On the General Equation of the Second Degree in Tetrahedral Coordinates. By Mr. HARRY HART.

[Read April 14th, 1881.]

1. Let the equation to the surface be

$$\phi(a, \beta, \gamma, \delta) \equiv u_{11}a^2 + u_{22}\beta^2 + u_{33}\gamma^2 + u_{44}\delta^2 + 2u_{12}a\beta + 2u_{13}a\gamma + 2u_{14}a\delta + 2u_{23}\beta\gamma + 2u_{24}\beta\delta + 2u_{34}\gamma\delta = 0 \dots (i.);$$

a, β, γ, δ , the coordinates of a point P , being the ratios of the tetrahedra $PBCD, PCDA, PDAB, PABO$ to the tetrahedron of reference $ABCD$, and consequently $a + \beta + \gamma + \delta = 1$.

Then, $\bar{a}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ being coordinates of the centre (the pole of the plane at infinity $a + \beta + \gamma + \delta = 0$), we have, as usual,

$$\left. \begin{aligned} u_{11}\bar{a} + u_{12}\bar{\beta} + u_{13}\bar{\gamma} + u_{14}\bar{\delta} + \epsilon &= 0 \\ u_{21}\bar{a} + u_{22}\bar{\beta} + u_{23}\bar{\gamma} + u_{24}\bar{\delta} + \epsilon &= 0 \\ u_{31}\bar{a} + u_{32}\bar{\beta} + u_{33}\bar{\gamma} + u_{34}\bar{\delta} + \epsilon &= 0 \\ u_{41}\bar{a} + u_{42}\bar{\beta} + u_{43}\bar{\gamma} + u_{44}\bar{\delta} + \epsilon &= 0 \\ \bar{a} + \bar{\beta} + \bar{\gamma} + \bar{\delta} &= 1 \end{aligned} \right\} \dots\dots\dots(ii.),$$

where $u_{11} = u_{12}$, &c., and ϵ is some constant.

2. Let H, h_1, h_2, h_3, h_4 be the discriminants of the surface and the four conics (real or imaginary), in which the surface is cut by the four faces of the tetrahedron of reference $ABCD$; also let B, b_1, b_2, b_3, b_4 be the above discriminants (expressed as determinants) bordered with 1's; thus

$$B \equiv \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & 1 \\ u_{21} & u_{22} & u_{23} & u_{24} & 1 \\ u_{31} & u_{32} & u_{33} & u_{34} & 1 \\ u_{41} & u_{42} & u_{43} & u_{44} & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}, \quad b_1 = \begin{vmatrix} u_{22} & u_{23} & u_{24} & 1 \\ u_{32} & u_{33} & u_{34} & 1 \\ u_{42} & u_{43} & u_{44} & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}, \text{ \&c.,}$$

so that $b_1 \dots$ are the minors of $u_{11} \dots$ in B . Lastly, let k_1 be the minor of the 1 which is situated in the same row or column as u_{11} , $k_2 = \&c.$; that is,

$$k_1 = \begin{vmatrix} u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

By a known property of determinants we have the following relations:—

$$\left. \begin{aligned} k_1^2 &= Hb_1 - Bh_1 \\ k_2^2 &= Hb_2 - Bh_2 \\ k_3^2 &= Hb_3 - Bh_3 \\ k_4^2 &= Hb_4 - Bh_4 \end{aligned} \right\} \dots\dots\dots(iii.)$$

COR.—If $H = 0$, h_1, h_2, h_3, h_4 are all of opposite sign to B ; and if $B = 0$, b_1, b_2, b_3, b_4 have all the same sign as H .

3. Solving equations (ii.),

$$\bar{a} = \frac{k_1}{B}, \quad \bar{\beta} = \frac{k_2}{B}, \quad \bar{\gamma} = \frac{k_3}{B}, \quad \bar{\delta} = \frac{k_4}{B}, \quad \epsilon = \frac{H}{B};$$

and, multiplying equations (ii.) by $\bar{a}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, -\epsilon$, respectively, and adding, we get $\phi(\bar{a}\bar{\beta}\bar{\gamma}\bar{\delta}) = -\epsilon = -\frac{H}{B}$;

thus the equation to the asymptotic cone is (since it is satisfied by the coordinates of the centre of the surface)

$$\begin{aligned} \phi(a\beta\gamma\delta) &= \text{constant} \\ &= \phi(\bar{a}\bar{\beta}\bar{\gamma}\bar{\delta}) \\ &= -\frac{H}{B}(a+\beta+\gamma+\delta)^2, \end{aligned}$$

or $B\phi(a\beta\gamma\delta) + H(a+\beta+\gamma+\delta)^2 = 0$ (iv.);

but, if equation (i.) represent a cone, $H = 0$, and the equation becomes

$$\phi(a, \beta, \gamma, \delta) = 0,$$

i.e., the asymptotic cone is the surface itself.

