The Algebra of Multi-linear Partial Differential Operators.

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§ 1.

In Vol. xviii., p. 61, *Proc. Lond. Math. Soc.*, I discussed a linear partial differential operator which was defined by

$$(\mu, \nu; m, n) \equiv \sum_{s=0}^{s=\infty} (\mu + s\nu) A_{s,m} \partial_{a_{n+s}};$$

where

$$A_{s,m} = \Sigma \frac{(m-1)!}{\kappa_0!\,\kappa_1!\,\kappa_2!\,\kappa_3!\,\dots} a^{\kappa_0} b^{\kappa_1} c^{\kappa_2} d^{\kappa} \dots \begin{pmatrix} \Sigma\kappa = m \\ \Sigma t\kappa_t = s \end{pmatrix}$$

These operators were shown to form an alternating group, in that the alternant of any two of them resulted in another operator of the same class.

It will be convenient to call the successive operation of two operators P and Q their outer multiplication, and to write it

$$(P)\,(Q),$$

also their symbolic algebraic multiplication may be called their inner multiplication, and may be written

$$(PQ),$$

and the explicit operation of P upon Q, the latter being considered as a function of symbols of quantity only, may, for reasons which will subsequently appear, be termed the symbolic addition of P and Q, and may be written $(P \dagger Q)$.

Of these three operations the second only is in general commutative. We have then

$$(P)(Q) = (PQ) + (P \dagger Q),$$

P and Q being any linear operators whatever, and the main theorem (*loc. cit.*) was that, P and Q being any members of the multi-linear class above defined,

$$(P)(Q) - (Q)(P) = (P \dagger Q) - (Q \dagger P),$$

$$= \text{an operator of the same class.}$$

This result expresses that the alternant of P and Q, viz.

$$(P)(Q)-(Q)(P),$$

is another operator of the same general class, and hence the characteristic property of these operators, that they form an alternating group.

The multiplication theorem which was auxiliary to this result was

$$(\mu', \nu'; m', n')(\mu, \nu; m, n) = \{(\mu', \nu'; m', n').(\mu, \nu; m, n)\}$$
$$+\sum_{\kappa=0}^{\kappa=\infty}\left\{(m'+m-1)\frac{\mu'}{m'}+\kappa\nu'\right\}\{\mu+(n'+\kappa)\nu\}A_{\kappa, m'+m-1}\partial_{a_{n'+n+\kappa}},$$

which by analogy may be written

$$(\mu', \nu'; m', n')(\mu, \nu; m, n) = \{(\mu', \nu'; m', n').(\mu, \nu; m, n)\}$$
$$+\left\{(m'+m-1)\frac{\mu'}{m'}, \nu'; \mu+n'\nu, \nu; m'+m-1, n'+n\right\};$$

the operator last written is a multi-linear operator of six elements which arises from the theorem

$$(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)$$
$$=\left\{(m'+m-1)\frac{\mu'}{m'}, \nu'; \mu+n'\nu, \nu; m'+m-1, n'+n\right\}.$$

In the further development of the algebra, operators of 8, 10, 12, ... elements will arise; in fact

$$(\mu'', \nu''; m'', n'') \dagger \{(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)\}$$
$$=(\mu'', \nu''; m'', n'') \dagger \Big[\ldots+\sum_{s=0}^{s=\infty}\left\{(m'+m-1)\frac{\mu'}{m'}+(n''+s)\nu'\right\}$$
$$\times\{\mu+(n''+n'+s)\nu\}A_{n''+s, m'+m-1}\partial_{a_{n''+n'+n+s}}\Big]$$
$$=\sum_{s=0}^{s=\infty}\left\{(m''+m'+m-2)\frac{\mu''}{m''}+s\nu''\right\}\left\{(m'+m-1)\frac{\mu'}{m'}+n''\nu'+s\nu'\right\}$$
$$\times\{\mu+(n''+n')\nu+s\nu\}A_{s, m''+m'+m-2}\partial_{a_{n''+n'+n+s}};$$

or, finally, we have a formula introducing a multi-linear operator of 8 elements, viz.:

$$(\mu'', \nu''; m'', n'') \dagger \{(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)\}$$
$$=\Big[(m''+m'+m-2)\frac{\mu''}{m''}, \nu''; (m'+m-1)\frac{\mu'}{m'}+n''\nu', \nu';$$
$$\mu+(n''+n')\nu, \nu; m''+m'+m-2, n''+n'+n\Big].$$

The next formula involving an operator of 10 elements is, without difficulty, found to be

$$(\mu''', \nu'''; m''', n''') \dagger \left[(\mu'', \nu''; m'', n'') \dagger \{ (\mu', \nu'; m', n') \dagger (\mu, \nu; m, n) \} \right]$$

$$= \left\{ \begin{array}{c} (m'''+m''+m'+m-3)\frac{\mu'''}{m'''},\ \nu''' \\ (m''+m'+m-2)\frac{\mu''}{m''}+m'''\nu'',\ \nu'' \\ (m'+m-1)\frac{\mu'}{m'}+(n'''+n'')\nu',\ \nu' \\ \mu+(n'''+n''+n')\nu,\ \nu \\ m'''+m''+m'+m-3,\ n'''+n''+n'+n \end{array} \right\},$$

where for conciseness the pairs of elements on the right-hand side have been written underneath one another.

By induction, the law of formation of the successive pairs of elements is easily established.

Section 2.

Explicit operation of a six-element upon a four-element operator.

