Then the Boothian circle $\frac{c}{p} = 1 + m\xi + n\eta$ (ξ being the common radical axis) is transformed into $\frac{c}{p} = -1 - m\xi \pm n\eta$, and the common tangent of two circles is unaltered in length by Dual Inversion.]

Reciprocation in Statics. By Prof. GENESE, M.A.

[Read June 10th, 1886.]

If a system of complanar forces $P_1, P_2, \&c.$, be in equilibrium, and any transversal meet their lines of action in $I_1, I_2, \&c.$, at angles $\theta_1, \theta_2, \&c.$ (estimated with due regard to the directions of the forces and one direction of the transversal), then, resolving perpendicularly to the transversal, we have a system of parallel forces $P_1 \sin \theta_1, P_2 \sin \theta_2, \&c.$, at $I_1, I_2, \&c.$, in equilibrium.

The opposite to one of the above forces is defined as the resultant of the others. The cases of four and five forces led to the theorems in the right-hand column of the following

CONTRAST.

1. If I be the centre of parallel forces l, m, n at the points A, B, C , then forces lIA, mIB, nIC are in equilibrium. (Leibnitz.)

1. If δ be the line of action of the resultant of forces l, m, n , acting along the sides a, b, c , then parallel forces $l \sin (\delta a), m \sin (\delta b), n \sin (\delta c)$ are in equilibrium.

2. Therefore, if x, y, z denote the connectors IA, IB, IC ,

$$\frac{lIA}{\sin(yz)} = \frac{mIB}{\sin(zx)} = \frac{nIC}{\sin(xy)}.$$

3. If I' be a second point, the centre for forces l', m', n' ; the resultant of forces $l'IA, m'IB, n'IC$ acts along $I'I$, and

$$= (l+m+n) I'I.$$

Similarly for forces $l'IA, m'IB, n'IC$.

2. Therefore, if X, Y, Z denote the intersections $\delta a, \delta b, \delta c$,

$$\frac{l \sin(\delta a)}{YZ} = \frac{m \sin(\delta b)}{ZX} = \frac{n \sin(\delta c)}{XY}.$$

3. If δ' be a second line for forces l', m', n' ; the centre for parallel forces $l \sin(\delta a), m \sin(\delta b),$ &c., is $\delta\delta'$, and resultant

$$= \{l, m, n\} \sin(\delta\delta'),$$

where $\{l, m, n\}$ = resultant of l, m, n . Similarly for parallel forces $l' \sin(\delta a), m' \sin(\delta b), n' \sin(\delta c)$.

4. The above six forces are in equilibrium—

$$\text{if } l+m+n = l'+m'+n', \quad | \quad \text{if } \{l, m, n\} = \{l', m', n'\}.$$

These conditions are secured—

if l, m, n be the areal coordinates of I ,

if l, m, n be the quantities ap, bq, cr , where p, q, r are the distances of δ from vertices of abc (for the resultant of these

$$= \sqrt{a^2 p^2 + \dots - 2bcqr \cos A - \&c.} \\ = 2\Delta),$$

l', m', n' those of I' .

l', m', n' the quantities for δ' .

5. If II' meet BC at U ,

$$l : l' :: (UI) : (UI').$$

5. If $\delta\delta'$ connect with bc by u ,

$$l : l' :: \sin(u\delta) : \sin(u\delta').$$

Whence, taking the six forces in pairs,* and observing that the three

* Let the resultant of lIA and $l'IA$ meet II' at U' , then

$$U'I : U'I' :: \Delta U'IA : \Delta U'IA'.$$

And, by moments about U' ,

$$l \Delta U'IA = l' \Delta U'IA',$$

therefore

$$U'I : U'I' :: l : l' :: UI : UI',$$

or U' is harmonic conjugate of U with respect to II' .

resultants must meet in a point, the following theorems in Geometry may be obtained :—

If II' meet the sides of ABC in U, V, W , and U', V', W' be the harmonic conjugates of U, V, W with respect to I and I' ; then AU', BV', CW' meet in one point.