4. Through the centre of the surface draw a plane $B'O'D'$ parallel to BCD , and let the coordinates of a point P be $\alpha', \beta', \gamma', \delta'$, referred to the tetrahedron $AB'O'D'$, and $\alpha, \beta, \gamma, \delta$ referred to $ABCD$.

Let p_1, p_2, p_3, p_4 be the perpendiculars from A, B, C, D on the opposite faces of the tetrahedron $ABCD$, and p'_1, p'_2, p'_3, p'_4 the perpendiculars from A, B', C', D' on the opposite faces of the tetrahedron $AB'O'D'$; then $\alpha p_1, \alpha' p'_1$ are the perpendiculars from P on the planes BCD and $B'O'D'$, so that

$$\alpha p_1 - \alpha' p'_1 = p_1 - p'_1,$$

or $(\alpha - 1)p_1 = (\alpha' - 1)p'_1$;

but when $\alpha = \bar{a}$, $\alpha' = 0$,

so that $(\bar{a} - 1)p_1 = -p'_1$;

whence $\alpha = \alpha' + \bar{a}(1 - \alpha')$
 $= \alpha' + \bar{a}(\beta' + \gamma' + \delta').$

Also $\beta p_2 = \beta' p'_2$,

so that $\beta = \frac{p'_2}{p_2} \beta' = \frac{p'_1}{p_1} \beta' = (1 - \bar{a}) \beta'$;

similarly $\gamma = (1-\bar{a}) \gamma',$

and $\delta = (1-\bar{a}) \delta'.$

Thus, suppressing the accents, we may write the equation to the surface ($AB'CD'$ being the tetrahedron of reference),

$$\begin{aligned} u_{11} \{a + \bar{a} (\beta + \gamma + \delta)\}^2 + \{u_{22}\beta^2 + u_{33}\gamma^2 + u_{44}\delta^2\} (1-\bar{a})^2 \\ + 2 \{a + \bar{a} (\beta + \gamma + \delta)\} (1-\bar{a}) \{u_{12}\beta + u_{13}\gamma + u_{14}\delta\} \\ + 2 \{u_{23}\beta\gamma + u_{24}\beta\delta + u_{34}\gamma\delta\} (1-\bar{a})^2 = 0. \end{aligned}$$

5. Making now $a = 0$, the above becomes the equation (in triangular coordinates) to the conic F_1 made by the central section of the surface parallel to the face BCD of the original tetrahedron. This equation is

$$\begin{aligned} \lambda^2 u_{11} (\beta + \gamma + \delta)^2 + 2\lambda (\beta + \gamma + \delta) (u_{12}\beta + u_{13}\gamma + u_{14}\delta) + u_{22}\beta^2 + u_{33}\gamma^2 + u_{44}\delta^2 \\ + 2 \{u_{23}\beta\gamma + u_{24}\beta\delta + u_{34}\gamma\delta\} = 0, \end{aligned}$$

where λ is put for $\frac{\bar{a}}{1-\bar{a}}.$

The discriminant of this conic is

$$\begin{vmatrix} \lambda^2 u_{11} + 2\lambda u_{12} + u_{22} & \lambda^2 u_{11} + \lambda (u_{12} + u_{13}) + u_{23} & \lambda^2 u_{11} + \lambda (u_{12} + u_{14}) + u_{24} \\ \lambda^2 u_{11} + \lambda (u_{12} + u_{13}) + u_{23} & \lambda^2 u_{11} + 2\lambda u_{13} + u_{33} & \lambda^2 u_{11} + \lambda (u_{13} + u_{14}) + u_{34} \\ \lambda^2 u_{11} + \lambda (u_{12} + u_{14}) + u_{24} & \lambda^2 u_{11} + \lambda (u_{13} + u_{14}) + u_{34} & \lambda^2 u_{11} + 2\lambda u_{14} + u_{44} \end{vmatrix},$$

