after reading the book his opinion was modified. He thought Professor Geddes had strengthened the case for Grote's theory, though not to the extent of proving an affirmative.

Dr Donaldson concurred with the views of Professor Blackie.

Professor Sellar, while having the greatest admiration for Professor Geddes's knowledge and ingenuity, felt that he had done nothing in the way of enlightening those interested in the subject.

Professor Blackie, in replying, observed that the Society were under obligations to Professor Geddes for having raised the question.

2. The Principles of the Algebra of Logic. Part III.—Application to certain Problems in the Theory of Probability. By Dr Alexander Macfarlane.

The Algebra of Logic, being the science of Necessity and Probability, supplies a variety of methods of great power for solving problems in the theory of Probability. I propose to bring before the Society a few examples extracted from my work on the "Principles of the Algebra of Logic," which is about to be published. A large class of problems, some of which have created considerable diversity of opinion among mathematicians of eminence, can be solved by means of a single theorem. It consists in finding the arithmetical value of $\frac{xy}{x}$. The meaning of this expression is shown by the diagram—

The collection of individuals forming the subject of discourse is represented by the part of the page within the square, those which have a character x by the part inside the one circular line, and those which have the character y by the part inside the other

circular line; those having both x and y by the part inside both lines, those having neither by the part outside.

$$\text{Now} \quad xy = xy \quad \therefore y = \frac{xy}{x}.$$

Expand $\frac{xy}{x}$ in terms of the parts formed by means of xy and x .

$$\frac{xy}{x} = a xy x + b xy (1 - x) + c(1 - xy)x + d(1 - xy)(1 - x).$$

$$\text{Let } xy = 1 \text{ and } x = 1. \quad \text{Then, } a = \frac{1}{1} = 1.$$

$$xy = 1 \text{ and } x = 0, \text{ then } b = \frac{1}{0}.$$

$$xy = 0 \text{ and } x = 1, \text{ then } c = \frac{0}{1} = 0.$$

$$xy = 0 \text{ and } x = 0, \text{ then } d = \frac{0}{0}.$$

$$\therefore \frac{xy}{x} = xy x + 0(1 - xy)x + \frac{0}{0}(1 - xy)(1 - x),$$

and also $xy(1 - x) = 0$, as evidently ought to be the case.

By putting in

$$x^2 = x, \text{ or } x(1 - x) = 0,$$

$$\text{we get} \quad \frac{xy}{x} = xy + 0 x(1 - y) + \frac{0}{0}(\overline{1 - x}),$$

that is, what is y is identical with what is x and y , together with no part of what is x and not y , together with an indefinite part of what is not x . The truth of this is evidenced by the diagram. Since every logical equation is true arithmetically,

$$\overline{y} = \overline{xy} + \frac{0}{0}(1 - x);$$

where the bar denotes that the arithmetical value of the symbol is taken.

Applications of the above Theorem.

(1.) The probability that it thunders upon a given day is p , the probability that it both thunders and hails is q , but of the connection of the two phenomena of thunder and hail, nothing further is supposed to be known. Required the probability that it hails on the proposed day.

Let U = a succession of states of the atmosphere at a given place, an individual state being of the length of a day.

x = containing a thunderstorm,

y = containing a hailstorm.

Then the data are—

$$x = \bar{p}, \quad xy = \bar{q}.$$

Hence, by means of the theorem proved

$$\bar{y} = \bar{q} + \frac{0}{0}(\bar{1} - \bar{p}),$$

$$\therefore \bar{y} > \bar{q},$$

$$\text{and } < \bar{q} + \bar{1} - p.$$

(2.) A says that B says that a certain event took place; required the probability that the event did take place, p_1 and p_2 being A's and B's respective probabilities of speaking the truth.

The solution of this problem recently gave rise to a great amount of discussion in the "Educational Times." No fewer than four different solutions are given, viz. :—

Todhunter—

$$p_1 p_2 + (1 - p_1)(1 - p_2).$$

Artemas Martin—

$$p_1 \{ p_1 p_2 + (1 - p_1)(1 - p_2) \}.$$

Woolhouse and American mathematicians—

$$p_1 p_2.$$

Cayley—

$$p_1 p_2 + \beta(1 - p_1)(1 - p_2) + \kappa(1 - \beta)(1 - p_1),$$

where β is the chance, on the supposition of the incorrectness of A's statement, that B told A that the event did *not* happen, and $1 - \beta$ that he did not tell him at all. κ is the antecedent probability.

Let U = statements of A about B's statements about an event taking place.

x = which truly reported a statement by B,

y = which truly reported the event.

Cor. 3. Suppose that A_n always speaks falsely, then

$$x_n = \frac{0}{0} (\bar{1} - \bar{p}_1 \bar{p}_2 \dots \bar{p}_{n-1}).$$

Cor. 4. Suppose that any other than A_n always speaks falsely, then

$$x_n = \frac{0}{0};$$

that is, the probability is quite indefinite.

Cor. 5. Suppose that each $\bar{p} = \frac{1}{2}$, then,

$$x_n = \left(\frac{1}{2}\right)^n + \frac{0}{0} \left(1 - \left(\frac{1}{2}\right)^{n-1}\right);$$

which, when n is infinite is equal to $\frac{0}{0}$, that is, is perfectly indefinite.