If $\delta\delta'$ be joined to the vertices of abc by u, v, w , and u', v', w' be the harmonic conjugates of u, v, w with respect to δ and δ' ; then au', bv', cw' are in one straight line.

An example of analytical correspondence is given by the following :

If parallel forces $\omega_1, \omega_2, \omega_3, \omega_4$, acting at points $(l_1 m_1 n_1)$, &c., be in equilibrium, and

If forces P_1, P_2, P_3, P_4 , acting along lines $(l_1 m_1 n_1)$, &c., be in equilibrium, and

$$| 1, 2, 3 | \text{ denote } \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

then

$$\frac{\omega_1}{| 2, 3, 4 |} = \frac{-\omega_2}{| 3, 4, 1 |} = \&c.,$$

or, $\omega_1, -\omega_2$, &c. are proportional to the areas of the corresponding triangles with their proper signs.

then

$$\frac{P_1}{| 2, 3, 4 |} = \frac{-P_2}{| 3, 4, 1 |} = \&c.,$$

or $P_1, -P_2$, &c. are proportional to quotients of the form $\frac{\Delta}{R}$, where Δ is the area of the corresponding triangle, and R the radius of its circum-circle. (See Clifford's *Mathematical Papers*, p. 90.)

These theorems suggest the existence of a principle of reciprocation in Statics. This will now be presented synthetically.

It will be convenient henceforth to replace the term "parallel force" by "weight."

Let P_1, P_2 , &c. be any system of coplanar forces in equilibrium; A_1, A_2 , &c. the poles of their lines of action with respect to any circle, centre O , radius unity. The forces P_1, P_2 , &c., translated to O , would balance, and therefore also if rotated through -90° . They will then act along OA_1, OA_2 , &c., from or towards O according to the sign of the moments of the forces about O .

Hence, by Leibnitz' theorem, weights $\frac{P_1}{OA_1}, \frac{P_2}{OA_2}$, &c., at A_1, A_2 , &c., are in equilibrium. That is, weights equal to

the moments of the forces $P_1, P_2, \&c.$, about O , placed at $A_1, A_2, \&c.$, balance. Reversing one of the forces, we have the resultant of the rest. Hence, *if weights be placed at the poles of the lines of action of any system of complanar forces with respect to a circle centre O , proportional to the moments of these forces about O (with their proper signs), the resultant force will act along the polar of the centre of the weights; and its magnitude and direction may be found from the resultant weight.*

COR. 1. If the weights balance, the forces balance.

COR. 2. If the weight centre be at O , the system of forces is equivalent to a couple, and conversely.

The following illustrates the theory.

Let $ABC \dots X$ be any closed polygon, $ALMX$ a rectangle of double the area on the same side of AB . Let forces represented by $AB, BC \dots$ act along all the sides except AX . Then AX is the magnitude of the resultant. By taking moments round A , it will be seen that LM represents completely the resultant. This is true when the polygon becomes a curve. Reciprocating, we obtain the weight-centre of the reciprocal curve properly loaded. In particular, if the curve be an arc of circle, centre O , weight-centre G , taking moments about O ,

$$\begin{aligned} LM \times \frac{r^3}{OG} &= \text{sum of moments round } O \\ &= \text{twice sector } AOX, \\ &= \text{arc } AX \times r, \end{aligned}$$

whence
$$OG = \frac{AX \times r}{\text{arc } AX}.$$

Again, forces may be projected orthogonally according to the same laws as the lines representing them. For, suppose a system of forces in equilibrium to be represented by straight lines, then their moments about any point are represented by areas. These areas project orthogonally in a constant ratio. The conditions for equilibrium are that the sums of moments round the vertices of a triangle should severally vanish; but the moments for the given system vanish, therefore they do for the projected system.

The same is true for weights. Therefore the reciprocation theorem

which has been proved for the circle will hold also for any ellipse ; whence, also, using imaginary projection, for any hyperbola.