which is easily seen to be the same as

$$\begin{aligned} & - \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & -1 \\ u_{21} & u_{22} & u_{23} & u_{24} & \lambda \\ u_{31} & u_{32} & u_{33} & u_{34} & \lambda \\ u_{41} & u_{42} & u_{43} & u_{44} & \lambda \\ -1 & \lambda & \lambda & \lambda & 0 \end{vmatrix} \\ & = -\lambda^2 \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & 0 \\ u_{21} & u_{22} & u_{23} & u_{24} & 1 \\ u_{31} & u_{32} & u_{33} & u_{34} & 1 \\ u_{41} & u_{42} & u_{43} & u_{44} & 1 \\ 0 & 1 & 1 & 1 & 0 \end{vmatrix} + 2\lambda \begin{vmatrix} u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \\ 0 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} u_{22} & u_{23} & u_{24} \\ u_{32} & u_{33} & u_{34} \\ u_{42} & u_{43} & u_{44} \end{vmatrix} \\ & = -\lambda^2 \{B - 2k_1 - h_1\} + 2\lambda \{k_1 + h_1\} + h_1 \\ & = h_1 \{1 + \lambda\}^2 + 2\lambda (1 + \lambda) k_1 - B\lambda^2. \end{aligned}$$

But since $\lambda = \frac{\bar{a}}{1-\bar{a}} = \frac{k_1}{B-k_1},$

$$\frac{1+\lambda}{\lambda} = \frac{B}{k_1};$$

$$\begin{aligned}
 \text{hence the discriminant} &= \lambda^2 \left\{ h_1 \frac{B^2}{k_1^2} + 2B - B \right\} \\
 &= \lambda^2 B \cdot \frac{Bh_1 + k_1^2}{k_1^2} \\
 &= \lambda^2 B \cdot \frac{HB_1}{k_1^2} \text{ by (iii.)} \\
 &= \frac{BHB_1}{(B-k_1)^2} \dots\dots\dots (\text{v.})
 \end{aligned}$$

The bordered discriminant of the same conic

$$\begin{aligned}
 &= \begin{vmatrix} \lambda^2 u_{11} + 2\lambda u_{12} + u_{22}, & \lambda^2 u_{11} + \lambda(u_{12} + u_{12}) + u_{22}, & \lambda^2 u_{11} + \lambda(u_{12} + u_{14}) + u_{24}, & 1 \\ \lambda^2 u_{11} + \lambda(u_{12} + u_{12}) + u_{22}, & \lambda^2 u_{11} + 2\lambda u_{12} + u_{22}, & \lambda^2 u_{11} + \lambda(u_{12} + u_{14}) + u_{24}, & 1 \\ \lambda^2 u_{11} + \lambda(u_{12} + u_{14}) + u_{24}, & \lambda^2 u_{11} + \lambda(u_{12} + u_{14}) + u_{24}, & \lambda^2 u_{11} + 2\lambda u_{12} + u_{22}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \\
 &= \begin{vmatrix} u_{22}, & u_{22}, & u_{24}, & 1 \\ u_{22}, & u_{22}, & u_{24}, & 1 \\ u_{22}, & u_{22}, & u_{24}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = b_1 \dots\dots\dots (\text{vi.}) ;
 \end{aligned}$$

whence we obtain for the area of this conic the expression

$$2\pi\Delta_1' \frac{BHB_1}{\{B-k_1\}^2} \cdot \frac{1}{\{-b_1\}^{\frac{1}{2}}},$$

where Δ_1' = area of the triangle $B'C'D'$. If Δ_1 = area of BCD , then

$$\begin{aligned}
 \Delta_1' &= \Delta_1 \frac{p_1^2}{p_1^2} = \Delta_1 (1 - \bar{\alpha})^2 \\
 &= \Delta_1 \left\{ \frac{B-k_1}{B} \right\}^2 ;
 \end{aligned}$$

so that the area of the conic

$$\begin{aligned}
 &= 2\pi\Delta_1 \cdot \frac{HB_1}{B} \cdot \frac{1}{\{-b_1\}^{\frac{1}{2}}} \\
 &= \frac{2\pi\Delta_1 H}{B\sqrt{-b_1}} \dots\dots\dots (\text{vii.})
 \end{aligned}$$

6. Next, let p be the perpendicular from the centre of the surface on the tangent plane parallel to the plane of the above conic, i.e., parallel to $\alpha = 0$.