Denoting by P, Q, R, any three four-element operators, we have

$$\{(P \dagger Q) \dagger R\} = \{(P)(Q) \dagger (R)\} - \{(PQ) \dagger (R)\}$$
$$= \{P \dagger (Q \dagger R)\} - \{(PQ) \dagger R\},$$

showing that P, Q, and R are non-associative as regards symbolic addition, and this may be regarded as a theorem either for the expression of the explicit operation of the six-element operator $(P \dagger Q)$ upon the four-element operator R, as a result of explicit operations performed only upon R; or as the expression of the explicit operation of the symbolic product (PQ) upon R by means of explict operations without the prior performance of symbolic multiplication.

We may write the formula as follows,

$$(PQ \dagger R) = \{P \dagger (Q \dagger R)\} - \{(P \dagger Q) \dagger R\},$$

also $(R)(P \dagger Q) - (P \dagger Q)(R) = \{R \dagger (P \dagger Q)\} - \{(P \dagger Q) \dagger R\}$

$$= \{R \dagger (P \dagger Q)\} - \{P \dagger (Q \dagger R)\} + (PQ \dagger R);$$

so that, if R be lineo-linear, its alternant with $(P \dagger Q)$ is expressible by means of two operators of eight elements each.

SECTION 3.

The general multilinear operator may be expressed in terms of the lineo-linear operators

$$d_\lambda = (1, 0; 1, \lambda) = a_0 \partial_{a_\lambda} + a_1 \partial_{a_{\lambda+1}} + a_2 \partial_{a_{\lambda+2}} + \ldots,$$

for
$$\partial_{a_\lambda} = \frac{d_\lambda}{a_0} - \frac{h_1 d_{\lambda+1}}{a_0^2} + \frac{h_2 d_{\lambda+2}}{a_0^3} - \frac{h_3 d_{\lambda+3}}{a_0^4} + \ldots,$$

where h_s is the product of a_0^s, and the total symmetric function of weight s of the roots of the equation

$$u \equiv a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0 \quad (n = \infty);$$

and hence

$$\mu A_{0,m} \partial_{a_n} + (\mu+\nu) A_{1,m} \partial_{a_{n+1}} + (\mu+2\nu) A_{2,} \;\partial_{a_{n+2}} + \ldots$$

$$= \mu A_{0,m} \left\{ \frac{d_n}{a_0} - \frac{h_1}{a_0^2} d_{n+1} + \frac{h_2}{a_0^3} d_{n+2} - \ldots \right\}$$

$$+ (\mu+\nu) A_{1,m} \left\{ \frac{1}{a_0} d_{n+1} - \frac{h_1}{a_0^2} d_{n+2} + \frac{h_2}{a_0^3} d_{n+3} - \ldots \right\}$$

$$+ (\mu+2\nu) A_{2,m} \left\{ \frac{1}{a_0} d_{n+2} - \frac{h_1}{a_0^2} d_{n+3} + \frac{h_2}{a_0^3} d_{n+4} - \ldots \right\}$$

$$+ \ldots\ldots$$

$$= \mu \frac{A_{0,m}}{a_0} d_n + \left\{ (\mu+\nu) A_{1,m} \frac{1}{a_0} - \mu \frac{A_{0,m}}{a_0^2} h_1 \right\} d_{n+1}$$

$$+ \left\{ (\mu+2\nu) \frac{A_{2,m}}{a_0} - (\mu+\nu) \frac{A_{1,m}}{a_0^2} h_1 + \mu \frac{A_{0,m}}{a_0^3} h_2 \right\} d_{n+2}$$

$$+ \ldots\ldots$$

$$+ \left\{ (\mu+s\nu) \frac{A_{s,m}}{a_0} - (\mu+s\nu-\nu) \frac{A_{s-1,m}}{a_0^2} h_1 + \ldots + (-)^s \mu \frac{A_{0,m}}{a_0^{s+1}} h_s \right\} d_{n+s}$$

$$+ \ldots\ldots.$$

Now, since
$$\frac{1}{m} u^m \doteq A_{0,m} - A_{1,m} x + A_{2,m} x^2 - \ldots,$$

$$u^{-1} \doteq \frac{1}{a_0} + \frac{h_1}{a_0^2} x + \frac{h_2}{a_0^3} x^2 + \ldots$$

We find, by multiplication and comparison of the coefficients of x^s,

$$\frac{m-1}{m} A_{s,m-1} = \frac{1}{a_0} A_{s,m} - \frac{h_1}{a_0^2} A_{s-1,m} + \ldots (-)^s \frac{h_s}{a_0^{s+1}} A_{0,m},$$

further $$u^{m-1}u' = -A_{1,m}+2A_{2,m}x-3A_{3,m}x^2+\ldots,$$
$$u^{m-2}u' = -A_{1,m-1}+2A_{2,m-1}x-3A_{3,m-1}x^2+\ldots,$$

therefore

$$(-A_{1,m}+2A_{2,m}x-3A_{3,m}x^2+\ldots)\left(\frac{1}{a_0}+\frac{h_1}{a_0^2}x+\ldots\right)$$
$$=-A_{1,m-1}+2A_{2,m-1}x-3A_{3,m-1}x^2+\ldots,$$

therefore

$$sA_{s,m-1} = s\,\frac{1}{a_0}A_{s,m}-(s-1)\,\frac{h_1}{a_0^2}A_{s-1,m}+\ldots+(-)^{s-1}\frac{h_{s-1}}{a_0^s}A_{1,m};$$

whence the coefficient of d_{n+s} is

$$\left(\frac{m-1}{m}\mu+s\nu\right)A_{s,m-1}$$

and the operator becomes

$$\sum_{s=0}^{s=\infty}\left(\frac{m-1}{m}\mu+s\nu\right)A_{s,m-1}d_{n+s}.$$

The operator in the theory of pure reciprocants now takes the simpler and, in some respects, more convenient form

$$2ad_1+3bd_2+4cd_3+\ldots.$$

Section 4.