A corresponding theorem may be derived for the parabola ; the centre being at infinity, the perpendiculars from it vary as the sines of the inclination of the forces to the axis of the parabola. Hence, if forces $P_1, P_2, \&c.$ make angles $\theta_1, \theta_2, \&c.$ with one direction of the axis of a parabola, and weights $P_1 \sin \theta_1, P_2 \sin \theta_2, \&c.$ be placed at the poles of the lines of action with respect to the curve, then the resultant force acts along the polar of the weight-centre.

The preceding theorems admit of easy verification by the methods of Cartesian coordinates.

The method of the paper has an interesting application to Kinetics. If G be the centre of inertia of a plane laminal mass m , at rest, and a force P begin to act in the plane at a distance $GL = p$ from G , it generates a linear acceleration $\dot{v}$ of the mass perpendicular to GL , and an angular acceleration $\dot{\omega}$ about G , given by

$$m\dot{v} = P \dots\dots\dots(1),$$

$$mk^2\dot{\omega} = Pp \dots\dots\dots(2),$$

where k^2 is the radius of gyration about G .

If I be the centre of instantaneous rotation, an angular velocity $\dot{\omega}dt$ must give velocity $\dot{v}dt$ to G , i.e., $IG \cdot \dot{\omega}$ must be equal to $\dot{v}$. Hence

$$\text{the known relation} \quad IG \cdot p = k^2,$$

$$\text{or} \quad GI \cdot GL = -k^2,$$

i.e., I is the pole of the line of action of P with respect to a circle, centre G , radius $k\sqrt{-1}$.

Now, simultaneous small rotations may be compounded by the same rules as parallel forces, and for different forces are proportional to the moments Pp . Hence the following theorem :—

If any number of forces begin to act in the plane of a laminal mass, and weights, proportional to their moments about the centre of mass, be placed at the poles of the lines of action of the forces with respect to a circle about the centre of radius $\sqrt{-1}$ times the radius of gyration round it ; then the centre of the weights is the centre of instantaneous rotation.†*

* Of course a real circle of radius k may be substituted, and the radius to the pole reversed.

† One of the referees points out that if the laminal mass be considered not at rest, but at any time during its motion, then the theorem gives the point which for the moment has no acceleration.