(4.) A goes to hall p times in q consecutive days and sees B there r times. What is the most probable number of times that B was in the hall in the q days?—*Whitworth's Choice and Chance.*

U = the consecutive days ;

x = on which A goes to hall ;

y = on which B goes to hall.

The data are—

$$U = \bar{q};$$

$$Ux = \bar{p};$$

$$Uxy = \bar{r};$$

$\therefore$ by means of the theorem,

$$Uy = r + \frac{0}{0} (\bar{q} - \bar{p}).$$

To find the most probable value, make the assumption of independence, that is, that B is as likely to go to hall when A does not go, as when A does go.

Then the most probable value of Uy is

$$r + \frac{r}{p}(q - p) = \frac{rq}{p}.$$

Whitworth gives $\frac{(q+1)r}{p}$, or next lower integer.

Cor. Let $r = p = q$. Then $Uy = q$.

Problems discussed by Boole in the "Laws of Thought."

1. The probability that one or both of two events happen is $\bar{p}$, that one or both of them fail is $\bar{q}$. What is the probability that only one of these happens?

$$xy + x(1 - y) + (1 - x)y = \bar{p},$$

$$x(1 - y) + (1 - x)y + (1 - x)(1 - y) = \bar{q},$$

it is required to find

$$x(1 - y) + y(1 - x).$$

Let

$$x(1 - y) + y(1 - x) = a + b\{xy + x(1 - y) + (1 - x)y\} \\ + c\{x(1 - y) + (1 - x)y + (1 - x)(1 - y)\}.$$

$$\text{Let } x = 1 \quad y = 1 \text{ then } 0 = a + b$$

$$,, \quad x = 1 \quad y = 0 \quad ,, \quad 1 = a + b + c$$

$$,, \quad x = 0 \quad y = 1 \quad ,, \quad 1 = a + b + c$$

$$,, \quad x = 0 \quad y = 0 \quad ,, \quad 0 = a + c.$$

We have four equations, but two of them are identical. When solved—

$$a = -1, \quad b = 1, \quad c = 1,$$

$$\therefore x(1 - y) + y(1 - x) = -1 + p + \bar{q}.$$

This method by indeterminate coefficients serves to indicate whether a problem is determinate. For example, investigate the first problem by its means—

$$y = a + bx + cxy.$$

Then

$$1 = a + b + c,$$

$$0 = a + b,$$

$$\left. \begin{array}{l} 1 = a \\ 0 = a \end{array} \right\} ;$$

$$\therefore a = \frac{0}{0}, \quad b = -\frac{0}{0}, \quad c = 1.$$

$$\therefore y = \frac{0}{0} - \frac{0}{0}x + xy$$

$$= xy + \frac{0}{0}(1 - x).$$

2. The probabilities of two causes A_1 and A_2 are $\bar{a}$ and $\bar{b}$ respec-

tively. The probability that if the cause A_1 present itself, an event E will accompany it (whether as a consequence of the cause A_1 or not) is p_1 , and the probability that if the cause A_2 present itself, that event E will accompany it, whether as a consequence of it or not, is q . Moreover, the event E cannot appear in the absence of both the causes, A_1 and A_2 . Required the probability of the event E .

The data are—

$$x = \bar{a}, \quad y = \bar{b}, \quad xz = ap, \quad yz = \bar{b}\bar{q},$$

and

$$(1 - x)(1 - y)z = 0,$$

and $\bar{z}$ is required.

$$\text{Now } (1 - x)(1 - y)z = z - xz - yz + xyz,$$

$$\therefore \quad z = xz + yz - xyz$$

by the last datum ;

$$\therefore \quad z < xz + yz - x - yz + 1 \quad . \quad . \quad . \quad (1.)$$

$$< xz + yz - y - xz + 1 \quad . \quad . \quad . \quad (2.)$$

$$< xz + yz \quad . \quad . \quad . \quad (3.)$$

$$\therefore \quad z < \bar{1} - \bar{a} + \bar{a}p. \quad . \quad . \quad . \quad (1.)$$

$$< \bar{1} - \bar{b} + \bar{b}\bar{q}. \quad . \quad . \quad . \quad (2.)$$

$$< \bar{a}\bar{p} + \bar{b}\bar{q}. \quad . \quad . \quad . \quad (3.)$$

Also

$$(1 - x)yz = yz - xyz,$$

$$= yz + z - xz - yz$$

by the last datum ;

$$\therefore \quad z = xz + (1 - x)yz ;$$

$$\therefore \quad z > \bar{a}\bar{p}.$$

Also

$$z > \bar{b}\bar{q}.$$

This problem was discussed in the “*Philosophical Magazine*,” by Boole, Wilbraham, and Cayley. Cayley’s solution is different, applying to a modification of the problem. Boole goes further, and finds the most probable value of the probability. Wilbraham considers only mathematical probability, and maintains, quite rightly, that we cannot proceed further than above without making assumptions. He says that the disadvantage of Boole’s method is, that it does not show whether a problem is determinate. This desideratum is supplied by the method of indeterminate coefficients to which I have referred above.