The sub-group of Operators of Two Elements.

A special case of the general operator arises when the second element ν is zero; then we may without loss of generality put μ equal to unity, since it is common to every term, and we may represent the operator $(1, 0;\ m, n)$ by the shorter notation (m, n).

The multiplication theorem is

$$(m', n')(m, n) = \{(m', n')(m, n)\}+\frac{m'+m-1}{m'}(m'+m-1, n'+n),$$

an identity in which only operators of two elements occur.

This class thus constitutes an algebraic group in the sense that algebraic operations produce operators which may be always expressed by operators of the same class.

For the alternant of $(\mu', \nu';\ m', n')$ and (m, n), we find

$$(\mu', \nu';\ m', n')(m, n)-(m, n)(\mu', \nu';\ m', n')$$
$$=(\mu_1, \nu_1;\ m'+m-1, n'+n),$$

where $$\mu_1 = (m'+m-1)\left\{\frac{\mu'}{m'} - \frac{1}{m}(\mu'+n\nu')\right\},$$
$$\nu_1 = -\frac{m'-1}{m}\nu'.$$

This alternant will vanish, if $\mu_1 = \nu_1 = 0$.

CASE I.—If $\nu' = 0$, then
$$m' = m, \text{ or } m' = 1-m,$$
leading to the results
$$(m, n')(m, n) - (m, n)(m, n') = 0,$$
$$(1-m, n')(m, n) - (m, n)(1-m, n') = 0.$$

CASE II.—If $m' = 1$, then
$$\frac{\mu'}{\nu'} = \frac{n}{m-1},$$
leading to
$$(n, m-1;\ 1, n')(m, n) - (m, n)(n, m-1;\ 1, n') = 0.$$
Thus, in general, (m, n) has the three commutors
$$(m, n'),$$
$$(1-m, n'),$$
$$(n, m-1;\ 1, n');$$
the alternant is given by
$$\begin{vmatrix} (m', n'), & (m, n) \\ (m', n'), & (m, n) \end{vmatrix} = \frac{(m'+m-1)(m-m')}{m'm}(m'+m-1, n'+n),$$
to which may be added the result
$$\begin{vmatrix} (m'', n''), & (m', n'), & (m, n) \\ (m'', n''), & (m', n'), & (m, n) \\ (m'', n''), & (m', n'), & (m, n) \end{vmatrix}$$
$$= \frac{(m'+m-1)(m-m')}{m'm}\left\{(m'', n'')(m'+m-1, n'+n)\right\}$$
$$+ \frac{(m''+m'-1)(m'-m'')}{m''m'}\left\{(m, n)(m''+m'-1, n''+n')\right\}$$
$$+ \frac{(m+m''-1)(m''-m)}{mm''}\left\{(m', n')(m+m''-1, n+n'')\right\}.$$

SECTION 5.

Let π, ρ_0 be any two members of the alternating group, and further

$$(\pi)(\rho_0)-(\rho_0)(\pi)=\rho_1,$$
$$(\pi)(\rho_1)-(\rho_1)(\pi)=\rho_2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(\pi)(\rho_m)-(\rho_m)(\pi)=\rho_{m+1};$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

we have then the known theorem

$$(\pi)^n(\rho_0)=(\rho_0)(\pi)^n+n(\rho_1)(\pi)^{n-1}+\frac{n(n-1)}{2!}(\rho_2)(\pi)^{n-2}+\ldots.$$

If the subject of operation be previously operated upon by ρ_0, we have

$$(\pi)^n(\rho_0)^2=(\rho_0)^2(\pi)^n+n\left\{(\rho_0)(\rho_1)+(\rho_1)(\rho_0)\right\}(\pi)^{n-1}$$
$$+\frac{n(n-1)}{2!}\left\{(\rho_0)(\rho_2)+2(\rho_1)^2+(\rho_2)(\rho_0)\right\}(\pi)^{n-2}+\ldots,$$

which may be symbolically written

$$(\pi)^n(\rho_0)^2=(\rho+\rho)^0(\pi)^n+n(\rho+\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\rho+\rho)^2(\pi)^{n-2}+\ldots,$$

wherein $(\rho+\rho)^t$ denotes

$$(\rho_0)(\rho_t)+t(\rho_1)(\rho_{t-1})+\frac{t(t-1)}{2!}(\rho_2)(\rho_{t-2})+\ldots.$$

Let us assume

$$(\pi)^n(\rho_0)^s=(\overset{s}{\Sigma}\rho)^0(\pi)^n+n(\overset{s}{\Sigma}\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho)^2(\pi)^{n-2}+\ldots,$$

and then $(\pi)^n(\rho_0)^{s+1}=(\overset{s}{\Sigma}\rho)^0(\pi)^n(\rho_0)+n(\overset{s}{\Sigma}\rho)^1(\pi)^{n-1}(\rho_0)$

$$+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho)^2(\pi)^{n-2}(\rho_0)+\ldots,$$

or, by a previous theorem,

$$(\pi)^n(\rho_0)^{s+1}=(\overset{s}{\Sigma}\rho+\rho)^0(\pi)^n+n(\overset{s}{\Sigma}\rho+\rho)^1(\pi)^{n-1}$$
$$+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho+\rho)^2(\pi)^{n-2}+\ldots$$
$$=(\overset{s+1}{\Sigma}\rho)^0(\pi)^n+n(\overset{s+1}{\Sigma}\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\overset{s+1}{\Sigma}\rho)^2(\pi)^{n-2}+\ldots;$$

or, the law assumed true for the expansion of $(\pi)^n(\rho_0)^s$ is equally true for the expansion of $(\pi)^n(\rho_0)^{s+1}$.