The following presents were received in the Recess :—

- "Proceedings of the Royal Society," Vol. XL., Nos. 243—246.
- "Educational Times," July, August, September, October (1886).
- "Proceedings of the Physical Society of London," Vol. VIII., Part I., July, 1886.
- "Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. XLV.
- "Greek Geometry from Thales to Euclid," by G. J. Allman ; 8vo pamphlet. ("Hermathena," Vol. VI., No. XII., 1886.)
- "Appendix to Mathematical Questions and Solutions from the 'Educational Times,' Vol. XLIV.—Solutions of some Old Questions," by Âsûtos̃h Mukhopâdhyây ; 8vo pamphlet.
- "A Note on Elliptic Functions," by Âsûtos̃h Mukhopâdhyây ; 8vo pamphlet. (*Quar. Jour. of Math.*, No. 83, 1886.)
- "Smithsonian Report for 1884," 8vo ; Washington, 1885.
- "Journal of the Franklin Institute," Vol. XCII., No. 3, Sept., 1886.
- "Proceedings of the Canadian Institute," Toronto, Vol. XXI., No. 145, 1886.
- "Johns Hopkins University Circulars," Vol. V., Nos. 49, 50, 51.
- "On the Flexure of Meridian Instruments, and the means available for eliminating its effects from star places," by Prof. W. Harkness, 4to pamphlet ; Washington, 1886.
- "Bulletin des Sciences Mathématiques," T. X. ; August, Sept., and Oct., 1886.
- "Bulletin de la Société Mathématique de France," T. XIV., Nos. 3 and 4, 1886.
- "Annales de l'École Polytechnique de Delft," Livr. 1 and 2 ; Leide, 1886.
- "Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," I. to XXXIX., 1886.
- "Atti della Reale Accademia dei Lincei," Serie Quarta—"Rendiconti," Vol. II., Fasc. 12—14 ; 2° Semestre, Comunicazioni pervenute all' Accademia sino al 4 Luglio, 1886 ; Vol. II., Fasc. 1—5.
- "Atti della R. Accademia dei Lincei—Memorie," Serie 3, 1884 ; Vols. XVIII., XIX. ; Serie 4, Vol. II.
- "Acta Mathematica," VIII., 1—4.
- "Journal für die reine und angewandte Mathematik," Bd. C., Heft 2.
- "Jahrbuch über die Fortschritte der Mathematik," Bd. XV., Jahrgang 1883, Heft 3 ; Berlin, 1886.
- "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," I.—IV., 1886.
- "Beiblätter zu den Annalen der Physik und Chemie," Band X., St. 5—9.
- "Annali di Matematica," Tome XIV., Fasc. 2, 3.
- "Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XX., Liv. 5 ; Tome XXI., Liv. 1.
- "Jornal de Sciencias Mathematicas e Astronomicas," Vol. VII., No. 1.
- "Bollettino delle Pubblicazioni Italiane," ricevute per Diritto di Stampa dalla Biblioteca Centrale di Firenze, Nos. 11—18 ; Firenze, 1886.
- "Bollettino delle Opere Moderne Straniere," acquistate dalle Biblioteche pubbliche Governative del Regno d'Italia, Nos. 1, 2, 3 ; Roma, 1886.
- "Liste Alphabétique de la Correspondance de Christiaan Huygens qui sera publiée par la Société Hollandaise des Sciences à Harlem" ; 4to, Harlem.
- A number of pamphlets by M. Émile Lemoine.
- A number of pamphlets by M. Maurice D'Ocagno.
- "A History of the Theory of Elasticity and of the Strength of Materials from

Galilei to the Present Time," by the late Isaac Todhunter, edited and completed for the Syndics of the University Press, by Karl Pearson. Galilei to Saint Venant, 1639—1850, Vol. I., pp. x., xvi., 924; Cambridge, 1886. From the Editor.

"Sur les Surfaces Anallagmatiques," par J. Neuberg (Association Française pour l'Avancement des Sciences, Congrès de Grenoble, 1885.)

"Sur quelques Systèmes de Tiges Articulées, tracé mécanique des Lignes," par J. Neuberg (Université de Liège, Association des élèves des écoles spéciales), 25 Av., 1885, et 21 Mars, 1886; Liège, 1886.

"Sur le Point de Steiner," par J. Neuberg (Journal de Mathématiques Spéciales); Paris, 1886.

"American Journal of Mathematics," Vol. VIII., Nos. 3, 4; Baltimore, 1886.

APPENDIX.

Mr. Basset's paper (p. 122) is published in the *Philosophical Magazine* for August, 1886 (Vol. XXII., No. 135, pp. 140—144).

Prof. Neuberg's article, "Sur Quelques Systèmes de Tiges Articulées" (above), contains a good deal of information on Linkages.

The short communication by Mr. H. M. Taylor (p. 127) is embodied in an article entitled, "On a Geometrical Interpretation of the Algebraical Expression which equated to zero represents a Curve or a Surface," in the *Messenger of Mathematics* for July, 1886 (Vol. XVI., No. CLXXXIII., pp. 39—41).

Mr. J. Griffiths sends the following Note:

"In a paper published in the *Proceedings* of the Society (Vol. XIV., p. 196), it was shown by me that the integral relation

$$t = p + q \operatorname{sn} u \operatorname{sn} (u - u_0)$$

gives a differential equation

$$\frac{dt}{\sqrt{T}} = M du, \text{ where } T = (a_0, a_1, a_2, a_3, a_4)(t, 1)^4.$$

The object of the present note is to point out that the above leads to an easy method of finding the integral of

$$\frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}} = 0,$$