Let $\alpha_1, \beta_1, \gamma_1, \delta_1$ be the coordinates ($ABCD$ being the tetrahedron of reference) of the point at which this tangent plane is drawn. Then the equation of the plane through $\alpha_1, \beta_1, \gamma_1, \delta_1$, parallel to $\alpha = 0$, is

$$\alpha = \alpha_1 (\alpha + \beta + \gamma + \delta),$$

or $\mu\alpha_1 + \beta + \gamma + \delta = 0$, where $\mu = 1 - \frac{1}{\alpha_1}$, or $\alpha_1 = \frac{1}{1-\mu}$,

and

$$\begin{aligned} p &= p_1(a_1 - \bar{a}), \\ &= p_1 \left\{ \frac{1}{1-\mu} - \frac{k_1}{B} \right\} \\ &= p_1 \left\{ \frac{B - (1-\mu)k_1}{B(1-\mu)} \right\} \end{aligned}$$

But the condition that $\mu\alpha + \beta + \gamma + \delta = 0$
should touch the quadric $\phi(\alpha, \beta, \gamma, \delta) = 0$ is

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & \mu \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & 1 \\ \mu, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which becomes, as we have seen (§ 5),

$$0 = h_1 \left\{ 1 - \frac{1}{\mu} \right\}^2 - 2 \frac{1}{\mu} \left\{ 1 - \frac{1}{\mu} \right\} k_1 - B \frac{1}{\mu^2};$$

whence

$$B - 2k_1(1-\mu) = h_1(1-\mu)^2,$$

or

$$\begin{aligned} \{B - k_1(1-\mu)\}^2 &= (Bh_1 - k_1^2)(1-\mu)^2 \\ &= Hb_1(1-\mu)^2 \quad \text{by (iii.)}; \end{aligned}$$

hence

$$p = \pm \frac{p_1}{B} \sqrt{Hb_1} \dots\dots\dots \text{(viii.)}$$

And since the area of the parallel central section has been shown to be

$$= \frac{2\pi\Delta_1 H}{B\sqrt{-b_1}},$$

the volume of the quadric (if an ellipsoid)

$$\begin{aligned} &= \frac{4}{3} \cdot \frac{p_1}{B} \sqrt{Hb_1} \cdot \frac{2\pi\Delta_1 H}{\sqrt{-b_1}}, \\ &= 8\pi V \frac{\{-H\}^{\frac{3}{2}}}{B^2}, \end{aligned}$$

where V = volume of the tetrahedron of reference.

7. I have remarked (*Messenger of Mathematics*, Vol. x., Oct. 1880),
that $ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0$

being the equation (in triangular coordinates) of a conic, if the three
expressions $v + w - 2u'$, $w + u - 2v'$, $u + v - 2w'$

have not all the same sign, the conic is a hyperbola; if they have the
same sign, and the bordered discriminant be negative, the conic is
imaginary or real, according as the common sign is or is not the same
as that of the discriminant.

The three corresponding expressions for the conic F_1 (§ 5) are
 $\{\lambda^2 u_{11} + 2\lambda u_{12} + u_{22}\} + \{\lambda^2 u_{11} + 2\lambda u_{14} + u_{44}\} - 2\{\lambda^2 u_{11} + \lambda(u_{12} + u_{14}) + u_{22}\}$, &c.,

or $u_{23} + u_{44} - 2u_{24}, u_{44} + u_{23} - 2u_{43}, u_{23} + u_{43} - 2u_{22}.$

Hence, if the quantities $u_{11} + u_{22} - 2u_{12} (\equiv v_{12}), u_{11} + u_{33} - 2u_{13} (\equiv v_{13}),$ &c. are not all of the same sign, one at least of the central sections of the surface parallel to the faces of the tetrahedron of reference is a hyperbola.

8. The area of the conic

$$F_1 = 2\pi\Delta_1 \cdot \frac{H}{B\sqrt{-b_1}} = A_1,$$

the perpendicular from the centre of the surface on the parallel tangent plane

$$= \pm \frac{\pi_1}{B} \sqrt{Hb_1} = \pi_1,$$

and the discriminant $= \frac{BHb_1}{(B-k_1)^3} = D_1.$

Let the quadric $\phi = 0$ be—

(i.) *An ellipsoid.*

A_1, π_1 are both real, b_1 is negative, and v_{24}, v_{43}, v_{23} are all of the same sign, and unlike that of the discriminant $D_1.$

Hence H is negative, and v_{24}, v_{43}, v_{23} are of opposite sign to $B.$

(ii.) *A hyperboloid of one sheet.*

(a.) F_1 an ellipse, A_1 is real, π_1 imaginary, and b_1 negative.