Hence, by induction, the general law is established.

In particular, if $(\pi) = 0$,

$$(\pi)^n (\rho_0)^s \equiv (\overset{s}{\Sigma}\rho)^n.$$

We may write the result in the form

$$(\pi)^n (\rho_0)^s = (\overset{s}{\Sigma}\rho+\pi)^n;$$

and we easily reach the companion theorem

$$(\rho_0)^s (\pi)^n = (\pi-\overset{s}{\Sigma}\rho)^n$$

More generally

$$f(\pi)(\rho_0) = (\rho_0) f(\pi) + \frac{\rho_1}{1!} f'(\pi) + \frac{\rho_2}{2!} f''(\pi) + \ldots,$$

wherein $f(\pi)$ denotes any rational integral function of π; whence, proceeding as before, we find

$$f(\pi)(\rho_0)^s = f(\overset{s}{\Sigma}\rho+\pi);$$

and also $$(\rho_0)^s f(\pi) = f(\pi-\overset{s}{\Sigma}\rho).$$

Let now $$\phi(\rho_0) = \Sigma A_s (\rho_0)^s,$$

and then $$f(\pi)\,\phi(\rho_0) = \Sigma A_s f(\overset{s}{\Sigma}\rho+\pi);$$

or, if $$f_s = f(\overset{s}{\Sigma}\rho+\pi),$$

and $$\phi(f) = \Sigma A_s f_s,$$

then $$f(\pi)\,\phi(\rho_0) = \phi(f),$$

and $$\phi(\rho_0) f(\pi) = \phi(f'),$$

where $$f'_s = f(\pi-\overset{s}{\Sigma}\rho),$$

and $$\phi(f') = \Sigma A_s f'_s.$$

SECTION 6.

(P), (Q) denoting any two linear operators whatever, we have

$$(P)(Q) = (PQ)+(P \dagger Q),$$

and comparing this with the symmetric function relation

$$\Sigma\alpha^l\,\Sigma\alpha^m = \Sigma\alpha^l\beta^m + \Sigma\alpha^{l+m},$$

or, in the notation of partitions,

$$(l)(m) = (lm)+(l+m),$$

we see that, regarding the symbol † as expressing a symbolic addition, the linear operators (P), (Q) combine according to precisely the same law as single partition symmetric functions; the algebra of the operators is not, however, commutative, and we may in the first instance regard it as the algebra of symmetric functions freed from the restriction of being commutative in the two respects of outer multiplication and addition.

As regards three linear operators

$$(u_1),\ (u_2),\ (u_3),$$

we have the theorems

$$\begin{aligned}(u_1)(u_2)(u_3) &= (u_1u_2u_3)+(u_1 \dagger u_2,\ u_3)+(u_1 \dagger u_2 \dagger u_3)^{*}\\ &\quad +(u_1 \dagger u_3,\ u_2)\\ &\quad +(u_2 \dagger u_3,\ u_1),\\ (u_1)\,(u_2u_3) &= (u_1u_2u_3)+(u_1 \dagger u_2,\ u_3)\\ &\quad +(u_1 \dagger u_3,\ u_2),\\ (u_1u_2)\,(u_3) &= (u_1u_2u_3)+(u_1 \dagger u_3,\ u_2)+(u_1u_2 \dagger u_3)\\ &\quad +(u_2 \dagger u_3,\ u_1),\end{aligned}$$

wherein, in the expansion of $(u_1u_2)(u_3)$, the operator $(u_1u_2 \dagger u_3)$ is formed by multiplying (u_1) and (u_2) symbolically, and adding the result symbolically to (u_3).

It will be observed that, *quâ* the symbol †, the suffixes are in numerical order.

Comparing these with the corresponding relations in symmetric functions, we observe perfect coincidence of theory, except in the case of the term $(u_1u_2 \dagger u_3)$.

But, if (u_3) be lineo-linear, this operator vanishes, and there is no longer any exception.

In general, an exception occurs whenever an operator is formed by the explicit operation of a symbolic product of linear operators upon a linear operator.

Any outer multiplication of operators, each of which is either a single linear operator or a symbolic product (inner multiplication) of linear operators, may in general be expanded in a series of symbolic products, each component of which is a linear operator.

* Observe that $(u_1 \dagger u_2 \dagger u_3)$ means $u_1 \dagger (u_2 \dagger u_3)$ and not $(u_1 \dagger u_2) \dagger u_3$, and that u_1, u_2, u_3 are associative as regards outer multiplication.

