & -a_3\lambda^2 & -a_2\lambda \end{pmatrix};$$

and yet the latter identity is not included in the former.

2. Instead, however, of attempting a generalization of the new identity by making the elements of its left-hand member perfectly general and then seeking for the corresponding change in the right-hand member, we may adopt a reverse procedure, namely, view the elements on the right-hand as being changed from special to general, and then inquire what alteration is thereby necessitated on the left. Thus, instead of Mr. Roseveare's expression

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 - a_7\lambda^2 \\ . & a_0 & a_1 - a_7\lambda^3 & a_2 - a_7\lambda^2 \\ . & -a_7\lambda^4 & a_0 - a_6\lambda^3 & a_1 - a_5\lambda^2 \\ -a_7\lambda^5 & -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_4\lambda^2 \end{vmatrix} - \begin{vmatrix} a_0 & a_1 & a_2 & a_4 - a_6\lambda^2 \\ . & a_0 & a_1 - a_7\lambda^3 & a_3 - a_5\lambda \\ . & -a_7\lambda^4 & a_0 - a_6\lambda^3 & a_2 - a_4\lambda \\ -a_7\lambda^5 & -a_6\lambda^4 & -a_5\lambda^3 & a_1 - a_3\lambda \end{vmatrix} \cdot \lambda \\ + \begin{vmatrix} a_0 & a_1 & a_2 & a_3 - a_7\lambda^2 \\ . & a_0 & a_1 - a_7\lambda^3 & a_2 - a_6\lambda^2 \\ . & -a_7\lambda^4 & a_0 - a_6\lambda^2 & a_1 - a_5\lambda^2 \\ -a_7\lambda^5 & -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_4\lambda^2 \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_1 & a_3 - a_7\lambda^2 & a_4 - a_6\lambda \\ . & a_0 & a_2 - a_6\lambda^2 & a_3 - a_5\lambda \\ . & -a_7\lambda^4 & a_1 - a_5\lambda^2 & a_2 - a_4\lambda \\ -a_7\lambda^5 & -a_6\lambda^4 & a_0 - a_4\lambda^2 & a_1 - a_3\lambda \end{vmatrix} \cdot \lambda$$

which is a function of *nine* variables, let us pass to

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix}^2 - \begin{vmatrix} a_1 & a_2 & a_3 & a_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_5 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_5 \\ \delta_1 & \delta_2 & \delta_3 & \delta_5 \end{vmatrix}^2 \lambda + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_4 & a_5 \\ \beta_1 & \beta_2 & \beta_4 & \beta_5 \\ \gamma_1 & \gamma_2 & \gamma_4 & \gamma_5 \\ \delta_1 & \delta_2 & \delta_4 & \delta_5 \end{vmatrix} \lambda,$$

which is a function of *twenty-one* variables, and ask if there be a condensed expression equal to the latter.

A little examination of the determinants involved in it shows that all of them are 4-line minors of the array

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{vmatrix}.$$

and that in the first two columns all of them are identical. Not much more is then wanted to enable us to suggest the construction of the 8-line determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_4 & . & . \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & . & . \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & . & . \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_4 & . & . \\ . & . & \delta_4 & \lambda\delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\ . & . & \gamma_4 & \lambda\gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ . & . & \beta_4 & \lambda\beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ . & . & a_4 & \lambda a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix},$$

of which Laplace's expansion-theorem gives the equivalent

$$|a_1\beta_2\gamma_3\delta_4|^2 - |a_1\beta_2\gamma_3\delta_5|^2\lambda + |a_1\beta_2\gamma_3\delta_4| \cdot |a_1\beta_2\gamma_4\delta_5|\lambda$$

as desired.

It is a help here to note (a) in the first four rows of the 8-line determinant the repetition of the column with the suffix 4; (b) the mode of obtaining the last four rows from the first four; (c) the fact that but for the λ 's the determinant would be centrosymmetric.

3. That the result thus obtained includes Mr. Roseveare's is readily shown. As an example, let us take the next lower case of the extended theorem, namely,

$$|a_1\beta_2\gamma_3|^2 - |a_1\beta_2\gamma_4|^2\lambda + |a_1\beta_2\gamma_3| \cdot |a_1\beta_3\gamma_4|\lambda = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_3 & . \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_3 & . \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & . \\ . & . & \lambda\gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ . & . & \lambda\beta_4 & \beta_3 & \beta_2 & \beta_1 \\ . & . & \lambda a_4 & a_3 & a_2 & a_1 \end{vmatrix}.$$

Making in this the substitution

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{vmatrix} \equiv \begin{vmatrix} a_0 & a_1 & a_2 - a_6\lambda^2 & a_3 - a_5\lambda \\ . & a_0 - a_6\lambda^3 & a_1 - a_5\lambda^2 & a_2 - a_4\lambda \\ -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_4\lambda^2 & a_1 - a_3\lambda \end{vmatrix}$$

we of course at once find the left-hand side identical with one side of Mr. Roseveare's, and the right-hand side changed into

$$\begin{vmatrix} a_0 & a_1 & a_2 - a_6\lambda^2 & a_3 - a_5\lambda & a_2 - a_6\lambda^2 & . \\ . & a_0 - a_6\lambda^3 & a_1 - a_5\lambda^2 & a_2 - a_4\lambda & a_1 - a_5\lambda^2 & . \\ -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_4\lambda^2 & a_1 - a_3\lambda & a_0 - a_4\lambda^2 & . \\ . & . & a_1\lambda - a_3\lambda^2 & a_0 - a_4\lambda^2 & -a_5\lambda^3 & -a_6\lambda^4 \\ . & . & a_2\lambda - a_4\lambda^2 & a_1 - a_5\lambda^2 & a_0 - a_6\lambda^3 & . \\ . & . & a_3\lambda - a_5\lambda^2 & a_2 - a_6\lambda^2 & a_1 & a_0 \end{vmatrix}.$$