Hence H is positive, and v_{24}, v_{43}, v_{23} have sign opposite to D_1 , and therefore the same as $B.$

(β.) F_1 a hyperbola. A_1 is imaginary, π_1 real, and b_1 positive.

Hence H is positive.

(iii.) *A hyperboloid of two sheets.*

(a.) F_1 a hyperbola. A_1 is imaginary, π_1 real, and b_1 positive.

Hence H is negative.

(β.) F_1 imaginary. A_1 is real, π_1 real, and b_1 negative.

Hence H is negative, and v_{24}, v_{43}, v_{23} have the same sign as D_1 , and therefore as $B.$

(iv.) *Imaginary.*

F_1 is imaginary, A_1 real, π_1 imaginary, and b_1 negative.

Hence H is positive, and v_{24}, v_{43}, v_{23} have the same sign as D_1 , and therefore opposite to $B.$

(v.) *An elliptic paraboloid.*

The surface may be considered as an ellipsoid or hyperboloid of two sheets, having its centre at infinity.

Hence $B = 0$ and H is negative; this, by (iii.) § 2, necessitates b_1, b_2, b_3, b_4 all being negative.

(vi.) *A hyperbolic paraboloid.*

This is the limiting case of a hyperboloid of one sheet with its centre at infinity.

Hence $B = 0$ and H is positive; in this case b_1, b_2, b_3, b_4 are all positive.

(vii.) *A cone.*

$H = 0$, by § 3. Also h_1, h_2, h_3, h_4 are of opposite sign to B , and one at least not zero, since $\sqrt{-\frac{h_1}{B}}, \sqrt{-\frac{h_2}{B}}, \sqrt{-\frac{h_3}{B}}, \sqrt{-\frac{h_4}{B}}$ are the coordinates of the vertex.

(viii.) *A cylinder.*

$H = 0, B = 0$, since the surface has a line of centres. The cylinder will be

Parabolic, if $b_1, \&c.$ each $= 0$
 Elliptic „ „ be negative } ; but
 Hyperbolic „ „ be positive }

(ix.) *Two planes*, if in addition to

$H = 0, B = 0$, we have h_1, h_2, h_3, h_4 all $= 0$.

9. If, then, we wish to examine the nature of the surface

$$\phi(a, \beta, \gamma, \delta) = 0,$$

we must first find the values of the quantities

$$B, b_1, \dots, H, h_1 \dots v_{12}, \dots,$$

as defined in § 2 and 7.

Suppose (1.) $B = 0, H = 0$.

If $h_1, \&c.$ each $= 0$, the surface consists of two planes, which will be parallel if $b_1 \dots$ are also zero.

If $h_1, \&c.$ be not all $= 0$, the surface is a parabolic, elliptic, or hyperbolic cylinder, according as $b_1 \dots$ are zero, negative, or positive.

(2.) $B \neq 0, H = 0$.

The surface is a cone.

(3.) $B = 0, H \neq 0$.

The surface is an elliptic or hyperbolic paraboloid, according as H is negative or positive.

(4.) $B \neq 0, H \neq 0$.

The surface is central.

(i.) If $v_{12} \dots$ have not all the same sign, or if any of the quantities b_1, b_2, b_3, b_4 be positive, the surface has hyperbolic sections, and is a hyperboloid of one or two sheets according as H is positive or negative.

(ii.) If $v_{12} \dots$ have all the same sign, and $b_1 \dots$ be all negative, then, if H be positive and $Bv_{12}, \&c.$ be positive, the surface is a hyperboloid of one sheet; if H be positive and $Bv_{12}, \&c.$ be negative, the surface is imaginary; if H be negative and $Bv_{12}, \&c.$ be positive, the surface is a hyperboloid of two sheets; if H be negative and $Bv_{12}, \&c.$ be negative, the surface is an ellipsoid.

The condition that the surface may be a sphere is given in the ordinary text books.

Thursday, May 12th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

The following were elected members:—Lionel Rosenthal, B.A., T.C.D.; Fabian Franklin, Ph.D., Johns Hopkins University; Charles A. van Velzer, Johns Hopkins University; and Miss Christine Ladd, Johns Hopkins University.