Restricting ourselves, in the first place, to outer multiplications of two operators, we may calculate the set of relations

$$(u_1)(u_2) = (u_1u_2) + (u_1 \dagger u_2),$$

$$(u_1)(u_2u_3) = (u_1u_2u_3) + (u_1 \dagger u_2,\ u_3) + (u_1 \dagger u_3,\ u_2),$$

$$(u_1u_2)(u_3) = (u_1u_2u_3) + (u_1 \dagger u_3,\ u_2) + (u_1u_2 \dagger u_3) + (u_2 \dagger u_3,\ u_1),$$

$$(u_1)(u_2u_3u_4) = (u_1u_2u_3u_4) + (u_1 \dagger u_2,\ u_3u_4) + (u_1 \dagger u_3,\ u_2u_4) + (u_1 \dagger u_4,\ u_2u_3),$$

$$\begin{aligned}(u_1u_2)\,(u_3u_4) = {} & (u_1u_2u_3u_4) + (u_1 \dagger u_3,\ u_2u_4) + (u_1 \dagger u_3,\ u_2 \dagger u_4) + (u_1u_2 \dagger u_3,\ u_4) \\ & + (u_1 \dagger u_4,\ u_2u_3) + (u_1 \dagger u_4,\ u_2 \dagger u_3) + (u_1u_2 \dagger u_4,\ u_3) \\ & + (u_2 \dagger u_3,\ u_1u_4) \\ & + (u_2 \dagger u_4,\ u_1u_3),\end{aligned}$$

$$\begin{aligned}(u_1u_2u_3)(u_4) = {} & (u_1u_2u_3u_4) + (u_1 \dagger u_4,\ u_2u_3) + (u_1u_2 \dagger u_4,\ u_3) + (u_1u_2u_3 \dagger u_4) \\ & + (u_2 \dagger u_4,\ u_1u_3) + (u_1u_3 \dagger u_4,\ u_2) \\ & + (u_3 \dagger u_4,\ u_1u_2) + (u_2u_3 \dagger u_4,\ u_1),\end{aligned}$$

$$\begin{aligned}(u_1)(u_2u_3u_4u_5) = {} & (u_1u_2u_3u_4u_5) + (u_1 \dagger u_2,\ u_3u_4u_5) \\ & + (u_1 \dagger u_3,\ u_2u_4u_5) \\ & + (u_1 \dagger u_4,\ u_2u_3u_5) \\ & + (u_1 \dagger u_5,\ u_2u_3u_4),\end{aligned}$$

$$\begin{aligned}(u_1u_2)\,(u_3u_4u_5) = {} & (u_1u_2u_3u_4u_5) + (u_1 \dagger u_3,\ u_2u_4u_5) + (u_1 \dagger u_3,\ u_2 \dagger u_4,\ u_5) + (u_1u_2 \dagger u_3,\ u_4u_5) \\ & + (u_1 \dagger u_4,\ u_2u_3u_5) + (u_1 \dagger u_4,\ u_2 \dagger u_5,\ u_3) + (u_1u_2 \dagger u_4,\ u_3u_5) \\ & + (u_1 \dagger u_5,\ u_2u_3u_4) + (u_1 \dagger u_5,\ u_2 \dagger u_3,\ u_4) + (u_1u_2 \dagger u_5,\ u_3u_4) \\ & + (u_2 \dagger u_3,\ u_1u_4u_5) + (u_1 \dagger u_4,\ u_2 \dagger u_3,\ u_5) \\ & + (u_2 \dagger u_4,\ u_1u_3u_5) + (u_1 \dagger u_5,\ u_2 \dagger u_4,\ u_3) \\ & + (u_2 \dagger u_5,\ u_1u_3u_4) + (u_1 \dagger u_3,\ u_2 \dagger u_5,\ u_4),\end{aligned}$$

$$
\begin{aligned}
(u_1u_2u_3)(u_4u_5) = (u_1u_2u_3u_4u_5) &+ (u_1 \dagger u_4,\ u_2u_3u_5) + (u_1u_2 \dagger u_4,\ u_3u_5) \\
&+ (u_1 \dagger u_5,\ u_2u_3u_4) + (u_1u_2 \dagger u_5,\ u_3u_4) \\
&+ (u_2 \dagger u_4,\ u_1u_3u_5) + (u_1u_3 \dagger u_4,\ u_2u_5) \\
&+ (u_2 \dagger u_5,\ u_1u_3u_4) + (u_1u_3 \dagger u_5,\ u_2u_4) \\
&+ (u_3 \dagger u_4,\ u_1u_2u_5) + (u_2u_3 \dagger u_4,\ u_1u_5) \\
&+ (u_3 \dagger u_5,\ u_1u_2u_4) + (u_2u_3 \dagger u_5,\ u_1u_4),
\end{aligned}
$$

$$
\begin{aligned}
&+ (u_1 \dagger u_4,\ u_2 \dagger u_5,\ u_3) + (u_1u_2 \dagger u_4,\ u_3 \dagger u_5) + (u_1u_2u_3 \dagger u_4,\ u_5) \\
&+ (u_1 \dagger u_5,\ u_2 \dagger u_4,\ u_3) + (u_1u_2 \dagger u_5,\ u_3 \dagger u_4) + (u_1u_2u_3 \dagger u_5,\ u_4) \\
&+ (u_1 \dagger u_4,\ u_3 \dagger u_5,\ u_2) + (u_1u_3 \dagger u_4,\ u_2 \dagger u_5) \\
&+ (u_1 \dagger u_5,\ u_3 \dagger u_4,\ u_2) + (u_1u_3 \dagger u_5,\ u_2 \dagger u_4) \\
&+ (u_2 \dagger u_4,\ u_3 \dagger u_5,\ u_1) + (u_2u_3 \dagger u_4,\ u_1 \dagger u_5) \\
&+ (u_2 \dagger u_5,\ u_3 \dagger u_4,\ u_1) + (u_2u_3 \dagger u_5,\ u_1 \dagger u_4),
\end{aligned}
$$