By performing on this last the operations

$$\left. \begin{array}{l} \text{row}_1 - \lambda^{-1} \text{row}_5 \\ \text{row}_2 - \lambda^{-1} \text{row}_4 - \text{row}_6 \\ \text{row}_3 \quad \quad \quad - \text{row}_5 \end{array} \right\}, \quad \left. \begin{array}{l} \text{col}_4 + \lambda^{-1} \text{col}_2 \\ \text{col}_5 + \lambda^{-1} \text{col}_2 \\ \text{col}_6 + \quad \quad \text{col}_2 \end{array} \right\},$$

there is obtained

$$\left| \begin{array}{cccccc} a_0 & a_1 & a_4\lambda - a_6\lambda^2 & a_3 & a_2 & a_1 \\ . & a_0 - a_6\lambda^3 & . & . & . & . \\ -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_2\lambda & -a_3\lambda & -a_4\lambda^2 & -a_5\lambda^3 \\ . & . & a_1\lambda - a_3\lambda^2 & a_0 - a_4\lambda^2 & -a_5\lambda^3 & -a_6\lambda^4 \\ . & . & a_2\lambda - a_4\lambda^2 & a_1 - a_5\lambda^2 & a_0 - a_6\lambda^3 & . \\ . & . & a_3\lambda - a_5\lambda^2 & a_2 - a_6\lambda^2 & a_1 & a_0 \end{array} \right|.$$

which, when we have multiplied the 3rd row and divided the 3rd column by λ and have reversed the order of the last four columns, and thereafter of the last four rows, becomes

$$(a_0 - a_6\lambda^3) \left| \begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4\lambda - a_6\lambda^2 \\ . & a_0 & a_1 & a_2 - a_6\lambda^2 & a_3\lambda - a_5\lambda^2 \\ . & . & a_0 - a_6\lambda^3 & a_1 - a_5\lambda^2 & a_2\lambda - a_4\lambda^2 \\ . & -a_6\lambda^4 & -a_5\lambda^3 & a_0 - a_4\lambda^2 & a_1\lambda - a_3\lambda^2 \\ -a_6\lambda^4 & -a_5\lambda^3 & -a_4\lambda^2 & -a_3\lambda & a_0 - a_2\lambda \end{array} \right|,$$

as desired.

Similarly, in the next case, the determinant for transformation being in the notation of the original paper

$$\left| \begin{array}{cccccc} 0 & 1 & 2 & 37 & 46 & 37 & . & . \\ . & 0 & 17 & 26 & 35 & 26 & . & . \\ . & 7_4 & 06 & 15 & 24 & 15 & . & . \\ 7_5 & 6_4 & 5_3 & 04 & 13 & 04 & . & . \\ . & . & . & 13\lambda & 04 & 5_3 & 6_4 & 7_5 \\ . & . & . & 24\lambda & 15 & 06 & 7_4 & . \\ . & . & . & 35\lambda & 26 & 17 & 0 & . \\ . & . & . & 46\lambda & 37 & 2 & 1 & 0 \end{array} \right|$$

we perform the operations

$$\left. \begin{array}{l} \text{row}_1 - \lambda^{-1} \text{row}_7 \\ \text{row}_2 - \lambda^{-1} \text{row}_6 - \text{row}_8 \\ \text{row}_3 - \lambda^{-1} \text{row}_5 - \text{row}_7 \\ \text{row}_4 \quad \quad \quad - \text{row}_6 \end{array} \right\}, \quad \left. \begin{array}{l} \text{col}_8 + \text{col}_2 \\ \text{col}_7 + \text{col}_3 + \lambda^{-1} \text{col}_1 \\ \text{col}_6 \quad \quad \quad + \lambda^{-1} \text{col}_2 \\ \text{col}_5 \quad \quad \quad + \lambda^{-1} \text{col}_3 \end{array} \right\}$$

and obtain

$$\begin{vmatrix} 0 & 1 & 2 & 57\lambda & 4 & 3 & 2 & 1 \\ . & 0 & 17 & . & . & . & . & . \\ . & 7_4 & 06 & . & . & . & . & . \\ 7_5 & 6_4 & 5_3 & 02 & 3_1 & 4_2 & 5_3 & 6_4 \\ . & . & . & 13\lambda & 04 & 5_3 & 6_4 & 7_5 \\ . & . & . & 24\lambda & 15 & 06 & 7_4 & . \\ . & . & . & 35\lambda & 26 & 17 & 0 & . \\ . & . & . & 46\lambda & 37 & 2 & 1 & 0 \end{vmatrix},$$

which is readily seen to be equal to

$$\begin{vmatrix} 0 & 17 \\ 7_4 & 06 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 57 \\ . & 0 & 1 & 2 & 37 & 46 \\ . & . & 0 & 17 & 26 & 35 \\ . & . & 7_4 & 06 & 15 & 24 \\ . & 7_5 & 6_4 & 5_3 & 04 & 13 \\ 7_6 & 6_5 & 5_4 & 4_3 & 3_2 & 02 \end{vmatrix}$$