Prof. Charles Niven, F.R.S., was admitted into the Society.

The following communications were made:—

"On Ptolemy's Theorem," Mr. C. W. Merrifield.

"The Summation of certain Hyper-geometric Series," Rev. T. R. Terry.

"Quaternion Proof of Mr. S. Roberts's Theorem of Four Co-intersecting Spheres," Mr. J. J. Walker.

"Some Solutions of the '15-girl' Problem," Mr. E. Carpmael.

"Note on the Coordinates of a Tangent Line to the Curve of Intersection of two Quadrics," Mr. W. R. W. Roberts.

Short communications were made by the President, and Messrs. Cayley, Hart, and Walker.

The following presents were made to the Library:—

"Educational Times," May, 1881.

"Notice sur les travaux géométriques de M. Mannheim (à l'appui de sa candidature à l'Académie des Sciences)," from the Author; Paris, 1881.

"List of Members of the Institution of Civil Engineers," 30th March, 1881.

"Hyper-elliptische Functionen und die Kummer'sche Fläche," von K. Rohn (Math. Ann., Band. xv., pp. 1-40).

"Die verschiedenen Gestalten der Kummer'schen Fläche," von Karl Rohn (Math. Ann., Band xviii., pp. 99-159).

"Théorie des figures projectives sur une surface du second ordre," par H. G. Zeuthen, à Copenhague (Math. Ann., Band xviii., pp. 33-68).

"Konstruktion af det ottende Skærings Punkt, mellem de Fladen af anden Orden, som gaa gjennem syv givne Punkter," af H. G. Zeuthen.

"Construction du huitième point commun aux surfaces du second ordre qui passent par sept points donnés," par H. G. Zeuthen.

"Théorie des figures projectives sur une surface du second ordre," par H. G. Zeuthen.

"Grafisk Behandling af en Bjælkes bevægelige Belastning," af H. G. Zeuthen: Kjøbenhavn, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 9, 10; Roma, 1881.

"Monatsbericht," Dec., 1880.

"Proceedings of Royal Society," Vol. xxxi., No. 211.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 4; Leipzig, 1881.

"Crelle," Band 91, 1^{re} Heft; Berlin, 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iv., Nov., Dec., 1880.

"Proceedings of the Cambridge Philosophical Society," Vol. iii., Pt. vii., Oct. to Dec., 1879; Pt. viii., Feb. to May, 1880.

"Transactions of the Cambridge Philosophical Society," Vol. xiii., Pt. i., with Index to Vols. i. to xii.

"Proceedings of the Royal Society of Edinburgh," Session 1879–1880, Vol. x., pp. 315–781.

Note on the Coordinates of a Tangent Line to the Curve of Intersection of two Quadrics. By W. R. WESTROPP ROBERTS.

[Read May 12th, 1881.]

The condition that the line, the six coordinates of which are a, b, c, f, g, h , should touch the surface

$$U \equiv ax^3 + \beta y^3 + \gamma z^3 + \delta u^3,$$

is 1°. $\delta (af^3 + \beta g^3 + \gamma h^3) + \beta \gamma a^3 + \gamma ab^3 + a\beta c^3 = 0$.

I now determine the conditions that the line (a, b, c, f, g, h) should be a generator of U . Solving for x and y from the equations of the line, and substituting the resulting values in the equation of the quadric, we get a quadratic in z and u , each coefficient of which we must equate to zero. We have then three conditions which, joined to the universal equation $af + bg + ch = 0$, give us

$$\begin{cases} 2^\circ. \dots\dots \frac{a^3}{f^3} = \frac{a\delta}{\beta\gamma}, \quad \frac{b^3}{g^3} = \frac{\beta\delta}{\alpha\gamma}, \quad \frac{c^3}{h^3} = \frac{\gamma\delta}{\alpha\beta}, \\ 3^\circ. \dots\dots \alpha f^3 + \beta g^3 + \gamma h^3 = 0. \end{cases}$$

Now, if $V \equiv \alpha'x^3 + \beta'y^3 + \gamma'z^3 + \delta'u^3$; through any assumed point we can describe a quadric of the form $U + \lambda V$, the two generators of which, passing through the assumed point, are "lines through two points" of the curve UV .

Hence the line (a, b, c, f, g, h) will pass through two points of the