$$
\begin{aligned}
(u_1u_2u_3u_4)(u_5) &= (u_1u_2u_3u_4u_5) \\
&+ (u_1 \dagger u_5,\ u_2u_3u_4) + (u_1u_2 \dagger u_5,\ u_3u_4) + (u_1u_2u_3 \dagger u_5,\ u_4) + (u_1u_2u_3u_4 \dagger u_5) \\
&+ (u_2 \dagger u_5,\ u_1u_3u_4) + (u_1u_3 \dagger u_5,\ u_2u_4) + (u_1u_2u_4 \dagger u_5,\ u_3) \\
&+ (u_3 \dagger u_5,\ u_1u_2u_4) + (u_1u_4 \dagger u_5,\ u_2u_3) + (u_1u_3u_4 \dagger u_5,\ u_2) \\
&+ (u_4 \dagger u_5,\ u_1u_2u_3) + (u_2u_3 \dagger u_5,\ u_1u_4) + (u_2u_3u_4 \dagger u_5,\ u_1) \\
&+ (u_2u_4 \dagger u_5,\ u_1u_3) \\
&+ (u_3u_4 \dagger u_5,\ u_1u_2).
\end{aligned}
$$

In these expansions it will be observed that explicit operation only takes place upon a single linear operator.

It is easy to see that the outer multiplication of two operators, each of which is a symbolic product of linear operators, may be always so expanded.

Considering, in general, the product

$$(u_1u_2u_3 \dots u_m)(v_1v_2v_3 \dots v_n),$$

there will arise a batch of operators corresponding to every partition of m, and every lower number, into n or fewer parts.

If, for instance, we fix the attention upon the batches corresponding to the partitions of p ($p \overline{\overline{<}}\, m$) into s ($s \overline{\overline{<}}\, n$) parts, we see that the total number of operators which occur in these batches depends, firstly, upon the number of ways in which it is possible to pack up p things in exactly s parcels; and secondly, upon the number of ways in which s out of n things can be distributed amongst these parcels, one in each parcel.

The number of ways of choosing p out of m things is

$$\frac{m!}{p!\,(m-p)!},$$

and p things can be distributed into s parcels in a number of ways denoted by

$$\frac{1}{s!}\,\Delta^s(0^p),$$

in the notation of the calculus of finite differences.

Further, we can distribute s out of n things amongst these s parcels, one in each parcel in

$$\frac{n!}{(n-s)!}\text{ ways.}$$

Consequently, in the batches corresponding to the two numbers p and s, there will be a number of operators equal to

$$\frac{m!}{p!\,(m-p)!}\,\frac{1}{s!}\,\Delta^s(0^p)\,\frac{n!}{(n-s)!},$$

and in the aggregate of batches corresponding to the number p there will be

$$\frac{m!}{p!\,(m-p)!}\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p)$$

operators.

Giving p all values from 0 to m, we shall obtain the complete number of operators which appear in the expansion; this number thus is

$$\sum_{p=0}^{p=m}\frac{m!}{p!\,(m-p)!}\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p).$$

It will now be shown that this expression has the value

$$(n+1)^m$$

First consider the summation

$$\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p),$$

and write

$$\frac{n!}{(n-s)!}=A_n^s$$

$$\frac{1}{s!}\,\Delta\,(0^p)=K_p$$

(see M. Maurice d'Ocagne "Sur une Classe de Nombres remarquables," *American Journal of Mathematics*, Vol. IX., No. 4, p. 366);

and then $$\sum_{s=1}^{s=n} \frac{n!}{s!\,(n-s)!} \Delta^s (0^p) = \Sigma A_n^s K_p^s$$

$$= A_n^1 K_p^1 + A_n^2 K_p^2 + A_n^3 K_p^3 + \dots A_n^n K_p^n$$

$$= (1 + A_{n-1}^1) K_p^1 + (2A_{n-1}^1 + A_{n-1}^2) K_p^2 + (3A_{n-1}^2 + A_{n-1}^3) K_p^3 + \dots + (nA_{n-1}^{n-1}) K_p^n$$

$$= K_p^1 + A_{n-1}^1 (2K_p^2 + K_p^1) + A_{n-1}^2 (3K_p^3 + K_p^2) + \dots + A_{n-1}^{n-1} (nK_p^n + K_p^{n-1})$$

$$= K_{p+1}^1 + A_{n-1}^1 K_{p+1}^2 + A_{n-1}^2 K_{p+1}^3 + \dots + A_{n-1}^{n-1} K_{p+1}^n$$

$= n^p$ (*loc. cit.*).

Consequently the number we are in search of is

$$\sum_{p=0}^{p=m} \frac{m!}{p!\,(m-p)!} n^p$$

$$= (n+1)^m.$$

The theorem may be stated as follows:—

"The outer multiplication of two operators, the sinister and dexter being symbolic products of m and n linear operators respectively, may be expressed as a sum of $(n+1)^m$ operators, each of which is a symbolic product of linear operators."

In the case of the dexter being formed wholly of lineo-linear operators the theory is identical with that of the algebraic theory of symmetric functions.

Section 7.

Symbolic Addition of Operators.

Denoting by

$$u_1 u_2 u_3 \dots u_m, \quad v_1 v_2 v_3 \dots v_m,$$

operators of the m^{th} and n^{th} orders obtained by the symbolic multiplication of the linear operators $u_1, u_2 \dots, v_1, v_2 \dots$, we require the expansion of the operator $(u_1 u_2 \dots u_m \dagger v_1 v_2 \dots v_n)$ as a linear function of operators, each of which is a symbolic product of linear operators.

It will be shown that the number of operators occurring in the development is precisely n^m.

Consider a simple case of Leibnitz's theorem, viz., the continued performance of a single linear partial differential operation upon a product of two functions ϕ_1, ϕ_2.

If $(u)^m$ designates m successive operations of u, we have

$$\frac{(u)^m}{m!}\phi_1\phi_2 = \Sigma\frac{(u)^s\phi_1(u)^{m-s}\phi_2}{s!\,(m-s)!}.$$

It is to be proved that a perfectly valid theorem is obtained if herein we write (u^s) in place of $(u)^s$, where, as usual in this paper, (u^s) denotes the operator of the s^{th} order reached by raising u symbolically to the power s.

In fact, the theorem to be proved is

$$\frac{(u^m)}{m!}\phi_1\phi_2 = \Sigma\frac{(u^s)\,\phi_1\,(u^{m-s})\,\phi_2}{s!\,(m-s)!}.$$

In general, the most extended form of Leibnitz's theorem is capable of a similar dual interpretation, which may be established in the following manner:—

Suppose

$u_1, u_2, u_3, \ldots u_s$ to be any linear operators whatever,

and put

$$\Theta = \Theta_0\partial_a+\Theta_1\partial_b+\Theta_2\partial_c+\ldots = \lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\ldots+\lambda_su_s,$$

further let $\phi_1, \phi_2, \ldots \phi_m$ be any m functions of $a, b, c, d, \ldots$,

and put $\phi = \phi_1\phi_2\phi_3\ldots\phi_m$;

then

$$\phi_t(a+\Theta_0, b+\Theta_1, c+\Theta_2, \ldots) = \phi_t+\Theta\phi_t+\frac{(\Theta^2)}{2!}\phi_t+\frac{(\Theta^3)}{3!}\phi_t+\ldots,$$

and

$$\phi+\Theta\phi+\frac{(\Theta^2)}{2!}\phi+\frac{(\Theta^3)}{3!}\phi+\ldots = \prod_{t=1}^{t=m}\left(\phi_t+\Theta\phi_t+\frac{(\Theta^2)}{2!}\phi_t+\frac{(\Theta^3)}{3!}\phi_t+\ldots\right),$$

that is,

$$\begin{aligned}\phi+(\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\ldots+\lambda_su_s)\,\phi\qquad&\\ +\frac{1}{2!}(\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\ldots+\lambda_su_s)^2\,\phi+\ldots&\\ =\prod_{t=1}^{t=m}\Big\{\phi_t+(\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\ldots+\lambda_su_s)\,\phi_t\qquad&\\ +\frac{1}{2!}(\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\ldots+\lambda_su_s)^2\phi_t+\ldots\Big\}&.\end{aligned}$$

We now compare the coefficients of

$$\lambda_1^{\chi_1}\lambda_2^{\chi_2}\ldots\lambda_s^{\chi_s}$$

on the two sides of this identity, and obtain a result which may be written in the form :—

$$\frac{(u_1^{\chi_1} u_2^{\chi_2} \dots u_s^{\chi_s})\,\phi}{\chi_1!\,\chi_2!\dots\chi_s!\,\phi} = \Sigma\Sigma\,\frac{(u_1^{a_1} u_2^{a_2}\dots u_s^{a_s})\,\phi_1}{a_1!\,a_2!\dots a_s!\,\phi_1}\;\frac{(u_1^{\beta_1} u_2^{\beta_2}\dots u_s^{\beta_s})\,\phi_2}{\beta_1!\,\beta_2!\dots\beta_s!\,\phi_2}$$

$$\dots\frac{(u_1^{\mu_1} u_2^{\mu_2}\dots u_s^{\mu_s})\,\phi_m}{\mu_1!\,\mu_2!\dots\mu_s!\,\phi_m},$$

where $a_t+\beta_t+\gamma_t+\dots+\mu_t = \chi_t, \quad (t = 1, 2, \dots s),$

and the double summation is in regard to every positive integral solution of these s equations and to every permutation of the ϕ's.

When we compare this result with that of Leibnitz, viz. :—

$$\frac{(u_1)^{\chi_1}(u_2)^{\chi_2}\dots(u_s)^{\chi_s}\,\phi}{\chi_1!\,\chi_2!\dots\chi_s!\,\phi} = \Sigma\Sigma\,\frac{(u_1)^{a_1}(u_2)^{a_2}\dots(u_s)^{a_s}\,\phi_1}{a_1!\,a_2!\dots a_s!\,\phi_1}\cdot\frac{(u_1)^{\beta_1}(u_2)^{\beta_2}\dots(u_s)^{\beta_s}\,\phi_2}{\beta_1!\,\beta_2!\dots\beta_s!\,\phi_2}$$

$$\dots\frac{(u_1)^{\mu_1}(u_2)^{\mu_2}\dots(u_s)^{\mu_s}\,\phi_m}{\mu_1!\,\mu_2!\dots\mu_s!\,\phi_m},$$

we establish its dual character.

Comparing either of these formulæ with the ordinary multinomial theorem, we see at once that the number of terms in the development is

$$m^{\Sigma\chi}.$$

Applying the theorem to the case of symbolic addition, we find in particular

$$(u_1u_2 \dagger v_1v_2) = (v_1, u_1u_2 \dagger v_2) + (u_1 \dagger v_1, u_2 \dagger v_2)$$
$$+ (v_2, u_1u_2 \dagger v_1) + (u_1 \dagger v_2, u_2 \dagger v_1);$$

wherein, be it remembered, the operator

$$(v_1, u_1u_2 \dagger v_2)$$

is formed—

(i.) By multiplying u_1 and u_2 together symbolically.

(ii.) By then operating with this symbolic product upon v_2, considered as a function of the symbols of quantity only and not of the differential inverses.

(iii.) By finally multiplying the last result by v_1 symbolically.

Also

$$(u_1u_2u_3 \dagger v_1v_2) = (v_1,\, u_1u_2u_3 \dagger v_2) + (u_1 \dagger v_1,\, u_2u_3 \dagger v_2)$$
$$+ (v_2,\, u_1u_2u_3 \dagger v_1) + (u_1 \dagger v_2,\, u_2u_3 \dagger v_1)$$
$$+ (u_2 \dagger v_1,\, u_1u_3 \dagger v_2)$$
$$+ (u_2 \dagger v_2,\, u_1u_3 \dagger v_1)$$
$$+ (u_3 \dagger v_1,\, u_1u_2 \dagger v_2)$$
$$+ (u_3 \dagger v_2,\, u_1u_2 \dagger v_1).$$

In general, in writing down the expansion of

$$(u_1u_2 \dots u_m \dagger v_1v_2 \dots v_n),$$

we shall obtain a batch of operators corresponding to every partition of m into n or fewer parts, and, as before remarked, the total number of operators is n^m.

SECTION 8.

A very important example of the symbolic interpretation of Leibnitz's formula occurs in the theory of symmetric functions.

In the result

$$\frac{(u^m)}{m!}\,\phi_1\phi_2 \dots \phi_n = \Sigma\, \frac{(u^{s_1})\,\phi_1\,(u^{s_2})\,\phi_2 \dots (u^{s_{n-1}})\,\phi_{n-1}\,(u^{m-\Sigma s})\,\phi_n}{s!\; s! \dots s_{n-1}!\,(m-\Sigma s)!},$$

put
$$D_t = \frac{(u^t)}{t!} = \frac{1}{t!}\,(a\partial_b + b\partial_c + c\partial_d + \dots)^t,$$

thus obtaining

$$D_m\phi_1\phi_2 \dots \phi_n = \Sigma\, D_{s_1}\phi_1 D_{s_2}\phi_2 \dots D_{s_{n-1}}\phi_{n-1} D_{m-\Sigma s}\phi_n.$$

Supposing $\phi_1, \phi_2, \dots \phi_n$ to be symmetric functions of the roots of the equation

$$ax^n - bx^{n-1} + cx^{n-2} - \dots = 0,$$

expressed by means of partitions, the effect of operating with D_t upon any partition containing a symbolic number t, is to take away one such number t; further

$$D_t\,(t) = 1,$$

and any partition, not containing a number t, is obliterated.

Hence the operation of D_m upon a compound symmetric function is performed by picking out the different partitions of m in all possible ways from the partitions of $\phi_1, \phi_2, \phi_3, \dots \phi_n$, one part only at a time from each partition.

Thus $$\begin{aligned} D_6(541)(321)(21) = &\ (*\,41)(32\,*)(21) \\ &+(*\,41)(321)(2\,*) \\ &+(5\,*\,1)(3\,*\,1)(21) \\ &+(5\,*\,1)(321)(*\,1) \\ &+(5\,*\,1)(34\,*)(2\,*) \\ &+(54\,*)(*\,21)(*\,1), \end{aligned}$$

where the asterisks denote the partitions of 6, successively picked out.

The result then is

$$\begin{aligned} D_6(541)(321)(21) = &\ (41)(32)(21)+(41)(321)(2) \\ &+(51)(31)(21)+(51)(321)(1) \\ &+(51)(32)(2)+(54)(21)(1). \end{aligned}$$

From any symmetric function identity, we can, by repeating operations similar to the above, derive a number of other identities.

In particular, in the theory of invariants, we can from any syzygy between covariants derive a number of lower syzygies.

The operation of "decapitation," whether of a single or compound symmetric function of a degree θ, is seen to be merely the performance as above of the operation D_θ.

Also $$\begin{aligned} & D_\theta\, b^\beta c^\gamma d^\delta \ldots, \qquad (\beta+\gamma+\delta+\ldots=\theta), \\ &= D_\theta\,(1)^\beta\,(1^2)^\gamma\,(1^3)^\delta \ldots, \\ &= a^\beta\,(1)^\gamma\,(1^2)^\delta \ldots, \\ &= a^\beta b^\gamma c^\delta \ldots, \end{aligned}$$

showing that a symmetric function of degree θ belonging to the equation

$$bx^n - cx^{n-1} + dx^{n-2} - \ldots = 0$$

is transformed by the operation of D_θ into one appertaining to the equation

$$ax^n - bx^{n-1} + cx^{n-2} - \ldots = 0.$$